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<th>Block intersection numbers of block designs. II</th>
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1. Introduction

Let $t$, $v$, $k$, and $\lambda$ be positive integers with $v \geq k \geq t$. A $t$-$(v, k, \lambda)$ design is a pair consisting of a $v$-set $\Omega$ and a family $B$ of $k$-subsets of $\Omega$, such that each $t$-subset of $\Omega$ is contained in just $\lambda$ elements of $B$. Elements of $\Omega$ and $B$ are called points and blocks, respectively. A $t$-$(v, k, 1)$ design is often called a Steiner system $S(t, k, v)$. A $t$-$(v, k, \lambda)$ design is called nontrivial provided $B$ is a proper subfamily of the family of all $k$-subsets of $\Omega$, then $t < k < v$. In this paper we assume that all designs are nontrivial. For a $t$-$(v, k, \lambda)$ design $D$ we use $\lambda_i(0 \leq i \leq t)$ to represent the number of blocks which contain a given set of $i$ points of $D$. Then we have

$$\lambda_i = \binom{v}{t-i} \binom{k}{i} \frac{(v-i)(v-i-1)\cdots(v-t-1)}{(k-i)(k-i-1)\cdots(k-t-1)} \lambda \quad (0 \leq i \leq t).$$

A $t$-$(v, k, \lambda)$ design $D$ is called block-schematic if the blocks of $D$ form an association scheme with the relations determined by size of intersection (cf. [3]). Any Steiner system $S(2, k, v)$ ($t=2$) is block-schematic (cf. [2]). For a block $B$ of a $t$-$(v, k, \lambda)$ design $D$ we use $x_i(B)$ ($0 \leq i \leq k$) to denote the number of blocks each of which has exactly $i$ points in common with $B$. If, for each $i$ ($i=0, \ldots, k$), $x_i(B)$ is the same for every block $B$, we say that $D$ is block-regular and we write $x_i$ instead of $x_i(B)$. Any Steiner system $S(t, k, v)$ is block-regular (cf. [6]), and any block-schematic $t$-$(v, k, \lambda)$ design is also block-regular.

Atsumi [1] proved

**Result 1.** If a Steiner system $S(t, k, v)$ is block-schematic with $t \geq 3$, then $v \leq k^t \left( \begin{array}{c} k \\ 2 \end{array} \right)$ holds.

Yoshizawa [7] extended Result 1 and prove

**Result 2.** (a) For each $n \geq 1$ and $\lambda \geq 1$, there exist at most finitely many block-schematic $t$-$(v, k, \lambda)$ designs with $k-t=n$ and $t \geq 3$. 
(b) For each \(n \geq 1\) and \(\lambda \geq 2\), there exist at most finitely many block-schematic \(t-(v, k, \lambda)\) designs with \(k-t=n\) and \(t \geq 2\).

In §2 we first prove the following proposition, and we prove the following theorem related to the above results.

**Proposition.** \(x^2_{i-1} \geq x_0\) holds for any block-schematic Steiner system \(S(t, k, v)\) with \(k \geq 2(t-1)\).

**Theorem 1.** Let \(\varepsilon\) be a positive real number. Then for each \(t \geq 3\) there exist at most finitely many block-schematic Steiner systems \(S(t, k, v)\) with \(v < \lambda^{2-t}\), and for each \(t > \frac{2}{\varepsilon} + 2\) there exist at most finitely many block-schematic Steiner systems \(S(t, k, v)\) with \(v > \lambda^{2+t}\).

Yoshizawa [7] proved the following result about block-regular designs.

**Result 3.** Let \(c\) be a real number with \(c > 2\). Then for each \(n \geq 1\) and \(l \geq 0\), there exist at most finitely many block-regular \(t-(v, k, \lambda)\) designs with \(k-t=n\), \(v \geq ct\) and \(x_i \leq l\) for some \(i\) (0 \(\leq i \leq t-1\)).

In §3 we notice that the block-regularity of Result 3 is essentially unnecessary, and we prove

**Theorem 2.** Let \(c\) be a real number with \(c > 2\), and \(n, l\) be integers with \(n \geq 1\), \(l \geq 0\). Then there exist at most finitely many \(t-(v, k, \lambda)\) designs each of which satisfies the following conditions: (i) \(k-t=n\), (ii) \(v \geq ct\), (iii) there exist a block \(B\) and an integer \(i (0 \leq i \leq t-1)\) with \(x_i (B) \leq l\).

