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| Title | GORENSTEIN FLAT PREENVELOPES |
| Author(s) | Iacob, Alina |
| Citation | Osaka Journal of Mathematics. 2015, 52(4), p. 895-903 |
| Version Type | VoR |
| URL | https://doi.org/10.18910/57638 |
| rights | |
| Note | |

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Osaka J. Math.
**52** (2015), 895–903

# GORENSTEIN FLAT PREENVELOPES

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(Received January 21, 2014, revised May 12, 2014)

**Abstract**

We consider a two sided noetherian ring $R$ such that the character modules of Gorenstein injective left $R$-modules are Gorenstein flat right $R$-modules. We then prove that the class of Gorenstein flat right $R$-modules is preenveloping. We also show that the class of Gorenstein flat complexes of right $R$-modules is preenevloping in $Ch(R)$.

In the second part of the paper we give examples of rings with the property that the character modules of Gorenstein injective modules are Gorenstein flat. We prove that any two sided noetherian ring $R$ with $i.d._{R^{\mathrm{op}}}\, R < \infty$ has the desired property. We also prove that if $R$ is a two sided noetherian ring with a dualizing bimodule ${}_RV_R$ and such that $R$ is left $n$-perfect for some positive integer $n$, then the character modules of Gorenstein injective modules are Gorenstein flat.

## 1. Introduction

Gorenstein homological algebra is the relative version of homological algebra that uses Gorenstein injective, Gorenstein projective and Gorenstein flat resolutions instead of the classical injective, projective and flat resolutions. But while the existence of the classical resolutions over arbitrary rings is well known, things are a little different when it comes to Gorenstein homological algebra. The main open problems in this area concern the existence of the Gorenstein (injective, projective and flat) resolutions. This is the reason why the existence of the Gorenstein (injective, projective, flat) precovers and preenvelopes has been studied intensively in recent years. We consider here the existence of the Gorenstein flat preenvelopes. It is known that over a Gorenstein ring every module has a Gorenstein flat preenvelope ([8]). We prove that if $R$ is a two sided noetherian ring such that the character modules of Gorenstein injective left $R$-modules are Gorenstein flat, then the class of Gorenstein flat right $R$-modules is preenveloping. Then we prove that over these rings, every complex of right $R$-modules has a Gorenstein flat preenvelope.

In the second part of the paper, we give examples of rings with the property that the character module of any Gorenstein injective left $R$-module is a Gorenstein flat right $R$-module. More precisely we show that a two sided noetherian ring $R$ such that $i.d._{R^{op}}\, R < \infty$ has the desired property. We also prove that if $R$ is a two sided noetherian ring with a dualizing bimodule ${}_RV_R$ and such that $R$ is left n-perfect for

2010 Mathematics Subject Classification. 18G10, 18G25, 18G35.

some positive integer n, then the character modules of Gorenstein injective modules are Gorenstein flat.

## 2. Preliminaries

Throughout this section $R$ denotes an associative ring with unity.

DEFINITION 1 ([8], Definition 10.1.1). An $R$-module $M$ is Gorenstein injective if there exists an exact and Hom(Inj, –) exact sequence of injective $R$-modules

$$\cdots \to E_1 \to E_0 \to E_{-1} \to \cdots$$

such that $M = \mathrm{Ker}(E_0 \to E_{-1})$.

We will use the notation $\mathcal{GI}$ for the class of Gorenstein injective modules.

A stronger notion is that of strongly Gorenstein injective module:

DEFINITION 2 ([1], Definition 2.1). An $R$-module $M$ is strongly Gorenstein injective if there exists an exact and Hom(Inj, –) exact sequence

$$\cdots \to E \xrightarrow{f} E \xrightarrow{f} E \to \cdots$$

with $E$ injective and with $M = \mathrm{Ker}(f)$.

It is known ([1], Theorem on p. 3) that a module is Gorenstein injective if and only if it is a direct summand of a strongly Gorenstein injective one.

The Gorenstein flat modules are defined in terms of the tensor product.

