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ON t-DESIGNS

DIJEN K. RAY-CHAUDHURI* AND RICHARD M. WILSON**

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Introduction and preliminaries

An incidence structure is a triple $S=(X, \mathcal{A}, \mathcal{J})$ where $X$ and $\mathcal{A}$ are disjoint sets and $\mathcal{J} \subseteq X \times \mathcal{A}$. Elements $x \in X$ are called points and elements $A \in \mathcal{A}$ are called blocks of $S$. A point $x$ and a block $A$ are incident iff $(x, A) \in \mathcal{J}$. For any block $A$, $(A)$ will denote the set of points incident with $A$.

Let $v, k, t$ and $\lambda$ be integers with $v > k > t > 0$ and $\lambda > 1$. An $S_\lambda(t, k, v)$ (a $t$-design on $v$ points with block size $k$ and index $\lambda$) is an incidence structure $D=(X, \mathcal{A}, \mathcal{J})$ such that

(i) $|X| = v$,
(ii) $|(A)| = k$ for every $A \in \mathcal{A}$,
(iii) for every $t$-subset $T$ of $X$, there are exactly $\lambda$ blocks $A \in \mathcal{A}$ with $T \subseteq (A)$.

It is well known that every $S_\lambda(t, k, v)$ has exactly $b = \lambda \binom{v}{t} \binom{k}{t}$ blocks and more generally, for any $i$-subset $I$ of points ($0 \leq i \leq t$), the number of blocks $A$ of the design with $I \subseteq (A)$ is

$$b_i = \lambda \binom{v-i}{t-i} \binom{k-i}{t-i},$$

independent of the subset $I$ [2].

Abstract: We present the generalization (conjectured by A. Ja. Petrenjuk) of Fisher's Inequality $b \geq v$ for 2-designs and Petrenjuk's Inequality $b \geq \binom{v}{2}$ for 4-designs. The $t$-designs satisfying the inequality with equality may be considered as generalizations of the symmetric 2-designs ($b = v$) and have the property that there are exactly $\frac{1}{2} t$ possible values for the size of the intersection of two distinct blocks, these values being computable from the parameters.

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An $S_\lambda(t, k, v)$, say $D=(X, \mathcal{A}, \mathcal{J})$, is simple when the mapping $A \mapsto (A)$ from $\mathcal{A}$ into $\mathcal{P}_k(X)$ (the class of all $k$-element subsets of $X$) is injective; and $D$ is trivial when the mapping $A \mapsto (A)$ is (surjective and) $m$-to-one for some integer $m$, i.e. each $k$-subset "occurs as a block" exactly $m$ times. In this latter case, evidently $\lambda = m(v-t)k^t$.

The well known Fisher's Inequality (see [2]) asserts that the number $b$ of blocks of an $S_\lambda(2, k, v)$ is at least $v$, under the assumption $v \geq k+1$. A. Ja. Petrenjuk [4] proved in 1968 that $b \geq \binom{v}{2}$ for any $S_\lambda(4, k, v)$ with $v \geq k+2$ and conjectured that $b \geq \binom{v}{s}$ in any $S_\lambda(2s, k, v)$ with $v \geq k+s$. This conjecture is established in the following section.

This condition shows the nonexistence of certain $t$-designs. For example, Petrenjuk's Inequality shows that $S_\lambda(4, 22, 79)$ do not exist even though the $b_i$'s $(0 \leq i \leq 4)$ are integral. We might note that a hypothetical $S_\lambda(4, k, 2 + \frac{1}{2}(k-1)(k-2))$ would satisfy $b = \binom{v}{2}$ (and the $b_i$'s are integral when $k \equiv 1 \pmod{4}$), but no such designs exist by the corollary of Theorem 5 below. The inequality $b \geq \binom{v}{3}$ rules out the entire family of 6-designs with $v = 120m$, $k = 60m$, $\lambda = (20m-1)(15m-1)(12m-1)$, (for which the $b_i$'s are integral).