2. Proof of Theorem 1

Let \(D\) be a \(t-(v, k, \lambda)\) design. Let \(B_1, \ldots, B_{\lambda^0}\) be the blocks of \(D\), and \(A_h (0 \leq h \leq k)\) be the \(h\)-adjacency matrix of \(D\) of degree \(\lambda_0\) defined by

\[
A_h(i,j) = \begin{cases} 1 & \text{if } |B_i \cap B_j| = h, \\ 0 & \text{otherwise.} \end{cases}
\]

If \(D\) is block-schematic, then

\[
A_i A_j = \sum_{h=0}^{k} \mu(i, j, h) A_h(0 \leq i, j \leq k)\]

where \(\mu(i, j, h)\) is a non-negative integer defined by the following: When there exist blocks \(B_p\) and \(B_q\) with \(|B_p \cap B_q| = h\),

\[
\mu(i, j, h) = |\{B_r: |B_r \cap B_p| = i, |B_r \cap B_q| = j, 1 \leq r \leq \lambda_0\}|,
\]

and when there exist no blocks \(B_p\) and \(B_q\) with \(|B_p \cap B_q| = h\), \(\mu(i, j, h) = 0\). Let \(a\) be the all \(-1\) column vector of degree \(\lambda_0\). Then

\[
A_i A_j a = \sum_{h=0}^{k} \mu(i, j, h) A_h a.
\]
Lemma 1. For a block-schematic t-(v, k, λ) design, \( x_i x_j = \sum_{h=0}^{k} \mu(i, j, h) x_h \) holds (0 \( \leq i, j \leq k \)).

Remark. Lemma 1 is essentially well-known (cf. [1], [7]).

Lemma 2. Let \( D \) be a Steiner system \( S(t, k, v) \) with \( t \geq 2 \) and \( k \geq 2(t-1) \). If \((t, k, v) = (4, 7, 23), (2, n+1, n^2+n+1) (n \geq 2), \) then there exist three blocks \( B_1, B_2 \) and \( B_3 \) of \( D \) such that \( |B_1 \cap B_2| = |B_1 \cap B_3| = t-1 \) and \( |B_2 \cap B_3| = 0 \).

Proof. By [6] we have

\[
\lambda_{t-1} = (\lambda_{t-1} - 1) \left( \frac{k}{t-1} \right) = \frac{v-k}{k-t+1} \left( \frac{k}{t-1} \right) > 0.
\]

Hence we may assume that there exist two blocks \( B_1 \) and \( B_2 \) with \( |B_1 \cap B_2| = t-1 \). Since \( k \geq 2(t-1) \), \( B_1 - B_2 \) has (distinct) \( t-1 \) points \( \alpha_i, \ldots, \alpha_{t-1} \). Let \( M_i \) \((=B_i)\), \( M_2, \ldots, M_{\lambda_{t-1}} \) be the blocks which contain \( \alpha_i, \ldots, \alpha_{t-1} \). If \( M_i \cap B_2 = \emptyset \) for some \( i \) \((2 \leq i \leq \lambda_{t-1})\), then \( |B_1 \cap M_i| = t-1 \) and \( |B_2 \cap M_i| = 0 \) hold. Let us suppose \( M_i \cap B_2 = \emptyset \) for \( i=2, \ldots, \lambda_{t-1} \). Then we have

\[
\frac{v-t+1}{k-t+1} \leq k-t+1.
\]

On the other hand by Theorems 3A. 3 and 4 in [4], we have \( v-t+1 \geq (k-t+2)(k-t+1) \), with equality only when \((t, k, v) = (2, n+1, n^2+n+1) (n \geq 2), (3, 4, 8), (3, 6, 22), (3, 12, 112), (4, 7, 23) \) or \((5, 8, 24)\). Hence by (1) and the assumption of Lemma 2, we have \((t, k, v) = (3, 4, 8), (3, 6, 22), (3, 12, 112) \) or \((5, 8, 24)\). But we can easily check that \( S(3, 4, 8), S(3, 6, 22), S(3, 12, 112) \) and \( S(5, 8, 24) \) satisfy the conclusion of Lemma 2 if \( S(3, 12, 112) \) exists (cf. [5, Corollary 1]).

Proof of Proposition. Let us suppose that \( D \) is a block-schematic Steiner system \( S(t, k, v) \) with \( k \geq 2(t-1) \). Then by Lemma 1, we have

\[
\lambda_{t-1} = \sum_{h=0}^{k} \mu(t-1, t-1, h) x_h.
\]

Now by Lemma 2, \( \mu(t-1, t-1, 0) > 0 \) or \( x_0 = 0 \) holds when \( k \geq 2(t-1) \) and \( t \geq 2 \) hold. Hence we have \( \lambda_{t-1} > x_0 \).