DEFINITION 3 ([8], Definition 10.3.1). A right $R$-module $G$ is Gorenstein flat if there exists an exact and $- \otimes$ Inj exact sequence of flat right $R$-modules

$$\cdots \to F_1 \to F_0 \to F_{-1} \to \cdots$$

such that $G = \mathrm{Ker}(F_0 \to F_{-1})$.

We will use the notation $\mathcal{GF}$ for the class of Gorenstein flat modules.

We recall that the character module of a right $R$-module $M$ is the left $R$-module $M^+ = \mathrm{Hom}_Z(M, Q/Z)$. We also recall that $M^+ \simeq \mathrm{Hom}_R(M, R^+)$.

It is known ([10], Theorem 3.6) that if the ring $R$ is coherent then a module $G$ is Gorenstein flat if and only if its character module $G^+$ is Gorenstein injective.

We also recall the definition of a (pre)envelope.

DEFINITION 4. Let $\mathcal{C}$ be a class of $R$-modules. A homomorphism $\phi\colon M \to C$ is a $\mathcal{C}$-preenvelope of $M$ if $C \in \mathcal{C}$ and if for any homomorphism $\phi'\colon M \to C'$ with $C' \in \mathcal{C}$, there exists $u \in \mathrm{Hom}_R(C, C')$ such that $\phi' = u\phi$.

A $\mathcal{C}$-preenvelope $\phi$: $M \to C$ is a $\mathcal{C}$-envelope if any endomorphism $u \in \mathrm{Hom}_R(C,C)$ such that $\phi = u\phi$ is an automorphism of $C$.

(Pre)covers and covers are defined dually.

When $\mathcal{C} = \mathcal{GF}$ we obtain the definition of a Gorenstein flat (pre)envelope. Their existence allows us to define the right Gorenstein flat resolutions:

DEFINITION 5. A module $M$ has a right Gorenstein flat resolution if there exists a Hom(–, $\mathcal{GF}$) exact sequence $0 \to M \to G_0 \to G_{-1} \to G_{-2} \to \cdots$ with each $G_j$ Gorenstein flat.

We note that this is equivalent to $M \to G_0$ and each $\mathrm{coker}(G^{i-1} \to G^i) \to G^{i+1}$ for $i \geq 1$ being Gorenstein flat preenvelopes.

## 3. Gorenstein flat preenvelopes

In [7] we proved that the class $\mathcal{GI}$ is closed under direct limits, and so it is covering, over any commutative noetherian ring $R$ such that the character modules of the Gorenstein injective modules are Gorenstein flat. We also showed that $\mathcal{GI}$ is an enveloping class over such a ring. Then in [12] we considered a two sided noetherian ring $R$ such that the character module of any Gorenstein injective left $R$-module is a Gorenstein flat right $R$-module. We proved that over such a ring the class of Gorenstein injective left $R$-modules is still closed under direct limits, and therefore it is covering. We also proved in [12] that the class of Gorenstein injective left modules is enveloping over such rings.

We now consider the same type of rings. We prove that the class of Gorenstein flat right modules is preenveloping over such a ring. Then we show that every complex of right $R$-modules has a Gorenstein flat preenvelope.

**Proposition 1.** *Let $R$ be a two sided noetherian ring such that the character module of any left Gorenstein injective module is a Gorenstein flat right $R$-module. Then the class of Gorenstein flat right $R$-modules is preenveloping.*

Proof. By [3] p. 78 it suffices to prove that the class of Gorenstein flat right $R$-modules is closed under direct products.

Let $(G_i)_{i\in I}$ be a family of Gorenstein flat right $R$-modules. Then each $G_i^+$ is a left Gorenstein injective module. Since the class of Gorenstein injective left $R$-modules over such a ring is closed under arbitrary direct sums, it follows that $\oplus G_i^+$ is still Gorenstein injective. Then by hypothesis we have that $(\oplus G_i^+)^+$ is a Gorenstein flat module. That is, $\prod G_i^{++}$ is Gorenstein flat.

For each $i \in I$ we have a pure exact sequence: $0 \to G_i \to G_i^{++} \to Y_i \to 0$. Thus we have an exact sequence $0 \to \prod G_i \to \prod G_i^{++} \to \prod Y_i \to 0$.

We show that this sequence is also pure exact.