By a tight $t$-design ($t$ even, say $t = 2s$) we mean an $S_\lambda(t, k, v)$ with $v \geq k+s$ and $b = \binom{v}{s}$. As examples, we have the trivial designs $S_\lambda(2s, k, k+s)$ where $\lambda = \binom{k-s}{k-2s}$. An example of a tight 4-design is the well known $S_\lambda(4, 7, 23)$ where $b = 253 = \binom{23}{2}$. N. Ito [3] has recently shown, using Theorem 5 below, that the only nontrivial tight 4-designs are the $S_\lambda(4, 7, 23)$ and its complement, an $S_{st}(4, 16, 23)$. Tight $t$-designs with $t \geq 4$ seem to be very rare.

Our proof of Petrenjuk's conjecture uses only elementary linear algebra and the observation that the number of blocks of an $S_\lambda(t, k, v)$ which are incident with some $i$ points and not incident some other $j$ points is constant (i.e., depends only on $i, j$, and the parameters; not the particular sets of points) whenever $i+j \leq t$.

**Proposition 1.** Let $(X, \mathcal{A}, \mathcal{J})$ be an $S_\lambda(t, k, v)$. Let $i$ and $j$ be nonnegative integers with $i+j \leq t$. Then for any subsets $I, J \subseteq X$ with $|I| = i$, $|J| = j$, 

\[ \binom{v-t}{k-t} \]
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$I \cap J = \phi$, the number of blocks $A \in \mathcal{A}$ such that $I \subseteq (A)$ and $J \cap (A) = \phi$ is exactly

$$b^J_t = \lambda \frac{\binom{v-i-j}{k-i}}{\binom{k-t}{k-t}}.$$  

Proof. By inclusion-exclusion,

$$b^J_t = \sum_{r=0}^{j} (-1)^r \binom{j}{r} b_{t+r}.$$  

In view of the above expression for $b_i$, we have $b^J_t = \lambda c$ where

$$c = \sum_{r=0}^{j} (-1)^r \binom{j}{r} \binom{v-i-r}{t-i-r} \binom{k-i-r}{t-i-r}^{-1}.$$  

But in the case of the trivial design $(X, \mathcal{P}_k(X), \subseteq)$, $\lambda^* = \binom{v-t}{k-t}$ and $b^J_t = \binom{v-i-j}{k-i}$, from which we deduce the simpler expression $c^* = \binom{v-i-j}{k-i} \binom{v-t}{k-t}^{-1}$.

As a corollary, the complement $(X, \mathcal{A}, (X \times \mathcal{A}) - \mathcal{J})$ of an $S_\lambda(t, k, v)$ is an $S_{\lambda^*}(t, v-k, v)$ with

$$\lambda^* = b^*_t = \lambda \binom{v-t}{k} \binom{v-t}{k-t}^{-1},$$  

(unless $v < k + t$, in which case the original $S_\lambda(t, k, v)$ is evidently trivial).

2. Generalizations of Fisher's inequality

For any set $Y$, we denote by $V(Y)$ the free vector space over the rationals generated by $Y$, i.e. $V(Y)$ consists of all formal sums $\alpha = \sum_{y \in Y} a_y y$ with rational coefficients $a_y$ and formal addition and scalar multiplication. The "unit vectors" $y, y^* \in Y$, by definition provide a basis for $V(Y)$.

Theorem 1. The existence of an $S_\lambda(t, k, v)$ with $t$ even, say $t = 2s$, and $\nu \geq k+s$ implies

$$b \geq \binom{\nu}{s},$$  

where $b$ is the number of blocks of the design. In fact, the number of distinct subsets $(A)$ is itself at least $\binom{\nu}{s}$.