Proof of Theorem 1. First let us suppose that \( D \) is a block-schematic Steiner system \( S(t, k, v) \) with \( t \geq 3 \) and \( v < k^3 r^4 \). By Theorems 3A. 3 and 4 in [4], we have \( v \geq (k-t+2)(k-t+1)+t-1 \), where the right hand of this
inequality is a polynomial in $k$ of degree two. Hence there exists a positive number $N(\varepsilon, t)$ with $k < N(\varepsilon, t)$, where $N(\varepsilon, t)$ depends only on $\varepsilon$ and $t$. Hence by Result 1, $v$ is bounded above by a function of $\varepsilon$ and $t$.

Next suppose that $D$ is a block-schematic Steiner system $S(t, k, v)$ with $t \geq \frac{2}{\varepsilon} + 2$ and $v \geq k^{t+1}$.

By [6] we have

$$x_{i-1}^2 = (n_{i-1} - 1)^2 \left( \frac{k}{t-1} \right)^2 = \frac{(v-k)^2}{(k-t+1)^2} \left( \frac{k}{t-1} \right)^2 \tag{2}$$

By [5, Lemma 6] (or [7, Lemma 5]) we have

$$x_0 = \left\{ \left( \frac{v-k}{k} \right) + (-1)^{i+1} \sum_{s=0}^{t-1} \frac{t-1+q}{q} \left( \frac{v-k+q}{k-t} \right) \right\} \left( \frac{v-t}{k-t} \right),$$

$$x_0 \geq \left( \frac{v-k}{k-t} \right) \left( \frac{v-t}{k-t} \right) \left( \frac{k-2}{k-t-1} \right) \left( \frac{k}{k-t} \right) \left( \frac{v-t}{k-t} \right). \tag{3}$$

Hence by (2) and (3) we have

$$x_0 - x_{i-1}^2 \geq \frac{(v-k) \cdots (v-2k+1)}{(v-t) \cdots (v-k+1) k \cdots (k-t+1)} - (k-t)(k-2)^{i-1} \frac{(v-k)^2}{k-t} \left( \frac{k}{t-1} \right)^2,$$

$$x_0 - x_{i-1}^2 \geq \frac{(v-2k)^i}{\nu^{k-t} k^t} - 4v^2 k^{2t-4} \geq \frac{(v-2k)^i}{\nu^{k-t} k^t} - 5v^2 k^{2t-4}.$$  

Since we may assume $k \geq 2t$ by Result 1, we have

$$x_0 - x_{i-1}^2 \geq \frac{(v-2k)^i}{\nu^{k-t} k^t} - 4v^2 k^{2t-4} \geq \frac{(v-2k)^i}{\nu^{k-t} k^t} - 5v^2 k^{2t-4}.$$  

On the other hand by Proposition, $x_{i-1}^2 \geq x_0$ holds because of $k \geq 2t$. Thus we get

$$(v-2k)^i - 5v^{k-t+2} k^{2t-4} \leq 0. \tag{4}$$

Since $v > k^3$, we have

$$\frac{(v-2k)^i}{5v^{k-t+2} k^{2t-4}} = \frac{\left( 1 - \frac{2k}{v} \right)^{k-t+2}}{5k^{2t-4}} \geq \frac{\left( 1 - \frac{1}{k} \right)^{k-t+2}}{5 \left( 1 - \frac{1}{k} \right)^{k-t} k^{2t-4}}. \tag{5}$$

Since $\lim_{n \to \infty} \left( \frac{1}{n} \right)^n = \frac{1}{e}$, where $e$ is the Napier number, there is a positive num-
ber $M_i(t)$ which depends only on $t$, such that \( \left( \frac{1}{n} - \frac{1}{n} \right)^n \left/ \left\{ 5 \left( \frac{1}{n} - \frac{1}{n} \right)^{t-2} \right\} \right. > M_i(t) \) holds for all integer $n \geq 2$. On the other hand, \( (v-2k)^{t-2}/k^{3t-4} \geq v^{t-2}/(2^{t-2} k^{3t-4}) \) holds. Hence by (5), we have

\[
\frac{(v-2k)^{t}}{5v^{t-2} k^{3t-4}} \geq M_i(t) \frac{v^{t-2}}{k^{3t-4}},
\]

where $M_i(t)$ is a positive number which depends only on $t$. Since $v > k^{3+t}$ and $t > \frac{2}{\varepsilon} + 2$, there exists a positive number $M_3(\varepsilon, t)$ which depends only on $\varepsilon$ and $t$, such that

\[
\frac{(v-2k)^{t}}{5v^{t-2} k^{3t-4}} > 1 \text{ holds for any } k \geq M_3(\varepsilon, t).
\]

Hence by (4), we must have $k < M_3(\varepsilon, t)$. Hence by Result 1, $v$ is bounded above by a function of $\varepsilon$ and $t$.

3. Proof of Theorem 2

The proof of Theorem 2 is essentially similar to that of Theorem 2 in [7]. So we give its outline.

