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ON MOTION OF AN ELASTIC WIRE IN A RIEMANNIAN MANIFOLD AND SINGULAR PERTURBATION

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Abstract

R.E. Caflish and J.H. Maddocks analyzed the dynamics of a planar slender elastic rod. We consider a thin elastic rod γ in an N-dimensional riemannian manifold. The former model represents an elastic rod with positive thickness, and the equation becomes a semilinear wave equation. Our model represents an infinitely thin elastic rod, and the equation becomes a 1-dimensional semilinear plate equation. We prove the short time existence of solutions. We also discuss the behaviour of the solution when the resistance goes to infinity, and find that the solution converges to a solution of a gradient flow equation.

1. Introduction and preliminaries

Let $\gamma(x, t)$ be a closed curve in the N-dimensional euclidean space, parametrized by its arc length $0 \leq x \leq 1$. We define its potential energy by $\int_0^1 |\gamma_{xx}|^2 \, dx$ and kinetic energy by $\int_0^1 |\gamma_t|^2 \, dx$. The equation of motion derived by Hamilton's principle is a semilinear 1-dimensional plate equation: $\gamma_{tt} + \gamma_{xxxx} = (u\gamma_x)_x$. Here, $u = u(x, t)$ is the Lagrange multiplier determined by the constrained condition $|\gamma_x| \equiv 1$.

The existence of a short time solution of this equation was proved by the present author [9] using a perturbation to a composition of parabolic operators. Later, A. Burchard and L.E. Thomas [1] gave another proof using the contraction principle and Hasimoto's transformation in the 3-dimensional case.

In this paper, we generalize the above result into the case of riemannian manifolds by replacing the partial derivative γ_{xx} by the covariant derivative $\nabla_x \gamma_x$. The potential energy $E(\gamma)$ and kinetic energy $K(\gamma)$ becomes

$$E(\gamma) = \int_0^1 |\nabla_x \gamma_x|^2 \, dx, \quad K(\gamma) = \int_0^1 |\gamma_t|^2 \, dx. \tag{1.1}$$

Using Hamilton's principle, we will derive the equation of motion:

$$\begin{cases} \nabla_t \gamma_t + \nabla_x^3 \gamma_x + \mu \gamma_t = R(\gamma_x, \nabla_x \gamma_x)\gamma_x + \nabla_x(u\gamma_x), \\ |\gamma_x| \equiv 1, \end{cases} \tag{EW}$$

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with initial data $\gamma(x,0)=\gamma_0(x)$, $\gamma_t(x,0)=\gamma_1(x)$ satisfying $|\gamma_{0x}|=1$ and $g(\gamma_{0x},\nabla_x\gamma_1)=0$. Here, R is the curvature tensor, μ is a constant that represents the resistance, and $u=u(x,t)$ is the Lagrange multiplier. Note that the function u is unknown, and will be determined by the constrained condition $|\gamma_x|\equiv 1$. (EW) is a 1-dimensional semilinear plate equation. The constant μ is introduced to compare its solutions with that of the parabolic equation below.

The gradient flow equation for the potential energy $E(\gamma)$ becomes

$$\text{(EP)}\qquad \begin{cases} \eta_t + \nabla_x^3\eta_x = R(\eta_x,\, \nabla_x\eta_x)\eta_x + \nabla_x(u\eta_x), \\ |\eta_x| \equiv 1, \end{cases}$$

with initial data $\eta(x,\,0)=\eta_0(x)$ satisfying $|\eta_{0x}|=1$. By [8], if there are no closed geodesics of length 1, then equation (EP) has an infinite time solution. When the manifold is the euclidean space, we know the followings by [9].

(1) *Equation* (EW) *has a unique short time solution for any initial data.*

(2) *For* $\mu>0$, *we rescale the time variable* t *of equation* (EW) *to* $\tau=\mu^{-1}t$. *If* μ *is sufficiently large, then the solution of equation* (EW) *exists for sufficiently long time, and converges to the solution of equation* (EP) *when* $\mu\to\infty$.

In (2), the convergence is only in C^0 norm, because the rescaled initial velocity $d\gamma/d\tau=\mu\, d\gamma/dt=\mu\gamma_1$ diverges. Such phenomena is observed also in a hyperbolic-parabolic singular perturbation [7]. It clarifies the relation of completely different two differential equations, a plate equation and a 4-th order parabolic equation.

In this paper, we generalize these facts to the case of riemannian manifolds. The following results show that the 1-dimensional plate equation (EW) is stable under "riemannian perturbation".

Theorem 3.12. *Equation* (EW) *has a unique short time solution for any non-geodesic initial data.*

Theorem 4.12. *Assume that there are no closed geodesics of length* 1. *If* μ *is sufficiently large, then the solution of the rescaled equation of* (EW) *exists for sufficiently long time, and converges to the solution of equation* (EP) *when* $\mu\to\infty$.

We summarize notations and recall relevant basic facts from riemannian geometry for convenience. Let $(M,\,g)$ be a complete riemannian manifold.

We treat only C^∞-objects. A closed curve means a map from $S^1=\mathbf{R}/\mathbf{Z}$ into a manifold. The pointwise inner product of vectors is denoted by $g(*,\,*)$, and the norm is denoted by $|*|$.

For a map $z=z(u,\,v)\colon \mathbf{R}^2\to M$, $z_u=(\partial z^p/\partial u)(\partial/\partial x^p)$ is a vector field along the map z. The covariant derivative $\nabla_u X$ of a vector field $X=X^p\,\partial/\partial x^p$ along z for

u-direction is given by

$$\nabla_u X = (\nabla_u X)^p \frac{\partial}{\partial x^p} = \left\{ \frac{\partial X^p}{\partial u} + \Gamma_q{}^p{}_r(z) \frac{\partial z^q}{\partial u} X^r \right\} \frac{\partial}{\partial x^p}, \tag{1.2}$$

where $\Gamma_q{}^p{}_r$ are Christoffel's symbols. We see $\nabla_u z_v = \nabla_v z_u$ by definition, but higher covariant differentiations do not commute: $\nabla_v \nabla_u X - \nabla_u \nabla_v X = R(z_v, z_u)X$. The curvature tensor R has many symmetries, but we will not use them.

For functions on S^1 and vector fields along a closed curve in M, we use L_2-inner product $\langle *, * \rangle$ and L_2-norm $\| * \|$. Sobolev H^n-norm is denoted by $\| * \|_n$. For a tensor field along the closed curve, $\| * \|_n$ is defined using covariant derivatives. That is, $\|\zeta\|_n^2 = \sum_{i=0}^n \|\nabla_x^i \zeta\|^2$. As a special case, we set $\|\zeta\|_n = 0$ for $n < 0$.

In the riemannian case, we have several difficulties.

(1) In the euclidean case, we can convert the equation for γ to an equation for S^{N-1}-valued function $\xi(x,t) := \gamma_x(x,t)$. Where, the constrained condition $|\gamma_x| \equiv 1$ is automatically satisfied. Both of [9] and [1] used this transformation. In the riemannian case, this approach fails, because we cannot eliminate the original unknown functions γ. We are forced to solve equation (EW) directly, and have to manage a plate equation with third derivatives γ_{xxx}. Note that the plate equation is unstable under perturbation of third derivatives, and cannot be solved in general ([13, Section 11.7]).

(2) Since closed geodesics are singular points in the space of all closed curves of fixed length, we cannot extend the solution if the solution approaches to a closed geodesic. This is the reason why we exclude closed geodesics in Theorem 4.12.

There are other formulations of the motion of an elastic wire. When we consider a wire of radius r in the euclidean plain, the position vector is $\gamma + yJ\gamma_x$, where y is the coordinate orthogonal to x and J is the $\pi/2$ rotation. Here, the kinetic energy is $\|\gamma_t\|^2 + r^2\|\gamma_{xt}\|^2$, and the equation becomes a wave equation: $-\partial_x^2(r^2\gamma_{tt} - \gamma_{xx}) + \gamma_{tt} = (u\gamma_x)_x$. R.E. Caflisch and J.H. Maddocks [2] applied this approach to plane curves with $r > 0$ and obtained the global existence theorem. See also [5]. Equation (EW) is the limiting case: $r = 0$, and the equation becomes 1-dimensional plate equation. Note that the plate equation is more delicate than the wave equation. We find a linear version of equation (EW) in p. 246 of R. Courant and D. Hilbert [4].

The gradient flow equation (EP) is used in [6] for the euclidean case and in [8] for the riemannian case. In both cases, the global existence and convergence to elastica are proved, provided that there are no closed geodesics of given length. Such an equation is derived from a total Hilbert manifold and a riemannian metric on it. [6] and [8] use $\{\gamma\colon S^1 \to M \mid |\gamma_x| \equiv 1\}$ as the total manifold, and the standard L^2 metric on it. On the other hand, Y. Wen [14] uses $\{\gamma\colon S^1 \to M \mid \mathrm{Length}(\gamma) = 1\}$ and the L^2 metric for plane curves, and proves global existence and convergence to an elastica. Also, the Palais–Smale theoretical approach of J. Langer and D.A. Singer [12] for the riemannian case uses $\{\gamma : S^1 \to M \mid |\gamma_x| \equiv 1\}$ and the H^2 metric, and gives mountain pass lemma.

