The aim of this note is to complete the proof of the following theorem: Let $\mathcal{G}$ be a finite group which contains an element $P$ of prime order $p$ which commutes only with its own powers (condition (\#)) and assume that $\mathcal{G}$ is equal to its commutator-subgroup $\mathcal{G}'$ (condition (\##)). Then the order $g$ of $\mathcal{G}$ is expressed as $g=p(p-1)(1+np)/t$, where $1+np$ is the number of conjugate subgroups of order $p$ and $t$ is the number of classes of conjugate elements of order $p$. If $n<p+2$ and $t\equiv 0$ (mod. 2), then $p$ is of the form $2^a-1$ and $\mathcal{G}\cong LF(2, 2^a)$.

In [3], the theorem was proved for the case $n<p+2$ and $t\equiv 0$ (mod. 2): If $n<p+2$ and $t\equiv 0$ (mod. 2), under (\#) and (\##), then $p$ is of the form $2^a-1$ and $\mathcal{G}\cong LF(2, 2^a)$. In [4], the case $n=p+2$ and $t\equiv 0$ (mod. 2) are discussed, but the equation in p. 230, line 6 is not correct. This value should be $\omega^{u(p+1)}\cdot(-1)^t$. So the representation of degree $p+1$ may occur. Therefore, in this note, we shall assume that the irreducible representation of degree $p+1$ occurs besides the assumptions (\#), (\##), $n=p+2$ and $t\equiv 0$ (mod. 2). Under these assumptions we shall prove that such a group does not exist.

We shall use the same notations as Brauer [1]. First of all, we shall assume that $n=p+2=F(p, 1, 2)=F(p, u, 1)$ with positive integer $u$. For, if $n$ does not have the expression $F(p, u, 1)$ with positive integer $u$, then the character-relations in $B_i(p)$ yields a contradiction easily. Simple computations show that the possible values of the irreducible characters in $B_i(p)$ are $1, \chi + 1, \chi'/2, (u-1)p-1, (up+1)/t$ and $((u-1)p-1)/t$. In order to consider such characters, we shall prove following lemmas, essentially due to Brauer.

**Lemma 1.** Under assumptions (\#), (\##), $n=p+2$ and $t\equiv 0$ (mod. 2), if $\mathcal{G}$ has an irreducible character $A$ of degree $up+1(u>1)$, then for the element $I$ of order 2 in the normalizer $\mathcal{N}(\mathcal{P})$ of a $p$-Sylow subgroup $\mathcal{P}$...
\[ A(I) = \begin{cases} 
  u + 1, & \text{if } u \text{ is even.} \\
  0, & \text{if } u \text{ is odd.} 
\end{cases} \]

The normalizer \( \mathcal{N}(\mathfrak{B}) \) of a \( p \)-Sylow subgroup \( \mathfrak{B} \) is a metacyclic group \( \{P, Q\} \) of order \( pq = p(p-1)/t \) and has \( q \) linear characters \( \omega_\mu \) and \( t \) \( p \)-conjugate characters \( Y^{(\tau)} \) of degree \( q \). If we consider the character \( A \) in \( \mathcal{N}(\mathfrak{B}) \), then \( A \) is decomposed into two parts \( \hat{A} \) and \( A_0 \), where \( \hat{A} \) is a sum of \( u+1 \) linear characters \( \omega_\mu \) and \( A_0 \) is a linear homogeneous combination of \( Y^{(\tau)} \). Now, as \( u \geq 1 \), \( n = p+2 \) has an expression \( F(p, u, 1) = (up + u^2 + u + 1)/(u + 1) \). This implies \( p = u^2 - u - 1 \) and \( g = p \cdot q(up + 1)(p + u + 1)/(u + 1) \). Since \( gn(G)^{-1} \cdot D_{\mathfrak{B}} A^{-1} \cdot A(G) \) is an algebraic integer, where \( n(G) \) is the order of the normalizer of \( G \) in \( \mathfrak{S} \) and \( D_{\mathfrak{B}} A \) is the degree of \( A \), \( gn(Q^j)^{-1} \cdot (up + 1)^{-1} \cdot A(Q^j) \) is, and also \( A(Q^j)/(u + 1) \) is an algebraic integer. But \( A(Q^j) = \hat{A}(Q^j) \) for \( j \equiv 0 \pmod{q} \). Then applying Burnside's method, we have \( A(Q^j) = 0 \) or \( (u+1)\omega^j \): that is, \( A(Q^j) = 0 \) or \( (u+1)\omega^j \) for \( j \equiv 0 \pmod{q} \). Assume \( \hat{A} = \alpha_1 \omega_{\mu_1} + \alpha_2 \omega_{\mu_2} + \cdots + \alpha_r \omega_{\mu_r} \) is a decomposition of \( \hat{A} \) in \( \mathcal{N}(\mathfrak{B}) \). Let \( m \) be the least positive integer satisfying \( \hat{A}(Q^m) \equiv 0 \). Then \( m \) is a divisor of \( q \) and any integer \( x \) satisfying \( \hat{A}(Q^x) \equiv 0 \) is a multiple of \( m \). Now there exist \( q/m \) integers satisfying \( \hat{A}(Q^x) \equiv 0 \). From the orthogonality-relations, we have