Proof. Let $D = (X, \mathcal{A}, \mathcal{J})$ be an $S_\lambda(t, k, v)$ and put $V_s = V(\mathcal{P}_s(X))$, where $\mathcal{P}_s(X)$ is the class of all $s$-element subsets of $X$. For each block $A$ of $D$, define a vector $\tilde{A} \in V_s$ as the "sum" of all $s$-subsets of $(A)$, i.e.
\[ \hat{A} = \sum (S: S \in \mathcal{P}_s(X), S \subseteq \mathcal{A}) \]

We claim that the set of vectors \( \{ \hat{A}: A \in \mathcal{A} \} \) spans \( V_s \). Since \( V_s \) has dimension \( \binom{v}{s} \), the theorem follows immediately.

Let \( S_0 \in \mathcal{P}_s(X) \). To show \( S_0 \) belongs to the span of \( \{ \hat{A}: A \in \mathcal{A} \} \), we introduce the vectors

\[ E_i = \sum (S: S \in \mathcal{P}_s(X), |S \cap S_0| = s-i) \]

(so \( E_0 = S_0 \)) and

\[ F_i = \sum (A: A \in \mathcal{A}, |(A) \cap S_0| = s-i) \]

for \( i = 0, 1, \ldots, s \). Now for \( S_i \in \mathcal{P}_s(X) \) with \( |S_i \cap S_0| = s-i \), the coefficient of \( S_i \) in the sum \( F_r \) is the number of blocks \( A \) such that \( S_i \subseteq (A) \) and \( |(A) \cap S_0| = s-r \); and this number is \( \binom{i}{r} b_{s-r+i}^r \) with the notation of Proposition 1. Thus

\[ F_r = \sum_{i=0}^{s} \binom{i}{r} b_{s-r+i}^r E_i \quad (r = 0, 1, \ldots, s). \]

The above system of linear equations is triangular and the diagonal coefficients \( b_r^s \) (\( r = 0, 1, \ldots, s \)) are all nonzero under our hypothesis \( v \geq k+s \). Thus we can solve for the \( E_i \)'s (in particular, for \( E_0 = S_0 \)) as linear combinations of the \( F_r \)'s. Since the \( F_r \)'s are by definition in the span of \( \{ \hat{A}: A \in \mathcal{A} \} \), we have \( S_0 \in \text{span} \{ \hat{A}: A \in \mathcal{A} \} \) for every \( S_0 \in \mathcal{P}_s(X) \), and our claim is verified.

**Corollary.** The existence of an \( S_\lambda(t, k, v) \) with \( t \) odd, say \( t = 2s+1 \) and \( (v-1) \geq k+s \) implies the inequality

\[ b = \frac{\lambda \binom{v}{2s+1}}{\binom{k}{2s+1}} \geq \frac{\lambda \binom{v-1}{2s}}{\binom{k-1}{2s}} + \binom{v-1}{s} \geq 2 \binom{v-1}{s}. \]

**Proof.** Let \( D=(X, \mathcal{A}, \mathcal{J}) \) be an \( S_\lambda(t, k, v) \) and \( x \in X \). Let \( \mathcal{A}' \) be the class of blocks incident with \( x \) and \( \mathcal{A}'' \) be the class of blocks not incident with \( x \). Observe that both \( D'=(X', \mathcal{A}', \mathcal{J} \cap (X' \times \mathcal{A}')) \) and \( D''=(X', \mathcal{A}'', \mathcal{J} \cap (X' \times \mathcal{A}'')) \), where \( X'=X-\{x\} \), are 2s-designs and apply Theorem 1.

The above inequality also rules out infinitely many parameters for which \( b_i \)'s are integers, \( i = 0, 1, \ldots, t \).