2. The equations

To derive the equation of motion, we use Hamilton's principle. For a moving curve $\gamma = \gamma(x,t)$, the velocity energy is $\|\gamma_t\|^2$ and the elastic energy is $\|\nabla_x\gamma_x\|^2$. (By rescaling, we omit coefficients.) Therefore, real motions are stationary paths of the integral

$$L(\gamma) := \int_{t_1}^{t_2} \|\gamma_t\|^2 - \|\nabla_x\gamma_x\|^2\, dt. \tag{2.1}$$

Namely, the integral

$$\begin{aligned}\frac{1}{2}L' &:= \int_{t_1}^{t_2} \langle\gamma_t, \nabla_s\gamma_t\rangle - \langle\nabla_x\gamma_x, \nabla_s\nabla_x\gamma_x\rangle\, dt \\ &= \int_{t_1}^{t_2} \langle\gamma_t, \nabla_t\gamma_s\rangle - \langle\nabla_x\gamma_x, R(\gamma_s, \gamma_x)\gamma_x + \nabla_x^2\gamma_s\rangle\, dt\end{aligned} \tag{2.2}$$

should vanish for every $\gamma_s = \delta = \delta(x, t)$ satisfying the boundary condition $\delta(x, t_1) = \delta(x, t_2) = 0$ and the constrained condition $\partial_s\{g(\gamma_x, \gamma_x)\} = 2g(\gamma_x, \nabla_x\delta) \equiv 0$.

By integration by parts, we see

$$\frac{1}{2}L' = -\int_{t_1}^{t_2} \langle\nabla_t\gamma_t + \nabla_x^3\gamma_x - R(\gamma_x, \nabla_x\gamma_x)\gamma_x, \delta\rangle\, dt. \tag{2.3}$$

On the other hand, the L_2 orthogonal complement of the space $V := \{\delta \mid g(\gamma_x, \nabla_x\delta) \equiv 0\}$ at each time t is $\{\nabla_x(u\gamma_x) \mid u = u(x)\}$. Therefore, γ is stationary if and only if $\gamma_t \in V$ and $\nabla_t\gamma_t + \nabla_x^3\gamma_x - R(\gamma_x, \nabla_x\gamma_x)\gamma_x = \nabla_x(u\gamma_x)$ for some function $u = u(x, t)$.

In this paper, we treat an equation with resistance μ, which we call (EW).

$$\begin{cases}\nabla_t\gamma_t + \nabla_x^3\gamma_x + \mu\gamma_t = R(\gamma_x, \nabla_x\gamma_x)\gamma_x + \nabla_x(u\gamma_x), \\ |\gamma_x| \equiv 1,\end{cases} \tag{2.4}$$

with initial data $\gamma(x,0) = \gamma_0(x)$, $\gamma_t(x,0) = \gamma_1(x)$ satisfying $|\gamma_{0x}| = 1$ and $g(\gamma_{0x}, \nabla_x\gamma_1) = 0$. Its solutions satisfy the energy equality:

$$\frac{d}{dt}\{\|\gamma_t\|^2 + \|\nabla_x\gamma_x\|^2\} = -2\mu\|\gamma_t\|^2. \tag{2.5}$$

From $|\gamma_x| \equiv 1$, we derive an ODE for u as follows. Since

$$\begin{aligned}0 &= \partial_t^2|\gamma_x|^2 = 2g(\nabla_t^2\gamma_x, \gamma_x) + 2|\nabla_t\gamma_x|^2 \\ &= 2g(\nabla_x\nabla_t\gamma_t, \gamma_x) + 2g(R(\gamma_t, \gamma_x)\gamma_t, \gamma_x) + 2|\nabla_x\gamma_t|^2,\end{aligned} \tag{2.6}$$

the unknown u satisfies

$$\begin{aligned}&g(\nabla_x^4\gamma_x + \mu\nabla_x\gamma_t - \nabla_x\{R(\gamma_x, \nabla_x\gamma_x)\gamma_x\} - \nabla_x^2(u\gamma_x), \gamma_x) \\ &\quad = |\nabla_x\gamma_t|^2 + g(R(\gamma_x, \gamma_t)\gamma_x, \gamma_t).\end{aligned} \tag{2.7}$$

Using $|\gamma_x|^2 \equiv 1$, we have $g(\nabla_x\gamma_t, \gamma_x) = 0$, $g(\nabla_x^2\gamma_x\gamma_x) = -|\nabla_x\gamma_x|^2$, $g(\nabla_x^4\gamma_x, \gamma_x) = |\nabla_x^2\gamma_x|^2 - 2\partial_x^2|\nabla_x\gamma_x|^2$, and (2.7) becomes a linear ODE (EW′) for u:

$$\begin{aligned}&-u_{xx} + |\nabla_x\gamma_x|^2 u \\ &= 2\partial_x^2|\nabla_x\gamma_x|^2 - |\nabla_x^2\gamma_x|^2 + g(\nabla_x\{R(\gamma_x, \nabla_x\gamma_x)\gamma_x\}, \gamma_x) \\ &\quad + g(R(\gamma_x, \gamma_t)\gamma_x, \gamma_t) + |\nabla_x\gamma_t|^2.\end{aligned} \tag{2.8}$$

We write this equation as $-u_{xx} + |\nabla_x\gamma_x|^2 u = F_1(\nabla_x^3\gamma_x, \nabla_x\gamma_t)$. To compare solutions of equation EW (2.4) and solutions of equation (EP), we assume that $\mu > 1$ and change the time variable t in equation EW (2.4) to $\mu^{-1}t$. We call the resulting equation (EW^μ).

$$\begin{cases}\mu^{-2}\nabla_t\gamma_t + \nabla_x^3\gamma_x + \gamma_t = R(\gamma_x, \nabla_x\gamma_x)\gamma_x + \nabla_x(u\gamma_x), \\ |\gamma_x|^2 = 1,\end{cases} \tag{2.9}$$

with initial data $\gamma(x, 0) = \gamma_0(x)$, $\gamma_t(x, 0) = \mu\gamma_1(x)$ satisfying $|\gamma_{0x}| = 1$ and $g(\gamma_{0x}, \nabla_x\gamma_1) = 0$. Note that $\gamma_t(x, 0)$ diverges when $\mu \to \infty$. As equation EW′ (2.8), the condition $|\gamma_x|^2 = 1$ implies an ODE ($\mathrm{EW}^{\mu\prime}$) for u:

$$-u_{xx} + |\nabla_x\gamma_x|^2 u = F_1(\nabla_x^3\gamma_x, \mu^{-1}\nabla_x\gamma_t). \tag{2.10}$$

When $\mu \to \infty$, equation EW^μ (2.9) converges to an equation (EP)

$$\begin{cases}\eta_t + \nabla_x^3\eta_x = R(\eta_x, \nabla_x\eta_x)\eta_x + \nabla_x(w\eta_x), \\ |\eta_x|^2 \equiv 1,\end{cases} \tag{2.11}$$

with initial data $\eta(x, 0) = \eta_0(x)$ satisfying $|\eta_{0x}| = 1$. We will show that the solution of equation EW^μ (2.9) converges to the solution of equation EP (2.11). To prove it, we rewrite equation EP (2.11) as a parabolic version of equation EW^μ (2.9). The unknown w satisfies

$$-w_{xx} + |\nabla_x\eta_x|^2 w = F_1(\nabla_x^3\eta_x, 0). \tag{2.12}$$

It is derived from equation EW^μ (2.9) and equation $\mathrm{EW}^{\mu\prime}$ (2.10) by formal substitution: $\mu^{-1} = 0$. It is known that this equation has a unique long time solution.

Proposition 2.1 ([8, Theorem 4.1]). *Suppose that there are no closed geodesics of length* 1. *For any initial data* η_0 *satisfying* $|\eta_{0x}| = 1$, *there exists a unique solution of equation* EP (2.11) *on* $0 \le t < \infty$.

3. Short time existence

In this section, we prove Theorem 3.12 as follows. From equation EW′ (2.8), we see that u is comparable to γ_{xx}, and the right hand side of equation EW (2.4) implicitly contains the third derivative γ_{xxx}. This disturbs to apply the general method for plate equations. (See [13].) Therefore, we will perturb equation EW (2.4) to a parabolic equation EW^ε (3.3) below. Since it is standard to show the short time existence of solution γ^ε of equation EW^ε (3.3) for each $\varepsilon > 0$, the main step of proof of Theorem 3.12 is to show that the solutions γ^ε exist and are bounded on a time interval, which is uniform with respect to $\varepsilon > 0$. It implies that there exists a convergent subsequence γ^{ε_i} ($\varepsilon_i \to 0$), whose limit is a solution of equation EW (2.4).

In this section, we fix the resistance $\mu \in \mathbf{R}$ and the initial data γ_0, γ_1 satisfying $|\gamma_{0x}| = 1$ and $g(\gamma_{0x}, \nabla_x \gamma_1) = 0$.

3.1. Existence of a solution of perturbed equation. Observe that the principal part of equation EW (2.4) is a composition of two Schrödinger operators:

$$\gamma_{tt} + \partial_x^4 \gamma = \{\partial_t - \sqrt{-1}\partial_x^2\}\{\partial_t + \sqrt{-1}\partial_x^2\}\gamma, \tag{3.1}$$

which can be perturbed to a composition of parabolic operators:

$$\begin{aligned} &\gamma_{tt} - 2\varepsilon\gamma_{txx} + (1+\varepsilon^2)\partial_x^4\gamma \\ &= \{\partial_t - (\varepsilon + \sqrt{-1})\partial_x^2\}\{\partial_t - (\varepsilon - \sqrt{-1})\partial_x^2\}\gamma, \end{aligned} \tag{3.2}$$

where ε is a positive constant. Namely, we consider a 'parabolic' equation (EW^ε):

$$\begin{cases} \nabla_t\gamma_t - 2\varepsilon\nabla_x^2\gamma_t + (1+\varepsilon^2)\nabla_x^3\gamma_x = R(\gamma_x, \nabla_x\gamma_x)\gamma_x - \mu\gamma_t + \nabla_x(u\gamma_x), \\ |\gamma_x|^2 \equiv 1, \end{cases} \tag{3.3}$$

with initial data $\gamma(x,0) = \gamma_0(x)$, $\gamma_t(x,0) = \gamma_1(x)$ satisfying $|\gamma_{0x}| = 1$ and $g(\gamma_{0x}, \nabla_x\gamma_1) = 0$. We assume that the positive constant ε is smaller than 1.

