Title	ALGEBRAIC CURVES VIOLATING THE SLOPE INEQUALITIES
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Citation	Osaka Journal of Mathematics. 2015, 52(2), p. 423-437
Version Type	VoR
URL	https://doi.org/10.18910/57641
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Kato, T. and Martens, G.
Osaka J. Math.
52 (2015), 423–437

ALGEBRAIC CURVES VIOLATING THE SLOPE INEQUALITIES

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(Received December 4, 2013)

Abstract

The gonality sequence $(d_r)_{r\geq 1}$ of a curve of genus g encodes, for $r < g$, important information about the divisor theory of the curve. Mostly it is very difficult to compute this sequence. In general it grows rather modestly (made precise below) but for curves with special moduli some "unexpected jumps" may occur in it. We first determine all integers $g > 0$ such that there is no such jump, for all curves of genus g. Secondly, we compute the leading numbers (up to $r = 19$) in the gonality sequence of an extremal space curve, i.e. of a space curve of maximal geometric genus w.r.t. its degree.

1. Introduction

Let X denote a smooth irreducible projective curve of genus $g \geq 4$ defined over $\mathbb{C}$. The numbers $d_r = d_r(X) := \mathrm{Min}\{d\colon \exists g^r_d \text{ on } X\}$, $r = 1, 2, \dots$, form the *gonality sequence* of X (called so since d_1 is the *gonality* of X). We say that X *satisfies the slope inequalities* (for its gonality sequence) if $d_r/r \geq d_{r+1}/(r+1)$ for all $r = 1, 2, \dots$, i.e. if $d_{r+1} - d_r \leq d_r/r$ for all r. So the slope inequalities limit the growth of the gonality sequence, by virtue of shrinking upper bounds.

While the original interest in these inequalities came from attempts of extending the notion of Clifford index from line bundles to vector bundles on curves ([13]) we consider these inequalities here as a tool for the specification of curves with special moduli. In fact, if X does not satisfy the slope inequalities, i.e. if $d_r/r < d_{r+1}/(r+1)$ for some r, it is easy to see that the *Brill–Noether number* $\rho_g(d_r, r) := g - (r + 1)(g - d_r + r)$ is negative; consequently, by Brill–Noether theory ([1], V), a general curve X of genus g must satisfy the slope inequalities. But this is also true for "very special" curves (w.r.t. moduli) like hyperelliptic curves (i.e. $d_1 = 2$) or trigonal curves (i.e. $d_1 = 3$) or bi-elliptic curves (i.e. double coverings of elliptic curves). On the other hand ([12], 4.6), for every $g \equiv 0 \bmod 3$, $g > 3$ there are curves of gonality $d_1 = 4$ and genus g violating the slope inequalities. It seems to be a delicate problem to determine the curves violating the slope inequalities, by finding characteristic descriptions for them. As is indicated in [12], good candidates are smooth curves in $\mathbb{P}^r$ of a specific geometric significance resp. curves whose Clifford index is not computed by pencils

2010 Mathematics Subject Classification. Primary 14H45; Secondary 14H51.

The first named author expresses his thanks to Univ. Erlangen-Nürnberg for the hospitality during his stay.

only; to be more concrete: extremal curves in $\mathbb{P}^r$ resp. the normalizations of plane curves with only few double points provide examples ([12], 4.13; [2]). We will show, as the first topic of this paper (Section 3), that these examples suffice to answer the *Question* (cf. [12], Question 5.4) *What is the smallest integer g_0 such that for every integer $g \geq g_0$ there is a curve of genus g not satisfying the slope inequalities*?

Ballico ([2]) showed that g_0 exists and that $g_0 \leq 31$. We will prove that $g_0 = 14$, and we also identify the sporadic families of curves of genus $g < 14$ violating the slope inequalities.