**Theorem 2.** Let \( D=(X, \mathcal{A}, \mathcal{J}) \) be an \( S_\lambda(t, k, v) \) where \( t = 2s \) and \( v \geq k+s \). If there exists a partition \( \mathcal{A}=\mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_r \) such that each substructure \( (X, \mathcal{A}_i, \mathcal{J} \cap (X \times \mathcal{A}_i)) \) is an \( S_\lambda_i(s, k, v) \) for some positive integers \( \lambda_i \), then
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Proof. With the notation of Theorem 1, the vectors \( \{ A : A \in \mathcal{A} \} \) span \( V \). But observe that

\[
\sum \{ A : A \in \mathcal{A}_i \} = \lambda_i \sum (S : S \in \mathcal{P}(X)) = \lambda_i \tilde{X}, \text{ say.}
\]

So if we choose one block \( A_i \) from each \( \mathcal{A}_i \), then \( \{ A : A \in \mathcal{A} - \{ A_1, \ldots, A_s \} \} \cup \{ \tilde{X} \} \) spans \( V \). The stated inequality follows.

3. Tight t-designs

Recall that a tight \( t \)-design \((t=2s)\) is an \( S_\lambda(t, k, v) \) with \( v \geq k + s \) and

\[
b = \lambda \binom{v}{t} \binom{k}{s} = \binom{v}{s}.
\]

In view of Theorem 1, tight designs are simple. In this section we extend the well known result that two distinct blocks of a symmetric design (tight \( 2 \)-design) have exactly \( \lambda \) common incident points (see Theorem 4 below).

**Theorem 3.** Let \( X \) be a \( v \)-set and \( \mathcal{A} \) a class of \( k \)-subsets of \( X \) such that for distinct \( A, B \in \mathcal{A} \),

\[
|A \cap B| \in \{ \mu_1, \mu_2, \ldots, \mu_s \}
\]

where \( k > \mu_1 > \mu_2 \cdots > \mu_s > 0 \). Then

\[
|\mathcal{A}| \leq \binom{v}{s}.
\]

**Proof.** Let \( V = V(\mathcal{A}) \). For each \( S \in \mathcal{P}(X) \), define a vector

\[
\bar{S} = \sum (A : A \in \mathcal{A}, A \supseteq S).
\]

We claim that the vectors \( \{ \bar{S} : S \in \mathcal{P}(X) \} \) span \( V \). Since \( V \) has dimension \( |\mathcal{A}| \), the theorem will follow.

Write \( \mu_0 = k \). Let \( A_0 \in \mathcal{A} \) be given. Define

\[
H_i = \sum (B : B \in \mathcal{A}, |B \cap A_0| = \mu_i)
\]

for \( i = 0, 1, \ldots, s \) (note \( H_0 = A_0 \)). For \( r = 0, 1, \ldots, s \), we see that

\[
G_r = \sum (\bar{S} : S \in \mathcal{P}(X), |S \cap A_0| = r) = \sum_{i=0}^{s} \binom{\mu_i}{r} \binom{k-\mu_i}{s-r} H_i,
\]

by comparing the coefficient of each \( A \in \mathcal{A} \) on both sides of the equation. We now show that the coefficient matrix of this system of \( s+1 \) linear equations is
nonsingular, so that we can solve for the $H_i$'s in terms of the $G'_r$'s. In particular, we then have $H_0 = A_0 \in \text{span} \{G_0, G_1, \ldots, G_r\} \subseteq \text{span} \{S : S \in \mathcal{D}_s(X)\}$.

So consider the $s+1$ row vectors

\[ v_r = \left(\binom{\mu_0}{r} \binom{k-\mu_0}{s-r}, \binom{\mu_1}{r} \binom{k-\mu_1}{s-r}, \ldots, \binom{\mu_s}{r} \binom{k-\mu_s}{s-r}\right), \]

$r=0, 1, \ldots, s$. Suppose $c_0 v_0 + c_1 v_1 + \cdots + c_s v_s = 0$. This means that the polynomial

\[ p(x) = \sum_{r=0}^{s} c_r \binom{x}{r} \binom{k-x}{s-r} \]

of degree $\leq s$ has $s+1$ distinct roots $\mu_0, \mu_1, \ldots, \mu_s$ and hence is the zero polynomial. Now $p(0) = c_0 \binom{k}{s}$, so $c_0 = 0$; then $p(1) = c_1 \binom{k-1}{s-1}$, so $c_1 = 0$; and, inductively, $c_0 = c_1 = \cdots = c_s = 0$. That is, $v_0, \ldots, v_s$ are linearly independent. This completes the proof.