To prove the existence of solutions of equation EW^ε (3.3), we have to control the unknown function u. By calculation similar to equation EW' (2.8), we derive an ODE ($\mathrm{EW}^{\varepsilon\prime}$) for u:

$$-u_{xx} + |\nabla_x\gamma_x|^2 u = F_2(\nabla_x^3\gamma_x, \nabla_x^2\gamma_t), \tag{3.4}$$

where

$$\begin{aligned} &F_2(\nabla_x^3\gamma_x, \nabla_x^2\gamma_t) \\ &= -2\varepsilon\{2g(\nabla_x^2\gamma_t, \nabla_x\gamma_x) + g(\nabla_x\gamma_t, \nabla_x^2\gamma_x)\} \\ &\quad + (1+\varepsilon^2)\{-|\nabla_x^2\gamma_x|^2 + 2\partial_x^2|\nabla_x\gamma_x|^2\} \\ &\quad + g(\nabla_x\{R(\gamma_x, \nabla_x\gamma_x)\gamma_x\}, \gamma_x) + g(R(\gamma_x, \gamma_t)\gamma_x, \gamma_t) + |\nabla_x\gamma_t|^2. \end{aligned} \tag{3.5}$$

To control u, we apply the following lemmas to equation $\mathrm{EW}^{\varepsilon\prime}$ (3.4).

Lemma 3.1 ([6, Lemma 4.1, Lemma 4.2]). *The ODE*: $-u'' + pu = q$ *on* S^1, *where* $p, q \in L_1$, $p \geq 0$ *and* $\|p\|_{L_1} > 0$, *has a unique solution* u, *and* u *is bounded in* C^1 *as*

$$\max|u| \leq 2(1 + \|p\|_{L_1}{}^{-1})\|q\|_{L_1}, \quad \max|u'| \leq 2(1 + \|p\|_{L_1})\|q\|_{L_1}. \tag{3.6}$$

Moreover, if $\|p\|_{L_1} \geq 1$, *then higher derivatives are bounded as*

$$\|u\|_{n+2} \leq C(1 + \|p\|_n^B)\|q\|_n, \quad \|u\|_{C^{n+2}} \leq C(1 + \|p\|_{C^n}^B)\|q\|_{C^n}, \tag{3.7}$$

where the positive integer B *and the positive constant depends only on* n.

Lemma 3.2. *For any integer* $n \geq 4$ *and any positive constant* K, *there exists a positive constant* C *with the following property*: *if a map* $\gamma : S^1 \times (a, b) \to M$ *satisfies* $|\gamma_x| \equiv 1$ *and* $\|\gamma_t\|_{n-1}$, $\|\gamma_x\|_n$, $\|\nabla_x\gamma_x\|^{-1} \leq K$, *then the solution* u *of equation* $\mathrm{EW}^{\varepsilon\prime}$ (3.4) *satisfies*

$$\|u\|_i^2 \leq C\{1 + \|\gamma_t\|_i^2 + \|\gamma_x\|_{i+1}^2\} \tag{3.8}$$

for every $i \leq n$.

Proof. By Lemma 3.1, we know that

$$\|u\|_{C^1} \leq 2\{1 + \|\nabla_x\gamma_x\|^{-2} + \|\nabla_x\gamma_x\|^2\}\|F_2(\nabla_x^3\gamma_x, \nabla_x^2\gamma_t)\|_{L_1} \leq C_1. \tag{3.9}$$

Therefore, $\|u\|_{C^2}$ is bounded, too. For $2 < i \leq n$, we have

$$\begin{aligned} \|\partial_x^i u\| &\leq C_2\{\|u\|_{i-2} + \|F_2(\nabla_x^3\gamma_x, \nabla_x^2\gamma_t)\|_{i-2}\} \\ &\leq C_3\{1 + \|u\|_{i-2} + \|\gamma_t\|_i + \|\gamma_x\|_{i+1}\}. \end{aligned} \tag{3.10}$$

It implies the boundedness of $\|u\|_i$ by induction. □

Therefore, if the quantity $\|\nabla_x\gamma_x\|$ is bounded from below by a positive constant, then the left hand side of equation $\mathrm{EW}^{\varepsilon}$ (3.3) is bounded by lower derivatives. For the principal part of equation $\mathrm{EW}^{\varepsilon}$ (3.3), we know

Lemma 3.3 ([9, Lemma 3.4]). *We consider a linear PDE for* w:

$$w_{tt} - 2\varepsilon w_{txx} + (1 + \varepsilon^2)w_{xxxx} = f, \tag{3.11}$$

with initial data $w(x, 0) = w_0(x)$, $w_t(x, 0) = w_1(x)$. *If* $f \in C^{2\alpha}$, $w_0 \in C^{4+2\alpha}$, $w_1 \in C^{2+2\alpha}$, *then there is a unique solution* $w \in C^{4+2\alpha}$ *satisfying*

$$\|w\|_{C^{4+2\alpha}} \leq C\{\|f\|_{C^{2\alpha}} + \|w_0\|_{C^{4+2\alpha}} + \|w_1\|_{C^{2+2\alpha}}\}. \tag{3.12}$$

Here, $\|w\|_{C^{4+2\alpha}}$ *and* $\|f\|_{C^{2\alpha}}$ *means the weighted Hölder norm* (t*-derivatives are counted twice of* x*-derivatives*).

We can apply to equation $\mathrm{EW}^{\varepsilon}$ (3.3) the standard argument for parabolic equations ([10]), replacing the estimation of the solutions of the linear heat equation to Lemma 3.3. We refer to [6] for the details.

Proposition 3.4. *Suppose that γ_0 is not geodesic. Then, for each $\varepsilon > 0$, equation* $\mathrm{EW}^{\varepsilon}$ (3.3) *has a solution on some interval $0 \le t < T$.*

On the other hand, if $\nabla_x \gamma_{0x} \equiv 0$, i.e., if the initial closed curve is a geodesic, then we cannot expect the existence of solutions. For example, let (M, g) be the surface of revolution: $((1+p^2)\cos\theta, (1+p^2)\sin\theta, p)$, $\gamma_0(x) = (\cos x, \sin x, 0)$ and $\gamma_1(x) = (0, 0, 1)$. This initial data satisfies the condition: $|\gamma_{0x}| = 1$ and $g(\gamma_{0x}, \nabla_x \gamma_1) = 0$, but there are no maps $\gamma : S^1 \times (a, b) \to M$ satisfying the initial condition and $|\gamma_x| \equiv 1$.

3.2. Uniform boundedness of solutions of equation ($\mathbf{EW}^{\varepsilon}$). Now, we have to prove that $T = T(\varepsilon)$ is uniformly bounded from below by a positive constant, and that γ is uniformly bounded on the interval $[0, T)$. We prepare two lemmas which hold for general map $\gamma : S^1 \times (a, b) \to M$.

Lemma 3.5. *Let $\gamma(x, t)$ be an arbitrary map: $S^1 \times (a, b) \to M$. For any integer $n \ge 0$ and any positive constant K, there exists a positive constant C with the following property: If $\|\gamma_t\|_{C^0}$, $\|\gamma_x\|_{C^0}$, $\|\gamma_t\|_{n-1}$, $\|\gamma_x\|_n \le K$, then*

$$\|\nabla_t \nabla_x^n \gamma_t - \nabla_x^n \nabla_t \gamma_t\| \le C, \quad \|\nabla_t \nabla_x^{n+1} \gamma_x - \nabla_x^{n+2} \gamma_t\| \le C \|\gamma_t\|_n. \tag{3.13}$$

Proof. We have

$$\begin{aligned} \|\nabla_t \nabla_x^n \gamma_t - \nabla_x^n \nabla_t \gamma_t\| &= \left\| \sum_{i=0}^{n-1} \nabla_x^i \{R(\gamma_t, \gamma_x) \nabla_x^{n-1-i} \gamma_t\} \right\|, \\ \|\nabla_t \nabla_x^{n+1} \gamma_x - \nabla_x^{n+2} \gamma_t\| &= \left\| \sum_{i=0}^{n} \nabla_x^i \{R(\gamma_t, \gamma_x) \nabla_x^{n-i} \gamma_x\} \right\|, \end{aligned} \tag{3.14}$$

and the assertion for $n = 0$ is obvious. Suppose that $n > 0$. Since $\|\gamma_t\|_{C^i} \le C_1$ for $i < n-1$ and $\|\gamma_x\|_{C^i} \le C_2$ for $i < n$,

$$\begin{aligned} \left\| \sum_{i=0}^{n-1} \nabla_x^i \{R(\gamma_t, \gamma_x) \nabla_x^{n-1-i} \gamma_t\} \right\| &\le C_3 \sum_{i,j\ge 0,\, i+j<n} \| |\nabla_x^i \gamma_t| \, |\nabla_x^j \gamma_t| \| \\ &\le C_4 \{ \|\nabla_x^{n-1} \gamma_t\| \, \|\gamma_t\|_{C^0} + C_5 \} \le C_6, \\ \left\| \sum_{i=0}^{n} \nabla_x^i \{R(\gamma_t, \gamma_x) \nabla_x^{n-i} \gamma_x\} \right\| &\le C_7 \{ \|\gamma_t\|_n + \|\gamma_t\|_{C^0} \|\nabla_x^n \gamma_x\| \} \le C_8 \|\gamma_t\|_n. \end{aligned} \tag{3.15}$$

$\square$

By this lemma, the covariant differentiations ∇_x and ∇_t practically commute in estimations after this.

Lemma 3.6. *For any integer $n \geq 4$ and any positive constant K, there exists a positive constant C with the following property: if a map $\gamma\colon S^1 \times (a, b) \to M$ satisfies $|\gamma_x| \equiv 1$ and $\|\gamma_t\|_{n-1}$, $\|\gamma_x\|_n$, $\|\nabla_x \gamma_x\|^{-1} \leq K$, then the solution u of equation* $\mathrm{EW}^{\varepsilon\prime}$ (3.4) *satisfies*

$$|\langle \nabla_x^i \gamma_t, \nabla_x^{i+1}(u\gamma_x)\rangle| \leq C\{1 + \|\gamma_t\|_i^2 + \|\gamma_x\|_{i+1}^2\} \tag{3.16}$$

for every $i \leq n$.