As our second topic we compute, in Section 4, the leading numbers $d_1, d_2, \ldots, d_{19}$ of the gonality sequence of an *extremal space curve*, and we determine series "computing" these d_r (i.e. series $g^r_{d_r}$ for $r \leq 19$). In particular, we show that $d_3/3 < d_4/4$ for extremal space curves of degree at least 10; so for these curves the violation of the slope inequalities occurs much earlier than observed in [12], 4.13.

NOTATIONS. We basically adopt the notation of [1]; in particular, a g^r_d on X is a linear series of degree d and (projective) dimension r on X. We call a g^r_d on X *very special* if it is complete with $r \geq 1$ and if its index of speciality $g-d+r$ is at least 2; then the g^r_d and its dual series $|K_X - g^r_d|$ are both at least pencils. (Here $|K_X|$ denotes the canonical series of X.) A g^r_d on X is called *simple* if the rational map $X \to \mathbb{P}^r$ induced by it is birational onto its image X'; X' is then an integral curve of degree at most d in $\mathbb{P}^r$. We call X an *extremal curve* in $\mathbb{P}^r$ if it has a simple g^r_d of degree $d \geq 3r-1$ and if X has the maximal genus among all curves admitting such a linear series; then the g^r_d is very ample (i.e. X and X' are isomorphic), and the genus g of X attains Castelnuovo's bound (cf. [12], 2).

For a non-negative real number x we denote by $[x]$ its integer part.

2. Preliminaries

The following result is an useful complement to [12], 3.2.

Proposition 2.1. *Assume that X is not a smooth plane curve. Then we have $d_r = r+g-2$ for $g-d_2+2 \leq r \leq g-d_1$.*

Proof. For $r = g-d_1$ this is proved in [12], Remark 4.4. Let $g-d_2+2 \leq r < g-d_1$. Then we have $d_r \leq d_{g-d_1} - (g-d_1-r) = r+g-2$. Assume that $d_r = r+g-2-\varepsilon$ for some integer $\varepsilon > 0$. Then $\deg|K_X - g^r_{d_r}| = 2g-2-d_r = g-r+\varepsilon$ and $\dim|K_X - g^r_{d_r}| = g-1-d_r+r = 1+\varepsilon$, i.e. $d_{1+\varepsilon} \leq g-r+\varepsilon$. Hence $d_2 + (\varepsilon-1) \leq d_{1+\varepsilon} \leq g-(g-d_2+2)+\varepsilon = d_2-2+\varepsilon$, a contradiction. □

Corollary 2.2. *Assume that X is not a smooth plane curve. Then we have $d_r/r \geq d_{r+1}/(r+1)$ for $r \geq g-d_2+2$.* (*And we have equality here for $r = g-1$ only, provided that X is not hyperelliptic.*)

Note that a $g^r_{d_r}$ on X is very special if and only if $1 \leq r \leq g - d_1$ ([12], 3.2 (b) and 4.4). Recall that the Clifford index γ of X gives rise to the following non-existence statement: If $2r > d - \gamma$ there is no very special g^r_d on X. We call X *divisorial complete* if also the converse of this statement is true. Obviously the hyper- and the bi-elliptic curves are divisorial complete but there are also some other examples ([7]). These curves come out when the differences $d_r - d_{r-1}$, $r = 2, 3, \ldots,$ are "too long" constant:

Proposition 2.3. *Let $g \geq 9$. Then the differences $d_r - d_{r-1}$ are constant for $r = 2, \ldots, [(g - d_1)/2] + 1$ if and only if X is divisorial complete.*