**Theorem 4.** Let $D = (X, \mathcal{A}, \mathcal{B})$ be an $S_\lambda(t, k, v)$ with $t = 2s$ and $v \geq k + s$. Then there are at least $s$ distinct elements in the set

\[ \{(A) \cap (B) : A \in \mathcal{A}, B \in \mathcal{A}, A \neq B\}, \]

and there are exactly $s$ distinct elements if and only if $D$ is a tight $t$-design. 

Proof. In view of Theorems 1 and 3, it remains only to show that for any tight $t$-design, there exist $s$ integers $\mu_1, \mu_2, \ldots, \mu_s$ with $0 \leq \mu_i < k$ so that $|(A) \cap (B)| \in \{\mu_1, \ldots, \mu_s\}$ for distinct blocks $A$ and $B$. Let $D = (X, \mathcal{A}, \mathcal{B})$ be a tight $S_\lambda(t, k, v)$. With the notation of Theorem 1, the $b = \binom{v}{s}$ vectors

\[ \{A : A \in \mathcal{A}\} \]

must, since they span $V_s$, be a basis for $V_s$.

Fix $A_0 \in \mathcal{A}$ and for $B \in \mathcal{A}$, write $\mu_B = |(B) \cap (A_0)|$. For $i = 0, 1, \ldots, s$, define vectors

\[ M_i = \sum (S : S \in \mathcal{D}_s(X), |S \cap (A_0)| = i), \]

\[ N_i = \sum \binom{\mu_B}{i} B : B \in \mathcal{A}. \]

Now given $S \in \mathcal{D}_s(X)$ with $|S \cap (A_0)| = i$, the coefficient of $S$ in the sum $N_r$ is

\[ \sum \binom{\mu_B}{r} : B \in \mathcal{A}, S \subseteq (B), \]

i.e., the number of ordered pairs $(B, R)$ in $\mathcal{A} \times \mathcal{D}_r(X)$ such that $S \subseteq (B)$ and $R \subseteq (A_0) \cap (B)$. For any $r$-subset $R \subseteq (A_0)$ with $|R \cap S| = j$, the number of blocks $B$ such that $(B, R)$ satisfies the above conditions is $b_{s+r-j}$. Thus the coefficient of $S$ in $N_r$ is
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\[ c_i' = \sum_{j=0}^{s-1} \binom{i}{j} \binom{k-i}{r-j} b_{s+r-j}; \text{ and so} \]

\[ N_r = \sum_{i=0}^{s} c_i' M_i \quad (r = 0, 1, \ldots, s). \]

The \( s+1 \) vectors \( N_r - c_i' M_s \) are contained in the span of \( M_0, M_1, \ldots, M_{s-1} \); hence there exist rationals \( a_0, a_1, \ldots, a_s \), not all zero, such that

\[ \sum_{i=0}^{s} a_r (N_r - c_i' M_s) = 0, \quad \text{or} \]

\[ \sum_{i=0}^{s} a_r \sum_{B \in \mathcal{A}} \binom{\mu_B}{r} B - c_i' A_0 = 0. \]

Now \( \{A: A \in \mathcal{A}\} \) is a basis for \( V_s \), so for \( B \neq A_0 \), the coefficient

\[ \sum_{i=0}^{s} a_r \binom{\mu_B}{r} \]

of \( B \) must be 0. That is, for any \( B \neq A_0 \), the intersection number \( \mu_B \) is a root of the polynomial

\[ f(x) = \sum_{i=0}^{s} a_r \binom{x}{r} \]

of degree at most \( s \). Finally, note that the coefficients \( c_i' \) are (and hence \( f(x) \) can be chosen to be) independent of the block \( A_0 \); all intersection numbers are roots of \( f(x) \).