Proof. We express $\nabla_x^{i+1}(u\gamma_x)$ as a linear combination of $\partial_x^j u \cdot \nabla_x^{i+1-j}\gamma_x$. For the terms with $j < i + 1$,

$$\|\partial_x^j u \cdot \nabla_x^{i+1-j}\gamma_x\| \leq C_1 \max\{\|\gamma_x\|_{i+1}, \|u\|_i\} \leq C_2\{1 + \|\gamma_x\|_{i+1} + \|\gamma_t\|_i\}, \tag{3.17}$$

and,

$$\begin{aligned} \|\nabla_x^i \gamma_t\| \|\partial_x^j u \cdot \nabla_x^{i+1-j}\gamma_x\| &\leq C_3 \|\nabla_x^i \gamma_t\| \{1 + \|\gamma_x\|_{i+1} + \|\gamma_t\|_i\} \\ &\leq C_4\{1 + \|\gamma_t\|_i^2 + \|\gamma_x\|_{i+1}^2\}. \end{aligned} \tag{3.18}$$

For the term $\partial_x^{i+1} u \cdot \gamma_x$, we have

$$\langle \nabla_x^i \gamma_t, \partial_x^{i+1} u \cdot \gamma_x\rangle = \langle g(\nabla_x^i \gamma_t, \gamma_x), \partial_x^{i+1} u\rangle = -\langle \partial_x\{g(\nabla_x^i \gamma_t, \gamma_x)\}, \partial_x^i u\rangle. \tag{3.19}$$

Since $g(\nabla_x \gamma_t, \gamma_x) \equiv 0$, we can express the term $g(\nabla_x^i \gamma_t, \gamma_x)$ by a linear combination of $g(\nabla_x^j \gamma_t, \nabla_x^{i-j}\gamma_x)$ $(0 < j < i)$. Therefore,

$$\begin{aligned} |\langle \nabla_x^i \gamma_t, \partial_x^{i+1} u \cdot \gamma_x\rangle| &\leq C_5 \|u\|_i \sum_{j=1}^{i} \| |\nabla_x^j \gamma_t| \, |\nabla_x^{i+1-j}\gamma_x| \| \\ &\leq C_6 \|u\|_i (\|\gamma_x\|_i + \|\gamma_t\|_i) \leq C_7\{1 + \|\gamma_t\|_i^2 + \|\gamma_x\|_{i+1}^2\}. \quad \square \end{aligned} \tag{3.20}$$

REMARK 3.7. In the above proof, we utilized the condition $g(\nabla_x \gamma_t, \gamma_x) \equiv 0$ to estimate norms by lower derivatives. Such an estimation is a key point to prove existence theorem for equations which are unstable under perturbation of lower orders.

Using these lemmas, we can control the norm of γ as follows.

Lemma 3.8. *For any integer $n \geq 4$ and any positive constant K, there exists a positive constant C with the following property: Let γ is a solution of equation* $\mathrm{EW}^{\varepsilon}$ (3.3) *such that $\|\gamma_t\|_{n-1}$, $\|\gamma_x\|_n$, $\|\nabla_x \gamma_x\|^{-1} \leq K$. Then, $X(t) := \|\nabla_x^i \gamma_t\|^2 + (1 + \varepsilon^2)\|\nabla_x^{i+1}\gamma_x\|^2$ satisfies $X'(t) \leq C\{1 + X(t)\}$ for each $i \leq n$.*

Proof. By Lemma 3.5 and Lemma 3.2,

$$
\begin{aligned}
\frac{1}{2}X'(t) &= \langle \nabla_x^i\gamma_t, \nabla_t\nabla_x^i\gamma_t\rangle + (1+\varepsilon^2)\langle \nabla_x^{i+1}\gamma_x, \nabla_t\nabla_x^{i+1}\gamma_x\rangle \\
&= \langle \nabla_x^i\gamma_t, \nabla_x^i\nabla_t\gamma_t\rangle + (1+\varepsilon^2)\langle \nabla_x^{i+1}\gamma_x, \nabla_x^{i+2}\gamma_t\rangle \\
&\quad + \langle \nabla_x^i\gamma_t, \nabla_t\nabla_x^i\gamma_t - \nabla_x^i\nabla_t\gamma_t\rangle \\
&\quad + (1+\varepsilon^2))\nabla_x^{i+1}\gamma_x, \nabla_t\nabla_x^{i+1}\gamma_x - \nabla_x^{i+2}\gamma_t\rangle \\
&\leq \langle \nabla_x^i\gamma_t, \nabla_x^i\{2\varepsilon\nabla_x^2\gamma_t + R(\gamma_x, \nabla_x\gamma_x)\gamma_x - \mu\gamma_t + \nabla_x(u\gamma_x)\}\rangle \\
&\quad + C_1\{1+X(t)\} \\
&\leq -2\varepsilon\|\nabla_x^{i+1}\gamma_t\|^2 + \langle \nabla_x^i\gamma_t, \nabla_x^{i+1}(u\gamma_x)\rangle + C_2\{1+X(t)\} \\
&\leq C_3\{1+X(t)\}.
\end{aligned}
\tag{3.21}
$$

□

Proposition 3.9. *For any initial data $\gamma_0(x)$, $\gamma_1(x)$ satisfying $|\gamma_{0x}| = 1$ and $g(\gamma_{0x}, \nabla_x\gamma_1) = 0$, there exists a positive constant T such that equation* EW^ε (3.3) *has a solution γ^ε on $0 \leq t < T$ for every $0 < \varepsilon < 1$. Besides, γ^ε are smoothly and uniformly bounded with respect to ε.*

Proof. Note that the constant C in Lemma 3.8 depends continuously on the constant K. Therefore, $X(t) := \|\gamma_t\|_4^2 + (1+\varepsilon^2)\|\nabla_x\gamma_x\|_5^2 + \|\nabla_x\gamma_x\|^{-2}$ satisfies an inequality $X'(t) \leq f(X(t))$, where f is a continuous function independent of ε. Let φ be the solution of the ODE: $\varphi'(t) = f(\varphi(t))$ with initial value $\varphi(0) = \|\gamma_1\|_4^2 + 2\|\nabla_x\gamma_{0x}\|_5^2 + \|\nabla_x\gamma_{0x}\|^{-2}$. Suppose that φ exists on $[0, T]$. Since $0 < \varepsilon \leq 1$, $X(t) \leq \varphi(t)$ holds on this interval, hence, $\|\gamma_t\|_4$, $\|\gamma_x\|_5$ and $\|\nabla_x\gamma_x\|^{-2}$ are bounded.

On the interval $[0, T]$, we inductively apply Lemma 3.8 for $n \geq 4$. At each step, $\|\gamma_t\|_n$ and $\|\gamma_x\|_{n+1}$ are bounded. Therefore, the short time solution γ^ε of equation EW^ε (3.3) in Proposition 3.4 extends to the interval $[0, T]$. Besides, they are uniformly bounded with respect to ε. □

3.3. Existence and uniqueness of solutions of equation (EW).

Now, we can prove the short time existence.

Proposition 3.10. *Equation* EW (2.4) *has a short time solution for arbitrary smooth initial data satisfying $|\gamma_{0x}| = 1$ and $g(\gamma_{0x}, \nabla_x\gamma_1) = 0$.*

Proof. Let γ^ε be as in Proposition 3.9. Since they are uniformly bounded, there is a sequence $\varepsilon_i \to 0$ such that γ^{ε_i} converges C^∞-ly. The limit is a solution of equation EW (2.4). □

After the existence theorem is proved, the uniqueness of the solution can be proved by the standard argument.

Proposition 3.11. *Any two solutions of equation* EW (2.4) *on* $[0, T)$ *with same initial data identically coincide.*

Proof. Let $\{\gamma_0, \gamma_1\}$ be the initial data and (γ, u), $(\tilde{\gamma}, \tilde{u})$ are two solutions. Taking a tubular neighbourhood of γ_0, and taking its double covering if necessary, we may assume that γ_0 is in an open set U of $\mathbf{R}^N$. In U, we measure the difference between two solutions by $\zeta^p := \tilde{\gamma}^p - \gamma^p$ in coordinates and $v := \tilde{u} - u$. We regard $\zeta := (\zeta^p)(\partial/\partial x^p)$ as a vector field along γ.

We take a positive constant T', less than T in Proposition 3.10, such that the solutions γ and $\tilde{\gamma}$ stay in U for $t \le T'$. Note that γ and $\tilde{\gamma}$ are C^∞-ly bounded.

Since $(\nabla_x^3 \gamma_x)^p$ is expressed as $\partial_x^4 \gamma^p + 4\Gamma_q{}^p{}_r(\gamma)\gamma_x^q \partial_x^3 \gamma^r + \text{2nd order}$, the difference $(\nabla_x^3 \tilde{\gamma}_x)^p - (\nabla_x^3 \gamma_x)^p$ is expressed as

$$\partial_x^4 \zeta^p + 4\Gamma_q{}^p{}_r(\gamma)\gamma_x^q \partial_x^3 \zeta^r + \text{2nd order} = (\nabla_x^4 \zeta)^p + \text{2nd order}. \tag{3.22}$$

By a similar computation of v and rewriting them using covariant derivatives, we get

$$\begin{cases} \nabla_t^2 \zeta + \nabla_x^4 \zeta = v_x \gamma_x + H_1(\nabla_x^2 \zeta, \nabla_t \zeta, v), \\ -v_{xx} + |\nabla_x \tilde{\gamma}_x|^2 v = H_2(\nabla_x^4 \zeta, \nabla_x^2 \nabla_t \zeta). \end{cases} \tag{3.23}$$

Here, $H_1(\nabla_x^2 \zeta, \nabla_t \zeta, v)$ is a function of x, t, $\nabla_x^i \zeta$ $(i \le 2)$, $\nabla_t \zeta$, v such that

$$|H_1(\nabla_x^2 \zeta, \nabla_t \zeta, v)| \le C_1\{|\nabla_x^2 \zeta| + |\nabla_x \zeta| + |\zeta| + |\nabla_t \zeta| + |v|\}. \tag{3.24}$$

It implies that $\|H_1(\nabla_x^2 \zeta, \nabla_t \zeta, v)\|_i \le C_2\{\|\zeta\|_{i+2} + \|\nabla_t \zeta\|_i + \|v\|_i\}$. H_2 has similar property.