Proof. If X is divisorial complete we clearly have $d_r - d_{r-1} = 2$ for $r = 2, \ldots, g - d_1$. Let $g \geq 9$, and assume that $d_r - d_{r-1} = c$ with some constant $c > 0$, for $r = 2, \ldots, r_0 := [(g - d_1)/2] + 1$. This is certainly not true if X is a smooth plane curve; so we must have $d_2 > d_1 + 1$ whence $c \geq 2$. Assume that $c \geq 3$. Since $d_1 \leq (g + 3)/2$ and $g \geq 9$ we have $r_0 \geq 3$ unless $(g, d_1) = (9, 6)$ in which latter case $r_0 = 2$ and $d_2 = d_1 + c \geq 9$ contradicting $d_2 \leq 8$ ([12], 3.2 (c)). So $r_0 \geq 3$. In the (d, r)-plane, for given (d, r) let (d', r') be the "dual point" defined by $d' := 2g - 2 - d$, $r' := g - 1 - d + r$. Since the points (d_r, r), $r = 1, \ldots, r_0$, lie on a line l with slope $1/c$ the dual points $((d_r)', r')$ lie on a line l' with slope $1 - 1/c > 1/c$. For $r < r_0$ we have $d_{r+1} = d_r + c \geq d_r + 2$ which, by duality, is easily seen to imply that $d_{r'} = (d_r)'$ (i.e. $|K_X - g^r_{d_r}|$ is a $g^{r'}_{d_{r'}}$). We apply this for $r = r_0 - 1$. Note that $d_1 + 2(r_0 - 2)$ is $g - 2$ (resp. $g - 3$) if $g - d_1$ is even (resp. odd). So we have $d_{r_0-1} = d_1 + c(r_0 - 2) = d_1 + 2(r_0 - 2) + (c - 2)(r_0 - 2) \geq g - 3 + (c - 2)(r_0 - 2) \geq g - 2$ whence $(r_0 - 1)' = g - 1 - d_{r_0-1} + r_0 - 1 \leq r_0$. Thus the point $(d_{(r_0-1)'}, (r_0 - 1)') = ((d_{r_0-1})', (r_0 - 1)')$ lies on both l and l'. Since these lines meet on the line $d = g - 1$ we must have $(d_{r_0-1})' = g - 1$, and this implies that $(c - 2)(r_0 - 2) \leq 2$, i.e. $(r_0, c) = (3, 3), (3, 4), (4, 3)$. For $(r_0, c) = (4, 3)$ we have $g - d_1 = 7$, $d_{r_0} = d_4 = d_1 + 3c = d_1 + 9 = g + 2$ which, by duality, contradicts $d_2 = d_1 + c = g - 4$: $|K_X - g^2_{g-4}|$ is a g^5_{g+2}. Similarly, for $(r_0, c) = (3, 4)$ (resp. $(r_0, c) = (3, 3)$) we have $g - d_1 = 5$ (resp. $g - d_1 = 4$), i.e. $d_3 = g + 3$ (resp. $d_3 = g + 2$) contradicting $d_1 = g - 5$ (resp. $d_1 = g - 4$). Thus we obtain $c = 2$, for $g \geq 9$. Since $d_{r_0} = d_1 + 2(r_0 - 1) \geq g - 1$ and at least one $g^r_{d_r}$ with $d_r < g$ must compute the Clifford index γ of X (i.e. $d_r = \gamma + 2r$) all very special $g^r_{d_r}$ on X do. This implies that X is divisorial complete. □

In a sense, the next proposition indicates "how special" (w.r.t. moduli) a curve is which violates the slope inequalities.

Proposition 2.4. *$d_r/r < d_{r+1}/(r + 1)$ implies that $\rho_g(d_r, r) \leq -g/2$.*

Proof. Let i_s $(1 \leq s \in \mathbb{Z})$ denote the index of speciality of a $g^s_{d_s}$ (i.e. $i_s = g - d_s + s$) and let $D_s := (s + 1)d_s - sd_{s+1}$ (so $D_s < 0$ iff $d_s/s < d_{s+1}/(s + 1)$). One easily computes that

$$D_s = g - ((s+1)i_s - si_{s+1}) = \rho_g(d_s, s) + si_{s+1}.$$

By assumption, we have $D_r < 0$; then $1 < r < g - 1$. By [12], 3.2 (c) we have $i_{r+1} \geq [g/(r+2)]$. Hence we obtain

$$\rho_g(d_r, r) = D_r - ri_{r+1} \leq -1 - r\left[\frac{g}{r+2}\right].$$

We need the following numerical fact:

Claim. *Let* $1 < r < g - 1$. *Then* $r[g/(r+2)] \geq (g-3)/2$, *and we have equality here iff* $r = 2$ *and* $g \equiv 3$ *mod* 4, *or* $r = (g-3)/2$.