The polynomials \( f(x) \) described in the proof of Theorem 4 have been found explicitly by P. Delsarte [1]. As an example, we consider the case \( t=4 \). The equations of Theorem 4 are

\[ N_0 = b_2 M_0 + b_2 M_1 + b_2 M_2, \]

\[ N_1 = \frac{b_3 M_0 + (b_2 + (k-1)b_2) M_1 + (2b_2 + (k-2)b_2) M_2}, {2} \]

\[ N_2 = \left( \frac{k}{2} \right) b_4 M_0 + \left( \frac{k-1}{2} \right) b_4 (k-1) b_3 M_1 + \left( \frac{k-2}{2} \right) b_4 2k+2(k-2)b_3+b_3 \right) M_2. \]

Using the relation \( b_2 = \left( \frac{k}{2} \right) \) in a tight 4-design, one verifies that

\[ (b_2 - b_2) N_2 = (k-1)(b_3 - b_3) N_1 + (2b_2 (b_3 - b_3) - b_4 (b_2 - b_3)) N_0 \]

is a scalar multiple of \( M_2 = A_0 \). For a block \( B \neq A_0 \), the coefficient of \( B \) in the above expression must be zero, i.e.,

\[ \mu_B (\mu_B - 1) - \frac{2(k-1)(b_3 - b_3)}{(b_2 - b_2)} \mu_B + \frac{4b_2 (b_3 - b_3)}{(b_2 - b_2)} = 0. \]

Rewriting the coefficients in terms of \( v, k, \) and \( \lambda \), we have
Theorem 5. The two "intersection numbers" $\mu_1, \mu_2$ of a tight 4-design $S_4(4, k, v)$ are the roots of the polynomial

$$f(x) = x^2 - \left( \frac{2(k-1)(k-2)}{v-3} + 1 \right) x + \lambda \left( 2 + \frac{4}{k-3} \right).$$

Application of Theorem 5 yields the well known fact that any two distinct blocks of an $S_4(4, 7, 23)$ meet in 1 or 3 points.

Since $f(x)$ has integral roots, it must have integral coefficients, and we have the

Corollary. The existence of a tight 4-design $S_4(4, k, v)$ implies $v - 3$ divides $2(k-1)(k-2)$, and $k - 3$ divides $4\lambda$.

In [1], Delsarte observes that Theorems 4 and 5 are similar to Lloyd's Theorem on perfect codes. Indeed, Delsarte develops a theory of designs and codes (emphasizing a "formal duality") in the context of association schemes. Contained therein are results analogous to the above for orthogonal arrays of strength $t$, the analogue of Theorem 1 being Rao's bound.

We conclude with the following remarks.

Let $D=(X, \mathcal{A}, \mathcal{J})$ be a tight $S_4(t, k, v)$ with $t=2s$ and $v \geq k+s$. Let $f(s, v)$ denote the association scheme whose points are the $s$-element subsets of $X$ (see [1]). Let $N$ be a $(0-1)$-matrix whose rows are indexed by elements of $\mathcal{P}_s(X)$ and columns are indexed by the blocks of $D$. At the row corresponding to $S$ and column corresponding to a block $A$, the entry of $N$ is 1 iff $S \subseteq (A)$. The matrix $NN^T$ belongs to the Bose-Mesner algebra of the scheme $f(s, v)$. The matrix $NN^T$ is obviously rationally congruent to the identity matrix. Using the properties of the algebra of $f(s, v)$, it is possible to compute the Hasse-Minkowski invariant of $NN^T$ and obtain some more necessary conditions for the existence of tight 2s-designs. (See also [5].)

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References