\[
\sum_{j=0}^{q-1} \hat{A}(Q^j)\omega_{\mu_i}(Q^j) = \alpha_i \cdot q.
\]

On the other hand, \( \sum \hat{A}(Q^j)\omega_{\mu_i}(Q^j) = \sum_{j=0}^{q-1} \hat{A}(Q^j)\omega_{\mu_i}(Q^j) = (u + 1)\ q/m \). Hence \( \alpha_i q = (u + 1)\ q/m \). This means \( \alpha_i = (u + 1)/m \). Therefore \( \hat{A} = \frac{u+1}{m} (\omega_{\mu_1} + \omega_{\mu_2} + \cdots + \omega_{\mu_r}) \). Furthermore \( \hat{A}(Q^m) \equiv 0 \) implies \( \omega^{\mu_1 m} = \omega^{\mu_2 m} = \cdots = \omega^{\mu_r m} \). This means \( \mu_1 \equiv \mu_2 \equiv \cdots \equiv \mu_m \pmod{q/m} \). Then we can put \( \mu_1 = a, \mu_2 = a + q/m, \cdots, \mu_m = a + (m - 1)q/m \). Thus

\[
A = \frac{u+1}{m} (\omega_a + \omega_{a+q/m} + \cdots + \omega_{a+(m-1)q/m}) + A_0.
\]

Next consider its determinant for \( Q^j \) for \( j \equiv 0 \pmod{q} \). This value must be 1.

\[
\text{Det}(A(Q^j)) = \omega^{ja(u+1)}(-1)^{j(1-\frac{u+1}{m})}.
\]

Suppose \( (u+1)/m \equiv 0 \pmod{2} \), then \( \omega^{ja(u+1)} = (-1)^j \). For \( j = 1 \), we have \( a(u+1) \equiv q/2 \pmod{q} \), \( a(u+1) \equiv 0 \pmod{q} \). These yield \( u-2 \equiv 0 \pmod{2} \). This contradicts \( u+1 \equiv 0 \pmod{2} \). Now we have \( (u+1)/m \equiv 0 \pmod{2} \) and then \( a(u+1) \equiv 0 \pmod{q} \). From \( (D) \),
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\[ A(I) = \frac{u+1}{m}((-1)^a + (-1)^{a+q/m} + \cdots + (-1)^{a+(m-1)q/m}). \]

i) If \( u \) is even, then \( m \) is odd. And \( q/m \) must be even. From \( a(u+1) \equiv 0 \pmod{q} \), \( a \) is even. Thus \( A(I) = u+1 \).

ii) If \( u \) is odd, then \( \frac{q}{m} = \frac{u+1}{m} \frac{u-2}{t} \) is odd. Hence \( A(I) = 0 \). This proves lemma 1.

For other type of irreducible characters, similar results can be proved.

**Lemma 2.** Under the same assumptions as in Lemma 1, if \( \mathcal{G} \) has an irreducible character \( B \) of degree \( (u-1)p-1 \), then for an involution \( I \) (the element of order 2) in \( \mathbb{R}(\mathcal{G}) \)

\[ B(I) = \begin{cases} 0, & \text{if } u \text{ is even.} \\ u-2, & \text{if } u \text{ is odd.} \end{cases} \]

**Lemma 3.** Under the same assumptions as in Lemma 1, if \( \mathcal{G} \) has an irreducible character \( C \) of degree \( (up+1)/t \), then for an involution \( I \) of \( \mathbb{R}(\mathcal{G}) \)

\[ C(I) = \begin{cases} (u+1)/t, & \text{if } u \text{ is even.} \\ 0, & \text{if } u \text{ is odd.} \end{cases} \]

**Lemma 4.** Under the same assumptions as in Lemma 1, if \( \mathcal{G} \) has an irreducible character \( C \) of degree \( (u-1)\bar{p}-1 \), then for an involution \( I \) of \( \mathbb{R}(\mathcal{G}) \)

\[ C(I) = \begin{cases} 0, & \text{if } u \text{ is even.} \\ (u-2)/t, & \text{if } u \text{ is odd.} \end{cases} \]

2) If \( u-2=1 \), then the Burnside’s method yields nothing. But \( u-2=1 \) yields \( p=5 \). For \( p=5, \ g=5 \cdot 4 \cdot (1+7 \cdot 5)/t. \) Since \( t \) is odd, \( t=1 \). Then \( B_1(5) \) consists of the principal character, \( x \) characters of degree 6, \( y \) characters of degree 16 and \( z \) characters of degree 9. And we have \( 1+x+y+2=5 \) and \( 1+6x+16y=9z \). This is a contradiction. Hence \( u-2>1 \).