This claim shows that $\rho_g(d_r, r) \leq -(g-1)/2$, and equality is only possible for $r = 2$ or $r = (g-3)/2$. But $\rho_g(d_2, 2) = -(g-1)/2$ would imply that $6d_2 = 3g + 13$, and if $r = (g-3)/2$ equality would imply that $(r+1)(d_r - 3r) = -1$. Hence we obtain $\rho_g(d_r, r) \leq -g/2$. □

3. The number g_0

In this section we determine the smallest integer g_0 such that for every integer $g \geq g_0$ there is a curve of genus g not satisfying the slope inequalities. Ballico ([2]) proved that g_0 exists and $g_0 \leq 31$. We will show that $g_0 = 14$.

Proposition 3.1. *Let* Y *denote an integral plane curve of degree* $d \geq 6$ *whose singularities are* δ *ordinary double points. Assume that* $\delta \leq 2d - 12$. *Then we have* $d_3 \geq 2d - 4$ *for the normalization* X *of* Y.

Proof. Let $n := d_3$, and assume that $n \leq 2d - 5$. Then the g^3_n on X cannot be cut out on Y by conics. In fact, let $\mathbb{P}$ be a linear series of conics cutting out g^3_n on Y (in the sense of [4]). If $\mathbb{P}$ has a base curve then $\mathbb{P}$ splits off a line, and so g^3_n is already cut out on Y by lines which is impossible. Let $P_1, \ldots, P_r$ be the base points of $\mathbb{P}$ $(r \geq 0)$; including infinitely near points. Note that no 3 of these points are collinear since the line through 3 collinear points would be a base curve of $\mathbb{P}$. So $\mathbb{P}$ is contained in the linear series $\mathbb{P}' := |2l - P_1 - \cdots - P_r|$ of $\mathbb{P}^2$ with the assigned base points $P_1, \ldots, P_r$ where l denotes the class of a line in $\mathbb{P}^2$ ([8], V, 4), and we have $3 \leq \dim(\mathbb{P}') = 5 - r$ ([8], V, 4.2), i.e. $r \leq 2$. Since Y has merely double points this implies that $n \geq 2d - 2r \geq 2d - 4$ (and equality holds iff $r = 2$ and P_1, P_2 both are double points of Y which can happen for $\delta \geq 2$ only). This is a contradiction.

Since $n \leq 2d - 5$ and $\delta \leq 2d - 12$ we have $n + \delta < 4(d-4)$ which implies, according to the main lemma in [4], that g^3_n is cut out on Y by a linear series $\mathbb{P}$ of plane cubics. Since g^3_n is not cut out on Y by lines or conics $\mathbb{P}$ has no base curve. Again, let $P_1, \dots, P_r$ be the base points of $\mathbb{P}$. Then $\mathbb{P} \subseteqq \mathbb{P}' := |3l - P_1 - \cdots - P_r|$, and we have $3 \leq \dim(\mathbb{P}') = 9 - r$ ([8], V, 4.4), i.e. $r \leq 6$. Since Y has merely double points this implies that $n \geq 3d - 2r \geq 3d - 12 = (2d - 4) + (d - 8)$, a contradiction for $d \geq 8$. If $d = 6$ the plane sextic Y is smooth; then $d_3 = 2d - 2 = 10$ ([12], 4.3). If $d = 7$ the plane septic Y has, by hypothesis, at most two singular points; hence $n \geq 3d - 2.2 - 4.1 = 3d - 8 = 13 \geq 10 = 2d - 4$, again. □

Corollary 3.2. *In* Proposition 3.1 *let* $\delta \geq 2$. *Then* $d_3 = 2d - 4$.