We apply Lemma 3.1 to the ODE for v in (3.23), and get

$$\|v\|_2 \le C_3 \|H_2(\nabla_x^4 \zeta, \nabla_x^2 \nabla_t \zeta)\| \le C_4 Z(t)^{1/2}, \tag{3.25}$$

where $Z(t) = \|\nabla_t \zeta\|_2^2 + \|\zeta\|_4^2$. For $n \le 2$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\{\|\nabla_x^n \nabla_t \zeta\|^2 + \|\nabla_x^{n+2} \zeta\|^2\} \\
&= \langle \nabla_x^n \nabla_t \zeta, \nabla_t \nabla_x^n \nabla_t \zeta \rangle + \langle \nabla_x^{n+2} \zeta, \nabla_t \nabla_x^{n+2} \zeta \rangle \\
&= \langle \nabla_x^n \nabla_t \zeta, \nabla_x^n (\nabla_t^2 \zeta + \nabla_x^4 \zeta) \rangle + \langle \text{lower orders} \rangle \\
&\le \langle \nabla_x^n \nabla_t \zeta, \nabla_x^n \{v_x \gamma_x + H_1(\nabla_x^2 \zeta, \nabla_t \zeta, v)\} \rangle + C_5 Z(t) \\
&\le \langle g(\nabla_x^n \nabla_t \zeta, \gamma_x), \partial_x^{n+1} v \rangle + C_6 Z(t) \\
&\le \|v\|_n \, \|\partial_x \{g(\nabla_x^n \nabla_t \zeta, \gamma_x)\}\| + C_7 Z(t) \\
&\le \|g(\nabla_x^{n+1} \nabla_t \zeta, \gamma_x)\|^2 + C_8 Z(t).
\end{aligned} \tag{3.26}$$

The last expression is bounded by $C_9 Z(t)$ if $n = 0, 1$. For $n = 2$, since $g(\nabla_x^3 \gamma_t, \gamma_x)$ is expressed by $g(\nabla_x^j \gamma_t, \nabla_x^{3-j} \gamma_x)$ $(j < 3)$, we see $\|g(\nabla_x^3 \nabla_t \zeta, \gamma_x)\| \le C_{10} \|\nabla_t \zeta\|_2$. Therefore, $Z'(t) \le C_{11} Z(t)$. Since $Z(0) = 0$, we have $Z(t) \equiv 0$.

This proof is valid at t_0 whenever $\tilde{\gamma}(t_0) = \gamma(t_0)$ and $\tilde{\gamma}_t(t_0) = \gamma_t(t_0)$. Therefore, the set $\{t \ge 0 \mid \tilde{\gamma}(t) = \gamma(t)\}$ is open and closed in $[0, T)$, hence coincides with $[0, T)$. $\square$

Combining Proposition 3.10 and Proposition 3.11, we get the following

Theorem 3.12. *Equation* EW (2.4) *has a unique short time solution for arbitrary smooth initial data* γ_0, γ_1 *satisfying* $|\gamma_{0x}| = 1$ *and* $g(\gamma_{0x}, \nabla_x \gamma_1) = 0$.

REMARK 3.13. When we replace t by $-t$, equation EW (2.4) does not change its form. Therefore the result is time-invertible. Namely, a unique solution exists on some open time interval $(-T, T)$.

4. Singular perturbation

In this section we prove Theorem 4.12 as follows. To compare γ and η, we embed them into $\mathbf{R}^N$ as in the proof of Proposition 3.11, and put $\zeta := (\gamma^p - \eta^p)(\partial/\partial x^p)_\eta$ and $v := u - w$. We will find that ζ, v satisfy

$$\begin{cases} \mu^{-2} \nabla_t^2 \zeta + \nabla_x^4 \zeta + \nabla_t \zeta = H_1(\nabla_x^2 \zeta, \mu^{-1} \nabla_t \zeta, v) + v_x \eta_x + \mu^{-1} G_3, \\ -v_{xx} + |\nabla_x \gamma_x|^2 v = H_2(\nabla_x^4 \zeta, \mu^{-1} \nabla_x^2 \nabla_t \zeta) + \mu^{-1} G_4, \end{cases} \tag{4.1}$$

where we take covariant derivatives along η except $\nabla_x \gamma_x$.

Then, we can apply the following lemma to quantities $X := \|\nabla_x^n \zeta\|^2$, $Y := \|\nabla_x^{n+2} \zeta\|^2$, $Z := \|\nabla_x^n \nabla_t \zeta\|$ and get the desired estimation of $\|\nabla_x^n \zeta\|$.

Lemma 4.1 ([7, Lemma 1.5]). *For any* K_1, $K_2 > 0$ *and any* $T > 0$, *there are* $C > 0$ *and* $\mu_0 > 0$ *with the following property*:

If $\mu \ge \mu_0$ *and* $X(t)$, $Y(t)$, $Z(t)$ *are non-negative functions on* $[0, T)$ *such that*

$$X(0) \le K_1 \mu^{-2}, \quad |X'(0)| \le K_1, \quad Y(0) \le K_1, \quad Z(0) \le K_1 \mu^2, \tag{4.2}$$

and that

$$\begin{aligned} \mu^{-2} X''(t) + X'(t) &\le K_1 \{X(t) + \mu^{-2} Z(t) + \mu^{-2}\} - K_2 Y(t), \\ Y'(t) + \mu^{-2} Z'(t) &\le K_1 \{Y(t) + 1\} - K_2 Z(t), \end{aligned} \tag{4.3}$$

on $[0, T)$, *then they satisfy*

$$X(t) < C\mu^{-2}, \quad Y(t) < C \quad \text{and} \quad Z(t) < C\mu^2 \tag{4.4}$$

on $[0, T)$.

For estimation of higher derivatives, it will suffices to prove that $Z := \|\nabla_t^m \zeta\|^2 + \|\nabla_x^n \nabla_t^m \zeta\|^2$ satisfies $\mu^{-2} Z' + Z \le C(\mu^{2m-3} e^{-\mu^2 t/2} + \mu^{-1})^2$.

STEP 1. A priori uniform boundedness of γ^μ.

At first, we have to prove that γ are bounded uniformly with respect to $\mu > 1$. It implies that curvature tensor and its derivatives are uniformly bounded. The energy equality of equation EW^μ (2.9):

$$\frac{d}{dt}\{\|\mu^{-2}\gamma_t\|^2 + \|\nabla_x \gamma_x\|^2\} = -2\|\gamma_t\|^2 \tag{4.5}$$

implies that

$$\begin{aligned}
&\left\{\min_{x\in S^1} \int_0^T |\gamma_t(x,t)|\, dt\right\}^2 \\
&\le T \int_0^1 \int_0^T |\gamma_t(x,t)|^2\, dt\, dx \le T \cdot \int_0^T \|\gamma_t\|^2\, dt \\
&= -T \cdot [\|\mu^{-2}\gamma_t\|^2 + \|\nabla_x \gamma_x\|^2]_0^T \le T \cdot \{\|\mu^{-2}\gamma_t\|^2 + \|\nabla_x \gamma_x\|^2\}|_{t=0},
\end{aligned} \tag{4.6}$$

where T is any positive constant.

Therefore, if initial values $\|\mu^{-2}\gamma_t(x,0)\|^2$ and $\|\nabla_x \gamma_x(x,0)\|^2$ are uniformly bounded with respect to $\mu > 1$, the solutions stay in a compact set A^T of the manifold M. Moreover, if there are no closed geodesics of length 1, the energy $\|\nabla_x \gamma_x\|^2$ of closed curves in A^T are bounded from below by a positive constant.

4.1. Uniform existence of γ^μ. We prove that the solution γ exist on a uniform interval $[0, T)$ with respect to $\mu > 1$. We prepare two lemmas. The first one gives the bounds of u by means of γ, and the second one is an ordinal differential inequality of the norm of γ. The bounds of u is given as a modification of Lemma 3.5, Lemma 3.2 and Lemma 3.6. Constants T, K and C below are independent of μ.