3) Let \( u+1=1 \). If the irreducible character of degree \( up+1 \) occurs, then \( q=0 \pmod{(u+1)} \). And \( u=2 \). This contradicts (**). Therefore \( B_1(p) \) may consist of the 1-character, \( x \) characters of degree \( p+1 \), \( y \) characters of degree \( (u-1)p-1 \) and \( t \) characters of degree \( (up+1)/t \). Then \( 1+x+y=(p-1)/t=u-2 \) and \( x+1=(u-1)y \). This is also a contradiction. Hence \( (u+1)/t>1 \).

4) Let \( u-2=1 \). If the irreducible character of degree \( (u-1)p-1 \) occurs, then \( q=0 \pmod{(u-2)} \). And either \( u=5 \) or \( u=3 \). If \( u=5 \), then \( p=19 \). \( B_1(19) \) may consist of the 1-character, \( x \) characters of degree 20, \( y \) characters of degree 96, \( z \) characters of degree 75 and \( t \) characters of degree 25. Then \( 1+x+y+z=6 \) and \( x+5y=4z+1 \). This is a contradiction. If \( u=3 \), then \( p=5 \). So \( B_1(p) \) may consists of the 1-character, \( x \) characters of degree 6, \( y \) characters of degree 16 and \( z \) characters of degree 9. Then \( 1+x+y+z=5 \) and \( 1+6x+19y=9z \). This is a contradiction. If the irreducible character of degree \( (u-1)\bar{p}-1 \) does not occur, then \( B_1(p) \) may consist of the 1-character, \( x \) characters of degree \( p+1 \), \( y \) characters of degree \( up+1 \) and \( t \) characters of degree \( (u-1)\bar{p}-1/(u-2) \). Then \( 1+x+y=u+1 \) and \( x+uy=1 \). This is a contradiction. Hence \( (u-2)/t>1 \).
Lemma 5. Under the same assumptions as Lemma 1, let $X$ be an irreducible character of degree $p+1$, then for an involution $I$ of $\mathbb{R}^n$, \[ X(I) = \begin{cases} 0, & \text{if } q \equiv 0 \pmod{4}, \\ \text{either } +2 \text{ or } -2, & \text{if } q \equiv 0 \pmod{4}. \end{cases} \]

In the latter case, we denote by $y_1$ and $y_2$ respectively the numbers of characters whose values for $I$ are $+2$ and $-2$.

Now, we shall consider two cases.

Case I: $\mathcal{G}$ contains an irreducible character of degree $(up+1)/t$; Let $B_i(p)$ contain $x$ characters of degree $up+1$, $y$ characters of degree $p+1$, $z$ characters of degree $(u-1)p-1$. From the character-relations in $B_i(p)$, we have

\[
1 + x + y + z = (p-1)/t,
ux + y + (u+1)/t = (u-1)z,
p = u^2 - u - 1.
\]

The character-relation which holds for $p$-regular elements shows for an involution $I$ that

\[
1 + \sum A(I) + \sum X(I) + C(I) = \sum B(I).
\]

Eliminate $y$ and $p$, then $(u-1)x - (u-1)z + (u^2 - 1)/t = z + 1$. Put $z + 1 = \alpha(u-1)$.

Then $x = -(u+1)/t + \alpha u - 1$, $y = (u^2 - 1)/t - 2\alpha u + \alpha + 1$ and $z = \alpha u - \alpha - 1$. As $x \geq 0$, $\alpha \geq 1$. And $\alpha \geq 2$ for $t = 1$.

Now consider (I) for even $u$ and for odd $u$ separately.

Case Ia : Case where $u$ is even; From (I), none of $X(I)$ can be zero. Hence we have

\[
1 + x(u+1) + 2(y_1 - y_2) + (u+1)/t = 0.
\]

But $y_2 - y_1 \leq y$. Then $1 + x(u+1) + (u+1)/t \leq 2y$. Substitute above values for $x$ and $y$, then we have

\[
\alpha(u_z+5u-2) - u - 2 \leq (3u^2 + u - 2)/t.
\]

This inequality yields $\alpha = 0$ for $t = 1$ and $\alpha \leq 2$ for $t = 1$. Hence we have $t = 1$ and $\alpha = 2$.