Proof. X has two different base point free pencils g^1_{d-2}, and their sum is then a g^n_{2d-4} for some $n \geq 3$ (e.g., [1], III, ex. B-2). Hence $d_3 \leq 2d-4$, and so Proposition 3.1 proves the result. □

Corollary 3.3. *In* Proposition 3.1 *let* $\delta = 1$. *Then* $d_3 = 2d - 3$.

Proof. X has a base point free g^1_{d-2}; then $\dim|g^2_d + g^1_{d-2}| \geq 4$. Hence $d_3 \leq 2d-3$. If $d_3 < 2d - 3$ we have $d_3 + \delta < 3(d - 3)$, and according to the main lemma in [4] the $g^3_{d_3}$ is cut out on Y by conics. But then the arguments in the first part of the proof of Proposition 3.1 (with $\delta = 1$) give a contradiction. □

Corollary 3.4. *For* $g = 6, 10, 14, 15$ *and* $g \geq 20$ *there exists a curve of genus* g *such that* $d_2/2 < d_3/3$.

Proof. For $g = (d - 1)(d - 2)/2$ and $d \geq 5$ (note that this implies the cases $g = 6, 10, 15, 21$) we use [12], 4.3. For the remaining $g \geq 22$ we can write $g = (d - 1)(d-2)/2 - \delta$ with suitable $d \geq 9$, $1 \leq \delta \leq d - 3$ and apply Proposition 3.1 (note that $d - 3 \leq 2d - 12$ and $d_2 \leq d$). For $g = 14$ resp. $g = 20$ we apply Corollary 3.3 (for $d = 7$ resp. $d = 8$). □

With some effort one can extend Corollary 3.4 to $g = 18$ and $g = 19$; we don't need this fact.

Theorem 3.5. (i) *There is a curve of genus* g *violating the slope inequalities if and only if* $g \geq 14$, *or* $g \in \{6, 9, 10, 12\}$.
(ii) *More precisely, a curve of genus* $g < 14$ *violating the slope inequalities is an extremal curve, namely*

- *a smooth plane curve of degree* 5 *or* 6 ($g = 6$ *resp.* $g = 10$),
- *an extremal space curve of degree* 8 ($g = 9$; [12],4.7),
- *an extremal space curve of degree* 9 ($g = 12$; $d_1 = 4$, $d_2 = 8$, $d_3 = 9$, $d_4 = 12$, $d_5 = 13$, $d_6 = 16$),

- *an extremal curve of degree* 11 *in* $\mathbb{P}^4$ ($g = 12$; $d_1 = 4$, $d_2 = 7$, $d_3 = 10$, $d_4 = 11$, $d_5 = 14$, $d_6 = 15$).

Proof. (i) Assume that $g \geq 14$ or $g \in \{6, 9, 10, 12\}$. To observe the existence of a curve of genus g not satisfying the slope inequalities we first apply Corollary 3.4 for $g \geq 20$ and for $g = 6$, 10, 14, 15. For $g = 9$, 12, 16, 18, resp. $g = 19$ we apply [12], 2.1 and 4.13, resp. [12], 4.15.