Let $D$ be a $t-(v, k, \lambda)$ design, and $B$ be a block of $D$. Counting in two ways the number of the following set

\[
(B', \{\alpha_1, \cdots, \alpha_t\}): B' \text{ a block } (\not\equiv B), B' \cap B \not\equiv \alpha_1, \cdots, B' \cap B \not\equiv \alpha_t, \alpha_j \not\equiv \alpha_{j'} \text{ if } j \not\equiv j'
\]

gives

\[
x_i(B) + \binom{i+1}{i} x_{i+1}(B) + \cdots + \binom{t}{i} x_t(B) + \cdots + \binom{k-1}{i} x_{k-1}(B) = (\lambda - 1) \binom{k}{i},
\]

for $i = 0, \cdots, t-1$, and

\[
x_i(B) + \binom{i+1}{i} x_{i+1}(B) + \cdots + \binom{k-1}{i} x_{k-1}(B) \leq (\lambda - 1) \binom{k}{i},
\]

for $i = t, \cdots, k-1$. Let $w_i(B)$ \((t \leq i \leq k-1)\) be the left hand of the above inequality, where $w_i(B) = (\lambda - 1) \binom{k}{i}$.

By (6) and (7) we have

\[
x_i(B) = \sum_{j=0}^{i-1} \binom{i}{i} (\lambda_j - 1) \binom{k}{j} (-1)^{i+j} + \sum_{j=i}^{k-1} \binom{i}{i} w_j(B) (-1)^{i+j},
\]

for $i = 0, \cdots, t-1$ (cf. [7, Proof of Lemma 1]). By (8) we have that there exists
a positive number $C(k, l, t, i)$ which depends only on $k, l, t, i$, such that $x_i(B) - l > 0$ holds if $v \geq C(k, l, t, i)$ (cf. [7, Proof of Lemma 6]). Namely, $v < C(k, l, t, i)$ holds if $x_i(B) \leq l$. Hence we get

**Lemma 3.** For each $k \geq 2$ and $l \geq 0$, there exist at most finitely many $t$-$(v, k, \lambda)$ designs each of which satisfies that there exists a block $B$ and an integer $i$ \((0 \leq i \leq t-1)\) with $x_i(B) \leq l$.

Proof of Theorem 2. By Lemma 3 we may assume that \(t \geq \frac{2n + ((2n + 2)!)}{c-2} + 2n\). Let $D$ be a $t$-$(v, k, \lambda)$ design satisfying $v \geq ct$ and $t \geq \frac{2n + ((2n + 2)!)}{c-2} + 2n$. Set $v = mt$ \((m \geq c)\), where $m$ is not always integral.

By (8) we have

$$x_i(B) = \frac{\lambda(\binom{k}{i})}{\binom{v-t}{k-t}} \left[ (v-k) + (-1)^{i+1} \sum_{i=0}^{t-1} \binom{t-i-1+q}{q} (v-k+q) \right]$$

$$+ (\lambda-1) \sum_{j=1}^{t-1} \binom{k}{j} (-1)^{i+j} \sum_{j=1}^{t-1} \binom{j}{i} w_j(B) (-1)^{j+i},$$

where $x_i(B) \leq w_j(B) \leq (\lambda-1) \binom{k}{j} = (\lambda-1) \binom{t+n}{j} \quad (t \leq j \leq k-1)$

(cf. [7, Proof of Lemma 5]). Hence we get

$$x_i(B) \geq \frac{\lambda(\binom{t+n}{i})}{\binom{(m-1)t-n}{n}} \left[ (m-1) \binom{t-n}{n} + (-1)^{i+1} \sum_{i=0}^{t-1} \binom{t-i-1+q}{q} (m-1) \binom{t-n+q}{n} \right]$$

$$+ (\lambda-1) \sum_{j=1}^{t-1} \binom{j}{i} (-1)^{i+j} \sum_{j=1}^{t-1} \binom{j}{i} w_j(B) (-1)^{j+i},$$

for $i = 0, \ldots, t-1$. By the above equality and the condition on $t$, we have

$$x_i(B) \geq \frac{(c-1) \binom{t-n}{n}}{(n+1) t^2} \left( \frac{c-2}{c-1} \right)^n - 5n \quad \text{(cf. [7, pp. 797, 798])}.$$

We remark that the right hand of the above inequality does not depend on $i$.

Hence there exists a positive number $N(c, n, l)$ \(\geq \frac{2n + ((2n + 2)!)}{c-2} + 2n\) which depends only on $c, n, l$, such that $x_i(B) - l > 0$ holds for $i = 0, \ldots, t-1$ if $t \geq N(c, n, l)$. Namely, $t < N(c, n, l)$ holds if $x_i(B) \leq l$ holds for some $i$ \((0 \leq i \leq t-1)\). Hence by Lemma 3, we complete the proof.
References


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