Case Ib : Case where $u$ is odd; From (I), none of $X(I)$ can be zero. Hence we have

\[
1 + 2(y_1 - y_2) = (u-2)z.
\]

But $y_1 - y_2 \leq y$. Then $(u-2)z - 1 \leq 2y$. Substitute the above values for $y$ and $z$, then we have
Note on Brauer's Theorem II

\[ \alpha(u^2 + u) - u - 1 \leq 2(u^2 - 1)/t, \]
\[ \alpha u - 1 \leq 2(u - 1)/t. \]

This inequality yields \( \alpha = 0 \) for \( t = 1 \) and \( \alpha \leq 2 \) for \( t = 1 \). This is a contradiction.

Case II: \( \mathcal{G} \) contains an irreducible character of degree \( (u - 1)p - 1)/t \); Let \( B_i(p) \) contain \( x \) characters of degree \( up + 1 \), \( y \) characters of degree \( p + 1 \), \( z \) characters of degree \( (u - 1)p - 1 \). From the character-relations in \( B_i(p) \),

\[ 1 + x + y + z = (p - 1)/t, \]
\[ uz + y = (u - 1)z + (u - 2)/t, \]
\[ \bar{p} = u^2 - u - 1, \]
\[ 1 + \sum A(I) + \sum X(I) = \sum B(I) + C(I). \]

Eliminate \( y \) and \( \bar{p} \), then \( x + 1 =ux - uz + u(u - 2)/t \). Put \( x + 1 = \alpha u \). Then \( z = (u - 2)/t + \alpha u - \alpha - 1, y = u(u - 2)/t - 2\alpha u + \alpha + 1 \) and \( x = \alpha u - 1 \). Of course \( \alpha \) is a positive integer.

Case IIa: Case where \( u \) is even. As Case Ia, from (I'), we have \( 1 + x(u + 1) \leq 2y \). Substitute the above values for \( x \) and \( y \), then we have

\[ \alpha(u^2 + 5u - 2) - u - 2 \leq 2u(u - 2)/t. \]

This yields \( t = 1 \) and \( \alpha = 1 \).

Case IIb: Case where \( u \) is odd. We have from (I'),

\[ 1 + 2(y_1 - y_2) = (u - 2)z + (u - 2)/t. \]

As Case Ib, we have

\[ (u - 2)z + (u - 2)/t - 1 \leq 2y, \]
\[ \alpha(u^2 + u) - u - 1 \leq (u - 1)(u - 2)/t, \]
\[ \alpha u - 1 \leq (u - 2)/t. \]

This inequality yields \( \alpha \leq 1 \). This is a contradiction.

Combining the above cases, the only possible case occurs when \( t = 1 \) for even \( u \). In this case \( B_i(p) \) consists of the 1-character, \( u - 1 \) characters of degree \( up + 1 \), \( u^2 - 4u + 2 \) characters of degree \( p + 1 \) and \( 2u - 3 \) characters of degree \( (u - 1)p - 1 \).

Denote the sum of the elements in the conjugate class containing \( \mathcal{G} \) by \( \langle G \rangle \). Now we consider the coefficient of \( \langle I \rangle \) in the group ring of its center. From the orthogonality relations the coefficient \( a_p \) of \( \langle P \rangle \) is

\[ a_p = gn(I)^{-2} \sum(D_g X)^{-1}X(I)^2 \cdot \bar{X}(P), \]
where the summation ranges over all the irreducible characters of $\mathfrak{G}$. (cf. [2], §5). On the other hand this coefficient is equal to the number of pairs of conjugate elements $T$ and $S$ of $\langle I \rangle$ such that $TS=P$. If $TS=P$, then $TPT^{-1}=P^{-1}$. By condition (*), this number of pairs is $p$. Hence we get

$$p = gn(I)^{-1}\sum (D_gX)^{-1}X(I)^{-1}X(P),$$

where sum ranges over all irreducible characters of $\mathfrak{G}$.

Applying this, we have

$$n(I)^2p = g\{1 + (u_p+1)^{-1}(u+1)^2(u-1) + (p+1)^{-1}4(u^2-4u+2)\}.$$

$$n(I)^2 = 2u(u-2)^2(u-1)(3u-2)(u+1).$$

Since $n(I)$ is a multiple of $p-1=(u+1)(u-2)$ and $u$ is even, we have $u+1=5$. Hence $n(I)^2=5^22^3$. This number is not a square.

Thus for $n=p+2$ and $t\equiv 0 \pmod{2}$, such a group $\mathfrak{G}$ does not exist. This completes the proof of the theorem.

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**References**