To settle the remaining case $g = 17$ let S denote a general K3-surface in $\mathbb{P}^5$. Then Pic(S) is generated by (the class of) a hyperplane section H, $\deg(S) = H^2 = 8$, and S is known to be a complete intersection of three quadrics. Let X denote a smooth irreducible curve on S contained in the linear series $|2H|$ of S. Then X is a complete intersection of four quadrics in $\mathbb{P}^5$, of genus $g = 1 + (2H)^2/2 = 17$ and degree $d = 2H.H = 16$. The Clifford index γ of X is computed by $g^5_{16} := |H|_X|$ ([6], 3.2.6); hence $\gamma = 6$. We have $d_1 = 8$ ([6], 3.2.1) and $d_5 = 16$. We will show that $d_3 = 14$ which implies that $d_6 \geq 20$ (since $|K_X - g^6_{19}| = g^3_{13}$) and so $d_5/5 < d_6/6$. Assume that $d_3 < 14$. Then we have $d_3 = 12$ or $d_3 = 13$. First, let $d_3 = 13$. Then a g^3_{13} on X is base point free and simple, and so we have $\dim|g^5_{16} - g^3_{13}| \geq 2.5 + (3-1) - \dim|g^5_{16} + g^3_{13}|$, according to [1], III, ex. B-6. Since $d_1 > 3$ the series $|g^5_{16} + g^3_{13}|$ has dimension $(16 + 13) - g = 12$ or $(16 + 13) - g + 1 = 13$. In the first case we see that $\dim|g^5_{16} - g^3_{13}| \geq 0$. In the latter case we have $|g^5_{16} + g^3_{13}| \subset |K_X| = |2g^5_{16}|$, i.e. $\dim|g^5_{16} - g^3_{13}| \geq 0$, again. Hence any g^3_{13} on X is obtained by the projection of X into $\mathbb{P}^3$ with center a trisecant line of X. Similarly, for $d_3 = 12$ (note that a g^3_{12} on X computes γ and is therefore base point free and simple, [11]) an analogous argument shows that the g^3_{12} is obtained by the projection of X into $\mathbb{P}^3$ with center a quadrisecant line of X. But any tri- or quadrisecant line of X is contained in the four quadrics intersecting in X whence it is a part of X which is impossible. Hence we have $d_3 = 14$ (and $d_4 = 15$).

Conversely, let $g \leq 13$, $g \notin \{6, 9, 10, 12\}$; we have to show that every curve X of genus g satisfies the slope inequalities, then. We may assume that $g > 8$, $d_1 \geq 4$ and that X is not bi-elliptic ([12], Section 4). Hence $g = 11$ or $g = 13$, and we treat these two cases separately by brutal force (checking all possibilities for the gonality sequence of X without claiming that all these possibilities can actually be realized). To begin with, we state the

Claim. *Let g^2_d be a base point free and simple net on a curve X such that the induced plane model Y of X of degree d has* (*at least*) *two double points P, Q* (*i.e. points of multiplicity* 2). *Assume that P and Q are different points of $\mathbb{P}^2$ or that Q is infinitely near $P \in \mathbb{P}^2$. Then we have $d_3 \leq 2d - 4$.*

To prove the claim, if P, Q are different points of $\mathbb{P}^2$ the two projections $Y \to \mathbb{P}^1$ with center P resp. Q induce two different base point free pencils L_1, L_2 of degree $d - 2$ on X such that $\dim|L_1 + L_2| \geq 3$ (e.g., [1], III, ex. B-2). Based on this result, a semi-continuity argument implies: If $P \in \mathbb{P}^2$, Q is infinitely near to P on X and L is the base point free pencil on X defined by P then $\dim|2L| \geq 3$.

Let $g = 11$ or $g = 13$. Then X is not a smooth plane curve, and so $d_2 \geq d_1 + 2$. Furthermore, X has no g^2_6 (such a series implies that $g \leq 10$ or that X is hyper- or bi-elliptic or trigonal); hence $d_2 \geq 7$. Note that by duality, it is easy to compute the $d_r \geq g$ provided that all $d_r < g$ are already known. And it suffices to compute d_r up to $r \leq g - d_2 + 2$, by Corollary 2.2.

Let $g = 11$. By Brill–Noether theory ([1], V, (1.1)) we have $d_1 \leq 7$, $d_2 \leq 10$ and $d_3 \leq 12$. Moreover, $d_3 \geq 9$ (a g^3_8 implies that $g \leq 9$ or that X is hyper- or bi-elliptic), and $d_3 = 9$ implies $d_1 = 4$ since X—not being trigonal—is then birational equivalent to a space nonic lying on a quadric surface ([9], 3.13); so one of the (at most two) rulings of the quadric induces a g^1_4 on X. Keeping this in mind we obtain, for $d_1 \leq 5$, one of the following six possibilities for the gonality sequence of X; below the table we add some arguments.