Let $A_R$ be a finitely presented $R$-module. Then for each $i$ the sequence $0 \to \operatorname{Hom}(A, G_i) \to \operatorname{Hom}(A, G_i^{++}) \to \operatorname{Hom}(A, Y_i) \to 0$ is still exact. So we have an exact sequence

$$0 \to \prod \operatorname{Hom}(A, G_i) \to \prod \operatorname{Hom}(A, G_i^{++}) \to \prod \operatorname{Hom}(A, Y_i) \to 0.$$

Since $\prod \operatorname{Hom}(A, G_i) \simeq \operatorname{Hom}\big(A, \prod G_i\big)$, $\prod \operatorname{Hom}(A, G_i^{++}) \simeq \operatorname{Hom}\big(A, \prod G_i^{++}\big)$, and $\prod \operatorname{Hom}(A, Y_i) \simeq \operatorname{Hom}\big(A, \prod Y_i\big)$, it follows that for any finitely presented $A_R$, the sequence $0 \to \operatorname{Hom}\big(A, \prod G_i\big) \to \operatorname{Hom}\big(A, \prod G_i^{++}\big) \to \operatorname{Hom}\big(A, \prod Y_i\big) \to 0$ is exact.

So $\prod G_i$ is a pure submodule of the Gorenstein flat module $\prod G_i^{++}$. Then $\big(\prod G_i\big)^+$ is a direct summand of the Gorenstein injective module $\big(\prod G_i^{++}\big)^+$, and therefore it is Gorenstein injective. It follows that $\prod G_i$ is Gorenstein flat. $\square$

We prove that over the same type of rings every complex of right $R$-modules has a Gorenstein flat preenvelope.

We will use the following lemma:

**Lemma 1** ([9], Lemma 5.2.1). *Let $R$ be any ring and let $S$ be a subcomplex of right $R$-modules of a complex $G$. Then there is a subcomplex $S^*$ of $G$ with $S \subseteq S^* \subseteq G$ such that $S^* \subseteq G$ is pure and such that* $\operatorname{Card}(S^*) \leq \operatorname{Card}(S) \cdot \operatorname{Card}(R)$ *if either of* $\operatorname{Card}(S)$ *and* $\operatorname{Card}(R)$ *is infinite. If both are finite, there is an $S^*$ which is at most countable.*

**Theorem 1.** *Let $R$ be a two sided noetherian ring such that the character module of any left Gorenstein injective module is a right Gorenstein flat $R$-module. Then any complex of right $R$-modules has a Gorenstein flat preenvelope.*

Proof. We use a similar argument to that of [9], Theorem 5.2.2.

By [6], a complex is Gorenstein flat if and only it is a complex of Gorenstein flat modules. By the proof of Proposition 1, the class of Gorenstein flat modules is closed under direct products. So any direct product of Gorenstein flat complexes is also Gorenstein flat.

Given a complex $C$ let $\mathcal{N}_\beta$ be an infinite cardinal number such that

$$\operatorname{Card}(C) \cdot \operatorname{Card}(R) \leq \mathcal{N}_\beta.$$

Let $Y$ denote the class of all Gorenstein flat complexes $G$ such that $\operatorname{Card}(G) \leq \mathcal{N}_\beta$. Let $(G_i)_{i \in I}$ be a family of representatives of this class with the index set $I$. Let $H_i = \operatorname{Hom}(C, G_i)$ for each $i \in I$ and let $F = \prod G_i^{H_i}$. Then $F$ is a Gorenstein flat complex.

Define $\psi: C \to F$ so that the composition of $\psi$ with the projection map $F \to G_i^{H_i}$ maps $x \in F^k$ to $(h^k(x))_{h \in H_i}$. Then $\psi: C \to F$ is a map of complexes.