Lemma 4.2. *For any integer $n \ge 4$ and any positive constant K, there exists a positive constant C with the following property: If a map $\gamma\colon S^1 \times (a,b) \to A^T$ satisfies $|\gamma_x| \equiv 1$ and $\mu^{-1}\|\gamma_t\|_{n-1}$, $\|\gamma_x\|_n \le K$, then*

$$\mu^{-2}\|\nabla_t \nabla_x^i \gamma_t - \nabla_x^i \nabla_t \gamma_t\| \le C, \quad \|\nabla_t \nabla_x^{i+1}\gamma_x - \nabla_x^{i+2}\gamma_t\| \le C\|\gamma_t\|_i \tag{4.7}$$

hold for every $i \le n$. Besides, the solution u of equation $\mathrm{EW}^{\mu\prime}$ (2.10) *satisfies*

$$\begin{aligned}
&\|u\|_i \le C\{1 + \mu^{-1}\|\gamma_t\|_i + \|\gamma_x\|_{i+1}\}, \\
&|\langle \nabla_x^i \gamma_t, \nabla_x^{i+1}(u\gamma_x)\rangle| \le C\|\gamma_t\|_i \{1 + \mu^{-1}\|\gamma_t\|_i + \|\gamma_x\|_{i+1}\}.
\end{aligned} \tag{4.8}$$

Proof. We only check the last inequality.

$$
\begin{aligned}
|\langle \nabla_x^i \gamma_t, \partial_x^{i+1} u \cdot \gamma_x \rangle| &= |\langle \partial_x \{ g(\nabla_x^i \gamma_t, \gamma_x)\}, \partial_x^i u \rangle|, \\
\|\partial_x \{ g(\nabla_x^i \gamma_t, \gamma_x)\}\| &\le C_1 \{ \|\gamma_t\|_i + \|g(\nabla_x^{i+1}\gamma_t, \gamma_x)\|\} \\
&\le C_2 \{ \|\gamma_t\|_i + \sum_{j=1}^{i} \|g(\nabla_x^j \gamma_t, \nabla_x^{i+1-j}\gamma_x)\|\} \\
&\le C_3 \{ \|\gamma_t\|_i + \|\nabla_x \gamma_t\|_{C^0} \|\gamma_x\|_i \} \le C_4 \|\gamma_t\|_i. \qquad \square
\end{aligned}
\tag{4.9}
$$

Using this, we get the following differential inequality.

Lemma 4.3. *For any integer $n \ge 4$ and any positive constant K, there exists a positive constant C with the following property: Let γ be a solution of equation* EW^μ (2.9) *in A^T such that $\mu^{-1}\|\gamma_t\|_{n-1}$, $\|\gamma_x\|_n \le K$. Then, for every $i \le n$,*

$$
X(t) := \mu^{-2} \|\nabla_x^i \gamma_t\|^2 + \|\nabla_x^{i+1}\gamma_x\|^2 \tag{4.10}
$$

satisfies

$$
X'(t) \le C\{1 + X(t)\} + \|\gamma_t\|_{i-1}^2 - \|\nabla_x^i \gamma_t\|^2. \tag{4.11}
$$

Proof. By Lemma 4.2,

$$
\begin{aligned}
\frac{1}{2} X'(t) &= \mu^{-2} \langle \nabla_x^i \gamma_t, \nabla_t \nabla_x^i \gamma_t \rangle + \langle \nabla_x^{i+1}\gamma_x, \nabla_t \nabla_x^{i+1}\gamma_x \rangle \\
&= \mu^{-2} \langle \nabla_x^i \gamma_t, \nabla_x^i \nabla_t \gamma_t \rangle + \langle \nabla_x^{i+1}\gamma_x, \nabla_x^{i+2}\gamma_t \rangle \\
&\quad + \mu^{-2} \langle \nabla_x^i \gamma_t, \nabla_t \nabla_x^i \gamma_t - \nabla_x^i \nabla_t \gamma_t \rangle \\
&\quad + \langle \nabla_x^{i+1}\gamma_x, \nabla_t \nabla_x^{i+1}\gamma_x - \nabla_x^{i+2}\gamma_t \rangle \\
&\le \langle \nabla_x^i \gamma_t, \nabla_x^i \{ -\gamma_t + R(\gamma_x, \nabla_x \gamma_x)\gamma_x + \nabla_x (u \gamma_x)\}\rangle \\
&\quad + C_1 \|\gamma_t\|_i (1 + X(t))^{1/2} \\
&\le C_2 \|\gamma_t\|_i (1 + X(t))^{1/2} - \|\nabla_x^i \gamma_t\|^2 \\
&\le C_3 \{1 + X(t)\} + \frac{1}{2}\|\gamma_t\|_i^2 - \|\nabla_x^i \gamma_t\|^2 \\
&= C_3 \{1 + X(t)\} + \frac{1}{2}\|\gamma_t\|_{i-1}^2 - \frac{1}{2}\|\nabla_x^i \gamma_t\|^2. \qquad \square
\end{aligned}
\tag{4.12}
$$

Combining these lemmas, we get the following

Proposition 4.4. *For any positive number K, there exist positive constant T_1 and C with the following property: If the initial value of equation* EW^μ (2.9) *satisfies*

$\mu^{-1}\|\gamma_t(x,0)\|$, $\|\nabla_x \gamma_x(x,0)\| \le K$, then the solution exists on $[0, T_1]$. Besides, $\mu^{-1}\|\gamma_t\|_n$, $\|\gamma_x\|_n$ and $\|u\|_n \le C$ on $[0, T_1]$.

Proof. We see that the constant C in Lemma 4.3 continuously depends on K. Therefore,

$$X(t) := \sum_{i=0}^{3} 2^{-i}\{\mu^{-2}\|\nabla_x^i \gamma_t\|^2 + \|\nabla_x^{i+1}\gamma_x\|^2\} \tag{4.13}$$

satisfies $X'(t) \le f(X(t)) - (1/8)\|\gamma_t\|_3^2$ for some continuous function f independent of μ. Hence, $X(t)$ is bounded on some interval $[0, T_1]$. We inductively use Lemma 4.3 with $n = i \ge 4$, and get bounds of all $\|\gamma_x\|_n$ on $[0, T_1]$.

Since they are bounded, we can extend the solution on the interval $[0, T_1]$. □

4.2. Equations for the differences ζ, v. We derive a PDE for the difference of γ and η. To compare γ and η, we embed them into $\mathbf{R}^N$ as in the proof of Proposition 3.11. Since $\gamma(t)$ may jump when $\mu \to \infty$, we extend the riemannian metric to the whole $\mathbf{R}^N$, so that the metric is standard outside a compact set. In other words, we consider the solutions in $\mathbf{R}^N$ with metric tensor g_{pq} and Christoffel symbol $\Gamma_q{}^p{}_r$. In the coordinate expression, we have

$$\begin{aligned} \nabla_t \gamma_t &= \gamma_{tt}^p + \Gamma_q{}^p{}_r(\gamma)\gamma_t^q \gamma_t^r, \\ \nabla_x^3 \gamma_x &= \partial_x^4 \gamma^p + 4\Gamma_q{}^p{}_r(\gamma)\gamma_x^q \partial_x^3 \gamma^r + \text{lower derivatives.} \end{aligned} \tag{4.14}$$

Using these, we rewrite equation EW$^{\mu}$ (2.9) and equation EW$^{\mu\prime}$ (2.10) as

$$\begin{cases} \mu^{-2}\gamma_{tt} + \partial_x^4 \gamma + 4\Gamma(\gamma)\gamma_x \partial_x^3 \gamma + \gamma_t = G_1(\gamma_{xx}, \mu^{-1}\gamma_t, u) + (u\gamma_x)_x, \\ -u_{xx} + |\nabla_x \gamma_x|^2 u = G_2(\partial_x^4 \gamma, \mu^{-1}\gamma_{xxt}). \end{cases} \tag{4.15}$$

Similarly, the solution $\{\eta, w\}$ of equation EP (2.11) satisfies

$$\begin{cases} \partial_x^4 \eta + 4\Gamma(\eta)\eta_x \partial_x^3 \eta + \eta_t = G_1(\eta_{xx}, 0, w) + w_x \eta_x, \\ -w_{xx} + |\nabla_x \eta_x|^2 w = G_2(\partial_x^4 \eta, 0). \end{cases} \tag{4.16}$$

We measure the differences of these solutions by $\zeta := \gamma - \eta$ and $v := u - w$. Since we have to divide the given time interval to compare solutions, we cannot assume that $\zeta(x, 0) = 0$. For the initial data we make the following

ASSUMPTION 4.5. $\mu\|\zeta(x,0)\|_n$ and $\mu^{-1}\|\nabla_t \zeta(x,0)\|_n$ are uniformly bounded with respect to $\mu > 1$.

Let T_1 be as in Proposition 4.4. Then, the assumption implies that $\|\zeta\|_n$, $\mu^{-1}\|\nabla_t\zeta\|_n$ and $\|v\|_n$ are uniformly bounded with respect to $t \le T_1$ and $\mu > 1$. However, we do not have the bounds of $\mu\|\zeta\|_n$ at this step.

The differences ζ, v satisfy

$$\begin{cases} \mu^{-2}\zeta_{tt} + \partial_x^4\zeta + 4\Gamma(\eta)\eta_x\partial_x^3\zeta + \zeta_t \\ \quad = H_1(\zeta_{xx}, \mu^{-1}\zeta_t, v) + v_x\eta_x - \mu^{-2}\eta_{tt} + H_3(\mu^{-1}\eta_t), \\ -v_{xx} + |\nabla_x\gamma_x|^2 v = H_2(\partial_x^4\zeta, \mu^{-1}\zeta_{xxt}) + H_4(\mu^{-1}\eta_{xxt}). \end{cases} \tag{4.17}$$

Here, H_1 is a function such that $|H_1(\zeta_{xx},\mu^{-1}\zeta_t,v)| \le C\{|\zeta_{xx}|+|\zeta_x|+|\zeta|+\mu^{-1}|\zeta_t|+|v|\}$. It implies that $\|H_1(\zeta_{xx}, \mu^{-1}\zeta_t, v)\|_n \le C\{\|\zeta\|_{n+2} + \mu^{-1}\|\zeta_t\|_n + \|v\|_n\}$. Other H_p have similar property.

Regarding $\zeta = (\zeta^p)(\partial/\partial x^p)$ as a vector field along η, we can rewrite (4.17) using covariant derivatives:

$$\begin{cases} \mu^{-2}\nabla_t^2\zeta + \nabla_x^4\zeta + \nabla_t\zeta = H_1(\nabla_x^2\zeta, \mu^{-1}\nabla_t\zeta, v) + v_x\eta_x + \mu^{-1}G_3, \\ -v_{xx} + |\nabla_x\gamma_x|^2 v = H_2(\nabla_x^4\zeta, \mu^{-1}\nabla_x^2\nabla_t\zeta) + \mu^{-1}G_4, \end{cases} \tag{4.18}$$

where we take covariant derivatives along η except $\nabla_x\gamma_x$. $\|G_3\|_n$, $\|G_4\|_n$ are uniformly bounded with respect to $t \le T_1$ and $\mu > 1$.