	d_1	d_2	d_3	d_4	d_5	d_6
1	4	7	9	11	13	15
2	4	8	10 or 11	12	14	
3	5	7	10	12	13	15
4	5	8	10 or 11	12	14	
5	5	9	10 or 11	13		
6	5	10	12			

As to 1: By [12], 3.1 (d), $|g^1_4 + g^2_7| = g^4_{11} = |K_X - g^3_9|$; so $d_3 = 9$ whence a g^4_{10} on X would be very ample thus implying $g \leq 9$, by Castelnuovo's bound ([12], 2).

As to 2: $|K_X - g^2_8| = g^4_{12}$ (so $d_3 < d_4 \leq 12$). Assume that there is a g^3_9 on X. Then (see above) X is birationally equivalent to a space nonic Y on a quadric surface; if Y is singular we obtain a g^2_7 on X contradicting $d_2 = 8$. So Y is a smooth space curve of genus 11 on a quadric which is impossible since 11 is a prime number ([8], IV, 6.4.1).

As to 3: Recall that $d_3 = 9$ would imply $d_1 = 4$. So our claim (with $d = 7$) implies $d_3 = 10$.

If $d_1 = 6$ or $d_1 = 7$ there are no difficulties to compute the possible gonality sequences. Again, in all these cases the slope inequalities are satisfied.

Let $g = 13$. By Brill–Noether theory we have $d_1 \leq 8$, $d_2 \leq 11$ and $d_3 \leq 13$. Moreover, $d_3 \geq 10$ and $d_4 \geq 12$ (a g^3_9 or g^4_{11} implies, by Castelnuovo's bound, that $g \leq 12$, or that X is hyper- or bi-elliptic or trigonal).

We have, for $d_1 \leq 6$, the following possibilities for the gonality sequence of X; below the table we add some arguments.

	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8
1	4	8	10	12 or 13	14	16	18	
2	4	8	11	12 or 13	15	16	18	
3	4	8	12	14	15	16	18	
4	5	7	10	12	14	16	17	19
5	5	8	10	13	14	16	18	
6	5	8	11	13	15	16	18	
7	5	9	11	13	15	17		
8	5	9	12 or 13	14	15	17		
9	5	10	11	13	16			
10	5	10	12 or 13	14	16			
11	6	8	11	12 or 13	15	16	18	
12	6	8	12	14	15	16	18	
13	6	9	11	12 or 13	15	17		
14	6	9	12 or 13	14	15	17		
15	6	10	11	13	16			
16	6	10	12 or 13	14	16			
17	6	11	12 or 13	15				

As to 1, 2, 3 ($d_1 = 4$): Here $d_2 \leq 2d_1 = 8$ and $d_3 \leq 3d_1 = 12$. A g^2_7 induces a $g^4_{11} = |g^1_4 + g^2_7|$ ([12], 3.1 (d)), a contradiction. And $d_3 \leq 11$ means that $d_4 \leq 13$ (since $g^4_{13} = |K_X - g^3_{11}|$).

As to 4: Note that $\dim|2g^2_7| \geq 5$ and $\dim|g^1_5 + g^2_7| \geq 4$ ([12], 3.1 (d)).

As to 5, 6: For $d_1 = 5$, $d_2 \geq 8$ we have $d_4 > 12$, according to [15], Theorem 1. For $d_2 = 8$ we have $|g^1_5 + g^2_8| = g^4_{13} = |K_X - g^3_{11}|$; so $d_4 = 13$, $d_3 \leq 11$.

As to 7, 8: As before, $d_4 > 12$. If $d_3 = 10$ then $d_2 = 9$ implies that X is a smooth space curve of degree 10; for $g = 13$ it lies on a quadric surface ([9], 3.13) which is impossible since 13 is a prime number.

As to 9, 10: As before, $d_4 > 12$. Clearly, $d_2 \leq 2d_1 = 10$.

