Apollonius Points and Anharmonic Ratios

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Abstract. We give a characterization of Möbius transformation by use of Apollonius points introduced by Haruki and Rassias [2]. Our result is stronger than theirs.

1. Introduction

In their paper [2], Haruki and Rassias introduced a concept of Apollonius points for three distinct points \(z_1, z_2, z_3\) in the complex plane. \(z \in \mathbb{C}\) is called an Apollonius point of \(z_1, z_2, z_3\) if

\[ |z_1 - z_2| \cdot |z_3 - z| = |z_2 - z_3| \cdot |z_1 - z| = |z_3 - z_1| \cdot |z_2 - z|.\]

It is easy to see that this equation is equivalent to

\[ [z_1, z_2; z_3, z] = \frac{1 \pm \sqrt{3}i}{2}, \quad (1.1)\]

where the left hand side is the anharmonic ratio of \(z_1, z_2, z_3, z\). Namely, by definition,

\[ [z_1, z_2; z_3, z] = \frac{z_1 - z_3}{z_3 - z_2} \cdot \frac{z_2 - z}{z - z_1}.\]

Thus there are generally two Apollonius points for \(z_1, z_2, z_3\); one inside the circle through \(z_1, z_2, z_3\); and the other outside the circle.

Haruki and Rassias have proved that a complex analytic univalent function \(w = f(z)\) which preserves Apollonius points must be a Möbius transformation. Here we say that \(f\) preserves Apollonius points if \(f(z)\) is an Apollonius point of \(f(z_1), f(z_2), f(z_3)\) whenever \(z\) is an Apollonius point of \(z_1, z_2, z_3\). We extend this result and will prove the following.

Theorem. Let \(U \subset \mathbb{C}\) be a domain and \(f: U \to \mathbb{C}\) be a \(C^1\)-mapping (may not necessarily be complex analytic). If \(f\) preserves Apollonius points, then \(f\) is a Möbius transformation or its conjugate.

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2. Functions which preserve an anharmonic ratio

In this section we will prove the following theorem from which together with (1.1) Theorem in Introduction follows immediately.

**Theorem 2.1.** Let \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) be not a real number. Suppose \( f : U \rightarrow \mathbb{C} \) is a \( C^1 \)-mapping such that \([f(z_1), f(z_2); f(z_3), f(z_4)] = \lambda \) if \([z_1, z_2; z_3, z_4] = \lambda \). Then \( f \) is a Möbius transformation.

The proof of Theorem 2.1 is divided into two steps. One is the following.

**Proposition 2.2.** Let \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) be not a real number. Suppose \( f : U \rightarrow \mathbb{C} \) is a \( C^1 \)-mapping such that \([f(z_1), f(z_2); f(z_3), f(z_4)] = \lambda \) if \([z_1, z_2; z_3, z_4] = \lambda \). Then \( f \) is complex analytic.

The latter half is the following.

**Proposition 2.3.** Suppose \( \lambda \in \mathbb{C} \setminus \{0, 1\} \), and \( f : U \rightarrow \mathbb{C} \) is a complex analytic function such that \([f(z_1), f(z_2); f(z_3), f(z_4)] = \lambda \) if \([z_1, z_2; z_3, z_4] = \lambda \). Then \( f \) is a Möbius transformation.

**Proof of Proposition 2.2.** Choose \( a, b, c, d \in \mathbb{C} \) such that \( a, b, c \in \mathbb{R} \) and \([a, b; c, d] = \lambda \). The condition that \( \lambda \) is not real means that \( d \) is not real. Let \( z \in U \) and \( t \in \mathbb{C} \setminus \{0\} \) be small enough so that \( z + ta, z + tb, z + tc, z + td \in U \). We remark that \([z + ta, z + tb; z + tc, z + td] = \lambda \). From the Taylor development,

\[
[f(z + ta), f(z + tb); f(z + tc), f(z + td)] = [a, b; c, d] = \lambda.
\]

Hence we have

\[
[f(z + ta), f(z + tb); f(z + tc), f(z + td)] = \frac{\partial z f(z) t(a - c) + \tilde{\partial}_z f(z) \hat{t} (\tilde{\alpha} - \tilde{\beta})}{\partial z f(z) t(c - b) + \tilde{\partial}_z f(z) \hat{t} (\tilde{\beta} - \tilde{\alpha})} \cdot \frac{\partial z f(z) t(b - d) + \tilde{\partial}_z f(z) \hat{t} (\tilde{\beta} - \tilde{\alpha})}{\partial z f(z) t(d - a) + \tilde{\partial}_z f(z) \hat{t} (\tilde{\alpha} - \tilde{\beta})} + o(t).
\]

Since \( a, b \) and \( c \) are real, we obtain

\[
[f(z + ta), f(z + tb); f(z + tc), f(z + td)]
\]

\[
= \frac{(\partial z f(z) t + \tilde{\partial}_z f(z) \hat{t})(a - c)}{(\partial z f(z) t + \tilde{\partial}_z f(z) \hat{t})(c - b)} \cdot \frac{(\partial z f(z) t + \tilde{\partial}_z f(z) \hat{t})b - (\partial z f(z) t + \tilde{\partial}_z f(z) \hat{t})d}{(\partial z f(z) t + \tilde{\partial}_z f(z) \hat{t})a - (\partial z f(z) t + \tilde{\partial}_z f(z) \hat{t})b} + o(t)
\]

\[
= \left[\begin{array}{c} a, b; c, \frac{\partial z f(z) t + \tilde{\partial}_z f(z) \hat{t} d}{\partial z f(z) t + \tilde{\partial}_z f(z) \hat{t}} \end{array}\right] + o(t).
\]

From the assumption we see that the first term must converge as \( t \) goes to 0 and hence be equal to \( \lambda = [a, b; c, d] \). That is, we have

\[
\frac{\partial z f(z) t + \tilde{\partial}_z f(z) \hat{t} d}{\partial z f(z) t + \tilde{\partial}_z f(z) \hat{t}} = d.
\]
This implies $\bar{\partial}_z f(z) = 0$ because $\bar{d} \neq \bar{\bar{d}}$. Thus $f$ satisfies the Cauchy-Riemann equation. □

**Proof of Proposition 2.3.** Choose $a, b, c, d \in \mathbb{C}$ such that $[a, b; c, d] = \lambda$. The condition $\lambda \neq 1$ implies $a \neq b$ and $c \neq d$. The formula (11) of Ahlfors [1] says that for a complex analytic function $f$

$$[f(z + ta), f(z + tb); f(z + tc), f(z + td)]$$

$$= [a, b; c, d]\left(1 + \frac{1}{6}(a - b)(c - d)Sf(z)t^2 + o(t^2)\right),$$

where $Sf$ is the Schwarzian derivative of $f$ defined as

$$Sf = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

Therefore $[f(z + ta), f(z + tb); f(z + tc), f(z + td)] = \lambda$ yields $Sf(z) = 0$. This implies that $f$ is a linear fractional function. □

**References**


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