We show that $\psi: C \to F$ is a flat preenvelope. Let $\psi': C \to G$ be a map of complexes with $G$ a Gorenstein flat complex. By Lemma 1 above, the subcomplex $\psi'(C)$ can be enlarged to a pure subcomplex $G' \subseteq G$ with $\mathrm{Card}(G') \leq \mathcal{N}_\beta$. Since $G'$ is a pure subcomplex of the Gorenstein flat complex $G$ it follows that for each n, $G'_n$ is a pure submodule of $G_n$, and therefore each $G_n$ is Gorenstein flat. Then $G'$ is a Gorenstein flat complex. So $G'$ is isomorphic to one of the $G_i$. By the construction of the map $\psi$, $\psi'$ can be factored through $\psi$. □

## 4. Rings with the property that the character modules of Gorenstein injective left *R*-modules are Gorenstein flat

It is known ([11], Lemma 2.5 (b)) that any commutative noetherian ring $R$ with a dualizing complex has the desired property: the character modules of Gorenstein injective modules are Gorenstein flat. In the following we give examples of two sided noetherian rings with the property that the character modules of left Gorenstein injective modules are Gorenstein flat.

By Proposition 1 the class of Gorenstein flat right $R$-modules is preenveloping over such a ring and therefore any right $R$-module $M$ has a right Gorenstein flat resolution. Also, by [12] the class of Gorenstein injective left $R$-modules is both covering and enveloping over such a ring. Consequently any ${}_RN$ has both a left and a right minimal Gorenstein injective resolution.

**Theorem 2.** *Let $R$ be a two sided noetherian ring such that $\mathrm{i.d.}_{R^{\mathrm{op}}} R \leq n$ for some positive integer $n$. Then the character modules of Gorenstein injective left $R$-modules are Gorenstein flat.*

Proof. Let $M$ be a strongly Gorenstein injective left $R$-module, and let $I$ be any injective left $R$-module. By [13], Proposition 1, the flat dimension of $I$ is less than or equal to $n$. Then the right $R$-module $I^+$ has injective dimension $\leq n$. It follows that $\mathrm{Ext}^i(M^+, I^+) = 0$ for any $i \geq n+1$. So we have that $\mathrm{Ext}^i(I, M^{++}) \simeq \mathrm{Ext}^i(M^+, I^+) = 0$ for all $i \geq n+1$, for any injective ${}_RI$.

But $M$ is strongly Gorenstein injective, so there exists an exact sequence $0 \to M \to \bar{E} \to M \to 0$ with $\bar{E}$ injective. This gives an exact sequence $0 \to M^{++} \to E \to M^{++} \to 0$ with $E = \bar{E}^{++}$ an injective left $R$-module. It follows that $\mathrm{Ext}^k(-, M^{++}) \simeq \mathrm{Ext}^1(-, M^{++})$ for any $k \geq 1$. By the above, we have that $\mathrm{Ext}^k(I, M^{++}) = 0$ for any injective ${}_RI$, for any $k \geq 1$.

Then the exact sequence $\cdots \to E \to E \to M^{++} \to 0$ is also $\mathrm{Hom}(\mathrm{Inj}, -)$ exact. So $M^{++}$ has an exact left injective resolution. Since $\mathrm{Ext}^i(I, M^{++}) = 0$ for all $i \geq 1$, for any injective ${}_RI$ and $M^{++}$ has an exact left injective resolution it follows that $M^{++}$ is Gorenstein injective ([8], Proposition 10.1.3).

Let $G$ be a Gorenstein injective left $R$ module. By [1], there exists a strongly Gorenstein injective left $R$-module $M$ such that $M \simeq G \oplus H$. It follows that $G^{++}$ is isomorphic to a direct summand of the Gorenstein injective module $M^{++}$, and therefore it is Gorenstein injective.

Since the ring $R$ is right coherent and $(G^+)^+$ is Gorenstein injective it follows that $G^+$ is Gorenstein flat, for any Gorenstein injective ${}_RG$. □

We also prove that if $R$ is a two sided noetherian ring with a dualizing bimodule ${}_RV_R$ and such that $R$ is left $n$-perfect, then the character modules of Gorenstein injective modules are Gorenstein flat.

We recall the definition of a dualizing module:

DEFINITION 6 ([5], Definition 1). Let $R$ be a left and right noetherian ring and let ${}_RV_R$ be an $(R, R)$-bimodule such that $\mathrm{End}({}_RV) = R$ (naturally) and $\mathrm{End}(V_R) = R$ (naturally). Then $V$ is said to be a dualizing module if it satisfies the following three conditions:

(i) $\mathrm{id}({}_RV) \leq r$ and $\mathrm{id}(V_R) \leq r$ for some integer $r$;

(ii) $\mathrm{Ext}^i_R({}_RV, {}_RV) = \mathrm{Ext}^i_R(V_R, V_R) = 0$ for all $i \geq 1$;

(iii) ${}_RV$ and $V_R$ are finitely generated.