4.3. Smallness of the differences. We will prove this by applying Lemma 4.1 to norms of ζ. For it, we have to estimate v by means of ζ.

Lemma 4.6. *Let* $\{\zeta, v\}$ *be a solution of* (4.18) *satisfying* Assumption 4.5. *For any integer* $n \ge 0$, *there exists a positive constant* C *such that*

$$|\langle\nabla_x^n\zeta, \nabla_x^n(v_x\eta_x)\rangle| \le C\|\zeta\|_n\|v\|_n \tag{4.19}$$

holds on $[0, T_1]$. *For any integer* $n \ge 2$, *there exists a positive constant* C *such that*

$$\|v\|_n \le C\{\mu^{-1}\|\nabla_t\zeta\|_n + \|\zeta\|_{n+2} + \mu^{-1}\} \tag{4.20}$$

holds on $[0, T_1]$.

Proof. The second inequality comes from the ODE for v in (4.18). For the first inequality, we see

$$|\langle\nabla_x^n\zeta, \nabla_x^n(v_x\eta_x)\rangle| \le C_1\|\zeta\|_n\|v\|_n + |\langle g(\nabla_x^{n+1}\zeta, \eta_x), \partial_x^n v\rangle|. \tag{4.21}$$

Since $g(\nabla_x^n\gamma_x, \gamma_x)$ is expressed by lower derivatives, $g(\nabla_x^{n+1}\zeta, \eta_x)$ is expressed by a function of $\nabla_x^i\zeta$ $(i \le n)$, and $\|g(\nabla_x^{n+1}\zeta, \eta_x)\|_n \le C_2\|\zeta\|_n$ holds. □

We put $X_n := \|\nabla_x^n\zeta\|^2$ and $Z_n := \|\nabla_x^n\nabla_t\zeta\|$.

Lemma 4.7. *For any integer $n \geq 0$, there exists a positive constant C such that*

$$\mu^{-2}({Z_n}^2)' + ({X_{n+2}}^2)' + {Z_n}^2 \leq C \tag{4.22}$$

holds on $[0, T_1]$.

Proof. Note that $\|\zeta\|_i$, $\mu^{-1}\|\nabla_t\zeta\|_i$ and $\|v\|_i$ are already bounded.

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\{\mu^{-2}\|\nabla_x^n\nabla_t\zeta\|^2 + \|\nabla_x^{n+2}\zeta\|^2\} + \|\nabla_x^n\nabla_t\zeta\|^2 \\
&= \langle\nabla_x^n\nabla_t\zeta, \mu^{-2}\nabla_t\nabla_x^n\nabla_t\zeta + \nabla_x^n\nabla_t\zeta\rangle + \langle\nabla_t\nabla_x^{n+2}\zeta, \nabla_x^{n+2}\zeta\rangle \\
&\leq \langle\nabla_x^n\nabla_t\zeta, \mu^{-2}\nabla_x^n\nabla_t^2\zeta + \nabla_x^n\nabla_t\zeta + \nabla_x^{n+4}\zeta\rangle \\
&\quad + C_1\{\mu^{-2}\|\nabla_x^n\nabla_t\zeta\|\|\nabla_t\zeta\|_{n-1} + \|\nabla_x^{n+2}\zeta\|\,\|\zeta\|_{n+1}\} \\
&\leq \langle\nabla_x^n\nabla_t\zeta, \nabla_x^n\{H_1(\nabla_x^2\zeta, \mu^{-1}\nabla_t\zeta, v) + v_x\eta_x + \mu^{-1}G_3\}\rangle + C_2 \\
&\leq C_3\|\nabla_x^n\nabla_t\zeta\| + C_2 \leq \frac{1}{2}\|\nabla_x^n\nabla_t\zeta\|^2 + C_4.
\end{aligned} \tag{4.23}$$

□

Lemma 4.8. *For $n = 0, 1$, there exists a positive constant C such that*

$$\mu^{-2}({X_n}^2)'' + ({X_n}^2)' \leq C\{\|\zeta\|_4^2 + \mu^{-2}\|\nabla_t\zeta\|_2^2 + \mu^{-2}\} \tag{4.24}$$

holds on $[0, T_1]$. *For any integer $n \geq 2$, there exists a positive constant C such that*

$$\mu^{-2}({X_n}^2)'' + ({X_n}^2)' + {X_{n+2}}^2 \leq C\{\|\zeta\|_{n+1}^2 + \mu^{-2}\|\nabla_t\zeta\|_n^2 + \mu^{-2}\} \tag{4.25}$$

holds on $[0, T_1]$.

Proof. We use Lemma 4.6.

$$\begin{aligned}
&\frac{1}{2}\mu^{-2}\frac{d^2}{dt^2}\|\nabla_x^n\zeta\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla_x^n\zeta\|^2 + \|\nabla_x^{n+2}\zeta\|^2 \\
&= \mu^{-2}\langle\nabla_x^n\zeta, \nabla_t^2\nabla_x^n\zeta\rangle + \mu^{-2}\|\nabla_t\nabla_x^n\zeta\|^2 + \langle\nabla_x^n\zeta, \nabla_t\nabla_x^n\zeta\rangle + \|\nabla_x^{n+2}\zeta\|^2 \\
&\leq \langle\nabla_x^n\zeta, \mu^{-2}\nabla_x^n\nabla_t^2\zeta + \nabla_x^n\nabla_t\zeta + \nabla_x^{n+4}\zeta\rangle + \mu^{-2}\|\nabla_x^n\nabla_t\zeta\|^2 \\
&\quad + C_1\{\mu^{-2}\|\nabla_x^n\zeta\|\,\|\nabla_t\zeta\|_{n-1} + \mu^{-2}\|\zeta\|_{n-1}^2 + \|\nabla_x^n\zeta\|\,\|\zeta\|_{n-1}\} \\
&\leq \langle\nabla_x^n\zeta, \nabla_x^n\{H_1(\nabla_x^2\zeta, \mu^{-1}\nabla_t\zeta, v) + v_x\eta_x + \mu^{-1}G_3\}\rangle \\
&\quad + C_2\{\|\zeta\|_n^2 + \mu^{-2}\|\nabla_t\zeta\|_n^2\}.
\end{aligned} \tag{4.26}$$

For the terms containing v, we use Lemma 4.6. Put $k = \max\{n, 2\}$.

$$\begin{aligned}
\langle\nabla_x^n\zeta, \nabla_x^n\{H_1(0, 0, v) + v_x\eta_x\}\rangle &\leq C_3\|\zeta\|_n\|v\|_n \\
&\leq C_4\|\zeta\|_n\{\|\zeta\|_{k+2} + \mu^{-1}\|\nabla_t\zeta\|_k + \mu^{-1}\},
\end{aligned} \tag{4.27}$$

Therefore, the totality

$$
\begin{aligned}
&\le C_5\|\zeta\|_n\{\|\zeta\|_{k+2}+\mu^{-1}\|\nabla_t\zeta\|_k+\mu^{-1}\}+C_2\mu^{-2}\|\nabla_t\zeta\|_n^2 \\
&\le \frac{1}{2}\|\zeta\|_{k+2}^2+C_6\{\|\zeta\|_n^2+\mu^{-2}\|\nabla_t\zeta\|_k^2+\mu^{-2}\}. \qquad \square
\end{aligned} \tag{4.28}
$$

Proposition 4.9. *Let* $\{\zeta, v\}$ *be a solution of* (4.18) *satisfying* Assumption 4.5. *For any integer* $n \ge 0$, *there exists a positive constant* C *such that* $\|\zeta\|_n \le C\mu^{-1}$ *holds on* $[0, T_1]$.

Proof. We may assume that $n \ge 4$. From Lemma 4.8 and the log-convexity of the norm $\|*\|_i$, i.e., $\|\nabla_x^i\zeta\|^2 \le \|\nabla_x^{i-1}\zeta\|\,\|\nabla_x^{i+1}\zeta\|$, we have

$$
\begin{aligned}
&\mu^{-2}({X_0}^2)''+({X_0}^2)' \le C_1\{{X_0}^2+{X_n}^2+\mu^{-2}{Z_0}^2+\mu^{-2}{Z_n}^2+\mu^{-2}\}, \\
&\mu^{-2}({X_n}^2)''+({X_n}^2)'+\frac{1}{2}{X_{n+2}}^2 \\
&\le C_2\{{X_0}^2+{X_n}^2+\mu^{-2}{Z_0}^2+\mu^{-2}{Z_n}^2+\mu^{-2}\}.
\end{aligned} \tag{4.29}
$$

Put $X := {X_0}^2+{X_n}^2$, $Y := {X_2}^2+{X_{n+2}}^2$, $Z := {Z_0}^2+{Z_n}^2$. Then,

$$
\mu^{-2}X''+X' \le C_3\{X+\mu^{-2}Z+\mu^{-2}\}-\frac{1}{2}Y. \tag{4.30}
$$

Also, from Lemma 4.7, we have

$$
\mu^{-2}Z'+Y'+Z \le C_4. \tag{4.31}
$$

Therefore, the assumption of Lemma 4.1 is satisfied, and we get $X \le C_5\mu^{-2}$. It implies the desired estimate by the log-convexity. $\square$

4.4. Estimation of time derivatives of ζ. Now we estimate time derivatives of ζ. For it, we put $\|*\|_{n,m}^2 := \sum_{i=0}^m \|\nabla_t^i * \|_n^2$. An argument similar to Lemma 4.6 gives the following

Lemma 4.10. *Let* $\{\zeta, v\}$ *be a solution of* (4.18) *satisfying* Assumption 4.5. *Let* $m \ge 1$ *be arbitrary integer. For any integer* $n \ge 0$, *there exists a positive constant* C *such that*

$$
|\langle \nabla_x^n\nabla_t^m\zeta, \nabla_x^n\nabla_t^{m-1}(v_x\eta_x)\rangle| \le C\{\|\zeta\|_{n,m}+\|\zeta\|_{n+1,m-1}\}\|v\|_{n,m-1} \tag{4.32}
$$

holds on $[0, T_1]$. *For any integer* $n \ge 2$, *there exists a positive constant* C *such that*

$$
\|v\|_{n,m-1} \le C\{\mu^{-1}\|\zeta\|_{n,m}+\|\zeta\|_{n+2,m-1}+\mu^{-1}\} \tag{4.33}
$$

holds on $[0, T_1]$.

Using this, we get

Proposition 4.11. *Let* $\{\zeta, v\}$ *be a solution of* (4.18) *satisfying Assumption* 4.5. *For arbitrary integers* $n \geq 0$ *and* $m \geq 0$, *there exists a positive constant* C *such that*

$$\|\zeta\|_{n,m} \leq C\{\mu^{2m-1}e^{-\mu^2 t/2} + \mu^{-1}\} \tag{4.34}$$

holds for $t \leq T_1$ *and sufficiently large* μ.

Proof. The claim holds for $m = 0$ by Proposition 4.9. We put $K_m := \mu^{2m-1}e^{-\mu^2 t/2} + \mu^{-1}$. Suppose that the claim holds up to $m-1$.

$$\begin{aligned}
&\frac{1}{2}\mu^{-2}\frac{d}{dt}\|\nabla_x^n\nabla_t^m\zeta\|^2 + \|\nabla_x^n\nabla_t^m\zeta\|^2 \\
&= \langle\nabla_x^n\nabla_t^m\zeta, \mu^{-2}\nabla_t\nabla_x^n\nabla_t^m\zeta + \nabla_x^n\nabla_t^m\zeta\rangle \\
&\leq \langle\nabla_x^n\nabla_t^m\zeta, \nabla_x^n\nabla_t^{m-1}(\mu^{-2}\nabla_t^2\zeta + \nabla_t\zeta)\rangle + C_1\mu^{-2}\|\nabla_x^n\nabla_t^m\zeta\|\,\|\zeta\|_{n-1,m} \\
&\leq \langle\nabla_x^n\nabla_t^m\zeta, \nabla_x^n\nabla_t^{m-1}\{-\nabla_x^4\zeta + H_1(\nabla_x^2\zeta, \mu^{-1}\nabla_t\zeta, v) + v_x\eta_x + \mu^{-1}G_3\}\rangle \\
&\quad + \mu^{-2}\|\nabla_x^n\nabla_t^m\zeta\|^2 + C_2\mu^{-2}\|\zeta\|_{n-1,m}^2.
\end{aligned} \tag{4.35}$$

For the terms concerning v, we use Lemma 4.10. Let $k := \max\{2, n\}$.

$$\begin{aligned}
&\|\nabla_x^n\nabla_t^{m-1}\{H_1(\,\cdot\,,\,\cdot\,, v) + v_x\eta_x\}\|^2 \\
&\leq C_3\{\|v\|_{n,m-1}^2 + (\|\zeta\|_{n,m} + \|\zeta\|_{n+1,m-1})\|v\|_{n,m-1}\} \\
&\leq C_4\{K_{m-1}^2 + \mu^{-2}\|\zeta\|_{k,m}^2 + \|\zeta\|_{n,m}(\mu^{-1}\|\zeta\|_{k,m} + K_{m-1})\} \\
&\leq C_5\{K_{m-1}^2 + \mu^{-1}\|\zeta\|_{k,m}^2 + K_{m-1}\|\zeta\|_{n,m}\}.
\end{aligned} \tag{4.36}$$

Therefore, the totality

$$\begin{aligned}
&\leq \left(\frac{1}{4} + \mu^{-2}\right)\|\nabla_x^n\nabla_t^m\zeta\|^2 \\
&\quad + C_6\{K_{m-1}^2 + \mu^{-2}\|\zeta\|_{n,m}^2 + \mu^{-1}\|\zeta\|_{k,m}^2 + K_{m-1}\|\zeta\|_{n,m}\} \\
&\leq \left(\frac{1}{4} + \mu^{-2}\right)\|\nabla_x^n\nabla_t^m\zeta\|^2 \\
&\quad + C_7\{(1+A)K_{m-1}^2 + (\mu^{-1} + A^{-1})(\|\nabla_x^k\nabla_t^m\zeta\|^2 + \|\nabla_t^m\zeta\|^2)\}.
\end{aligned} \tag{4.37}$$

When $n = 0$, we have

$$\begin{aligned}
&\mu^{-2}\frac{d}{dt}\|\nabla_t^m\zeta\|^2 + \|\nabla_t^m\zeta\|^2 \\
&\leq \left(-\frac{1}{2} + 2\mu^{-2}\right)\|\nabla_t^m\zeta\|^2 \\
&\quad + C_8\{(1+A)K_{m-1}^2 + (\mu^{-1} + A^{-1})(\|\nabla_x^2\nabla_t^m\zeta\|^2 + \|\nabla_t^m\zeta\|^2)\},
\end{aligned} \tag{4.38}$$

and when $n \geq 2$, we have

$$
\begin{aligned}
& \mu^{-2}\frac{d}{dt}\|\nabla_x^n\nabla_t^m\zeta\|^2 + \|\nabla_x^n\nabla_t^m\zeta\|^2 \\
(4.39) \qquad & \leq \left(-\frac{1}{2} + 2\mu^{-2}\right)\|\nabla_x^n\nabla_t^m\zeta\|^2 \\
& \quad + C_9\{(1+A)K_{m-1}^2 + (\mu^{-1} + A^{-1})(\|\nabla_x^n\nabla_t^m\zeta\|^2 + \|\nabla_t^m\zeta\|^2)\}.
\end{aligned}
$$

By these two inequalities, $Z := \|\nabla_t^m\zeta\|^2 + \|\nabla_x^n\nabla_t^m\zeta\|^2$ satisfies

$$
(4.40) \qquad \mu^{-2}Z' + Z \leq \left(-\frac{1}{2} + 2\mu^{-2}\right)Z + C_{10}\{(1+A)K_{m-1}^2 + (\mu^{-1} + A^{-1})Z\}.
$$

We choose $A = 4C_{10}$. Then, for $\mu \geq 4(2+C_{10})$, we have $\mu^{-2}Z' + Z \leq C_{11}K_{m-1}^2$. From this, we see that $Z(t) := \|\nabla_t^m\zeta\|_n^2$ satisfies

$$
\begin{aligned}
(4.41) \qquad Z(t) & \leq Z(0)e^{-\mu^2 t} + C_{12}\{\mu^{4(m-1)}e^{-\mu^2 t} + \mu^{-2}\} \\
& \leq C_{13}\{(\mu^2\mu^{2m-3}e^{-\mu^2 t/2})^2 + \mu^{4(m-1)}e^{-\mu^2 t} + \mu^{-2}\}. \qquad \square
\end{aligned}
$$

4.5. Global convergence. We sum up these results, and get the following

Theorem 4.12. *Suppose that there are no closed geodesics of length* 1, *and let* γ_0 *be a closed curve satisfying* $|\gamma_{0x}| = 1$. *Then, for any* $T > 0$ *and any vector field* γ_1 *along* γ_0 *satisfying* $g(\eta_x, \nabla_x\gamma_1) = 0$, *there exists* $\mu_0 > 0$ *with the following property*: *If* $\mu > \mu_0$, *then the solution* γ^μ *of equation* EW$^\mu$ (2.9) *with initial data* $\gamma^\mu(x,0) = \eta_0(x)$, $\gamma_t^\mu(x,0) = \mu\gamma_1(x)$ *exists on* $0 \leq t \leq T$. *Besides, when* $\mu \to \infty$, γ^μ *uniformly converges to the solution* η *of equation* EP (2.11) *with initial data* $\eta(x,0) = \gamma_0(x)$. *More precisely,*

$$
(4.42) \qquad |\partial_t^m\partial_x^n(\gamma^p - \eta^p)| \leq C\{\mu^{-1} + \mu^{2m-1}e^{-\mu^2 t/2}\}
$$

holds on each local coordinate.

REMARK 4.13. Even if there is a closed geodesic of length 1, there exists a solution of equation EP (2.11) on some time interval $[0, T_0)$, provided that the initial curve η_0 is not a geodesic [8, Theorem 3.1]. In this case, Theorem 4.12 still holds restricting the time interval to $[0, T_0)$.

Proof of Theorem 4.12. T_1 in Proposition 4.11 is bounded from below by the initial data $\gamma(x,0)$, $\mu^{-1}\gamma_t(x,0)$. Therefore, we have constants $T_2 > 0$ and $\mu_0 > 0$ such that if $\gamma(*, t_0)$ is sufficiently close to $\eta(*, t_0)$ and if $\mu > \mu_0$, then γ can be extended to the interval $[t_0, t_0 + T_2]$. Besides, $\gamma(x, t_0 + T_2)$ converges to $\eta(x, t_0 + T_2)$ with order $O(\mu^{-1})$.

Since η is known to be bounded, $T_2 = T_2(t_0)$ is uniformly bounded from below and $\mu_0 = \mu_0(t_0)$ is uniformly bounded from above with respect to t_0. Let T_3 be the lower bounds of T_2. We apply Proposition 4.11 on each subinterval $[kT_3/2, (k+2)T_3/2]$, and get (4.42) for $m = 0$. Besides,

$$|\partial_t^m \partial_x^n (\gamma^p - \eta^p)| \le C_1\{\mu^{-1} + \mu^{2m-1} \exp\{-\mu^2(t - kT_3/2)/2\}\} \le C_2\mu^{-1} \tag{4.43}$$

holds on $(k+1)T_3/2 \le t \le (k+2)T_3/2$ for each $k \ge 0$. □

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