We will use the Auslander and Bass classes. They are defined in terms of the dualizing bimodule $V$.

DEFINITION 7 ([5], Definition 2). The right Auslander class (relative to $V$), $\mathcal{A}^r(R)$, is the class of right $R$-modules $M$ such that $\mathrm{Tor}^R_i(M, V) = 0$ and $\mathrm{Ext}^i_R(V, M \otimes V) = 0$ for all $i \geq 1$ and such that the natural morphism $M \to \mathrm{Hom}_R(V, M \otimes V)$ is an isomorphism.

The left Bass class (relative to $V$), $\mathcal{B}^l(R)$, is defined as those left $R$-modules $N$ such that $\mathrm{Ext}^i_R(V, N) = 0$, and $\mathrm{Tor}^R_i(V, \mathrm{Hom}_R(V, N)) = 0$ for all $i \geq 1$ and such that the natural morphism $V \otimes \mathrm{Hom}_R(V, N) \to N$ is an isomorphism.

In the following $R$ denotes a left and right noetherian ring with a dualizing module ${}_RV_R$. Also, $E$ will denote an $(R, R)$-bimodule such that ${}_RE$ and $E_R$ are both injective (for example $R^+$ is such a bimodule).

**Lemma 2.** *If* ${}_RN \in \mathcal{B}^l(R)$ *then* $\mathrm{Hom}(N, E) \in \mathcal{A}^r(R)$.

Proof. – Since $R$ is left noetherian, $V$ is finitely generated and $E$ is injective, we have that

$$\mathrm{Tor}^R_i(\mathrm{Hom}_R(N, E), V) \simeq \mathrm{Hom}_R(\mathrm{Ext}^i(V, N), E) = 0$$

(because $N \in \mathcal{B}^l(R)$, so $\mathrm{Ext}^i(V, N) = 0$).

– Since ${}_RV$ is finitely presented, and $E_R$ is injective, it follows that $\mathrm{Hom}_R(N, E) \otimes_R V \simeq \mathrm{Hom}_R(\mathrm{Hom}_R(V, N), E)$.

Then

$$\begin{aligned}\mathrm{Ext}^i_R(V, \mathrm{Hom}_R(N, E) \otimes V) &\simeq \mathrm{Ext}^i(V, \mathrm{Hom}_R(\mathrm{Hom}_R(V, N), E))\\ &\simeq \mathrm{Hom}_R(\mathrm{Tor}^R_i(\mathrm{Hom}_R(V, N), V), E) = 0\end{aligned}$$

(because $N \in \mathcal{B}^l(R)$ implies that $\mathrm{Tor}^R_i(\mathrm{Hom}_R(V, N), V) = 0$)).

– As above, we have $\mathrm{Hom}_R(N, E) \otimes_R V \simeq \mathrm{Hom}_R(\mathrm{Hom}_R(V, N), E)$. Then

$$\begin{aligned}\mathrm{Hom}_R(V, \mathrm{Hom}_R(N, E) \otimes V) &\simeq \mathrm{Hom}_R(V, \mathrm{Hom}_R(\mathrm{Hom}_R(V, N), E))\\ &\simeq \mathrm{Hom}_R(V \otimes \mathrm{Hom}(V, N)), E) \simeq \mathrm{Hom}_R(N, E)\end{aligned}$$

(because $N \in \mathcal{B}^l(R)$ implies that $V \otimes \mathrm{Hom}_R(V, N) \simeq N$).

Thus $\mathrm{Hom}_R(N, E) \in \mathcal{A}^r(R)$. □

**Lemma 3.** *If* $M \in \mathcal{A}^r(R)$ *then* $\mathrm{Hom}_R(M, E) \in \mathcal{B}^l(R)$.

Proof. – We have

$$\mathrm{Ext}^i_R(V, \mathrm{Hom}_R(M, E)) \simeq \mathrm{Hom}_R(\mathrm{Tor}^R_i(M, V), E) = 0$$

(because $M \in \mathcal{A}^r(R)$, so $\mathrm{Tor}^R_i(M, V) = 0$).

– We have $\mathrm{Hom}_R(V, \mathrm{Hom}_R(M, E)) \simeq \mathrm{Hom}_R(M \otimes V, E)$. Therefore

$$V \otimes_R \mathrm{Hom}_R(V, \mathrm{Hom}_R(M, E)) \simeq V \otimes \mathrm{Hom}_R(M \otimes V, E).$$

Since ${}_RV$ is finitely presented and $E_R$ is injective we have

$$V \otimes \mathrm{Hom}_R(M \otimes V, E) \simeq \mathrm{Hom}_R(\mathrm{Hom}_R(V, M \otimes V), E) \simeq \mathrm{Hom}_R(M, E)$$

(because $M \in \mathcal{A}^r(R)$ implies that $\mathrm{Hom}_R(V, M \otimes V) \simeq M$).

– As above we have $\mathrm{Hom}_R(V, \mathrm{Hom}_R(M, E)) \simeq \mathrm{Hom}_R(M \otimes V, E)$.

So $\mathrm{Tor}_i(V, \mathrm{Hom}(V, \mathrm{Hom}(M, E)) \simeq \mathrm{Tor}_i(V, \mathrm{Hom}_R(M \otimes V, E))$. Since $R$ is noetherian, and since $E$ is injective and $V_R$ is finitely generated we have that

$$\mathrm{Tor}_i(V, \mathrm{Hom}_R(M \otimes V, E)) \simeq \mathrm{Hom}_R(\mathrm{Ext}^i(V, M \otimes V), E) = 0$$

(because $M \in \mathcal{A}^r(R)$ implies that $\mathrm{Ext}^i_R(V, M \otimes V) = 0$).

So $\mathrm{Hom}_R(M, E) \in \mathcal{B}^l(R)$. □

Lemma 2 and Lemma 3 give the following:

**Corollary 1.** *If* $N \in \mathcal{B}^l(R)$, *then* $\mathrm{Hom}(\mathrm{Hom}(N, E), E) \in \mathcal{B}^l(R)$.

**Corollary 2.** *If* $N \in \mathcal{B}(R)$ *then* $N^{++} \in \mathcal{B}^l(R)$.

Proof. This follows from Corollary 1 when $E = R^+$. □

We recall ([4], Definition 2.1) that a ring $R$ is called *left n-perfect* if every left flat $R$-module has projective dimension less than or equal to n. Thus the left 0-perfect rings are the left perfect rings.

**Theorem 3.** *Let $R$ be a two sided noetherian ring such that ${}_RR$ is a left n-perfect ring. Assume also that ${}_RV_R$ is a dualizing module for the pair $(R,R)$. Then the character modules of Gorenstein injective left $R$-modules are Gorenstein flat right $R$-modules.*

Proof. Since $V$ is a dualizing module, there exists some positive integer $r$ such that $\mathrm{id}({}_RV) \leq r$ and $\mathrm{id}(V_R) \leq r$.

Let ${}_RN$ be a Gorenstein injective module. By [4] Theorem 3.17, there exists an exact sequence in $R - Mod$:

$$0 \to K \to E_{r+n} \to \cdots \to E_1 \to E_0 \to N \to 0$$

with each $E_i$ injective and with $K \in \mathcal{B}(R)$. This gives an exact sequence

$$0 \to K^{++} \to E_{r+n}^{++} \to \cdots \to E_1^{++} \to E_0^{++} \to N^{++} \to 0$$

with each $E_i^{++}$ injective and with $K^{++} \in \mathcal{B}(R)$ (by Corollary 2). By [4] Theorem 3.17 again, it follows that $N^{++}$ is a Gorenstein injective module.

Since $R$ is right coherent and $(N^+)^+$ is Gorenstein injective, it follows that $N^+$ is Gorenstein flat. □

ACKNOWLEDGMENT. The author was partially supported by the Fundacion Seneca grant 18934/JLI/13.

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