As to 11, 12: Since X has Clifford index 4 it cannot have a g^3_{10} ([11]). Hence we have $d_3 \geq 11$. We claim that $d_3 = 13$ is impossible. Assume that a g^2_8 on X (we are in the case $d_2 = 8$) is not simple. Then it is easy to see that it induces a double covering $X \to Y$ upon a smooth plane quartic Y. So X has infinitely many base point free pencils of degree 6. Taking two different of these pencils, L_1, L_2 say, we have $\dim|L_1 + L_2| \geq 3$ whence $d_3 \leq 12$. Assume that a g^2_8 on X is simple. Since $d_1 = 6$, X is birational equivalent to a plane octic with 8 double points. Applying our Claim we see that $d_3 \leq 16 - 4 = 12$.

As to 13, 14: As before, $d_3 \geq 11$.

As to 15: Since $d_5 = 16$ it is important in this case that $d_4 > 12$. But a g^4_{12} on X is very ample (since $d_3 = 11$) whence X is a smooth curve of degree 12 in $\mathbb{P}^4$; as such it has a trisecant line ([14], Lemma 4), and the projection with center this line gives us a g^2_9 on X contradicting $d_2 = 10$.

If $d_1 = 7$ or $d_1 = 8$ there are no difficulties to compute the possible gonality sequences. Again, the slope inequalities are satisfied.

(ii) The assertions (for $g = 6, 9, 10, 12$) follow from analogous considerations as in the proof of (i); we omit the details. □

4. Extremal space curves

In this section, X denotes an *extremal space curve* of degree d. We want to compute the first members of the gonality sequence of X. (By Theorem 3.5 we may assume that $d \geq 10$.) X has genus $g = [((d-2)/2)^2]$ and lies on a unique quadric surface; if this quadric surface is smooth X is of type $(d/2, d/2)$ resp. $((d-1)/2, (d+1)/2)$ on it if d is even resp. odd ([8], IV, 6.4, 6.4.1). We have $d_1 = [d/2]$, $d_2 = d - 1$ and $d_3 = d$ ([12], 4.8, 4.10).

Lemma 4.1. *Let $\nu \in \mathbb{N}$. If $\nu \leq (d-2)/2$ we have $\dim|\nu g^3_d| = \nu(\nu+2)$, $d_{\nu(\nu+2)} = \nu d$, and if $\nu < [(d-2)/2]$ we have $g^{\nu(\nu+2)}_{d_{\nu(\nu+2)}} = |\nu g^3_d|$ for the unique web g^3_d on X.*

Proof. The uniqueness of the g^3_d and the claim for $\nu < [(d-2)/2]$ follows from [12], 2.3. Let $\nu = [(d-2)/2]$. Then $\nu(\nu+2) \geq g$ and so ([12], 3.2 (a)) $d_{\nu(\nu+2)} = \nu(\nu+2) + g = \nu d$, and $|\nu g^3_d|$ is non-special of dimension $\nu d - g = \nu(\nu+2)$. □

If the quadric surface containing X is not smooth (i.e. is a quadric cone) then, according to [10], X is doubly covered by a smooth plane curve C of the same degree d; in that case we can try to relate the divisor theory of C ([5]) to that of X. To do so we recall from [5] the following notion:

DEFINITION. A base point free and very special g^r_n on a smooth plane curve C of degree d is called *trivial* if it is some multiple of the unique net g^2_d minus some points which impose independent linear conditions, i.e. if we have $g^r_n = |\mu g^2_d - E|$ for $\mu \in \mathbb{N}$ and an effective divisor E of C such that $r = \dim|\mu g^2_d| - \deg(E)$. (Note that the latter condition implies that $|\mu g^2_d|$ is special since g^r_n is; in particular, we have $\mu \leq d - 3$ and $\dim|\mu g^2_d| = \mu(\mu+3)/2$.)

Proposition 4.2. *Let X denote an extremal space curve of degree d lying on a quadric cone. Let $d \geq 21$, and for $\nu \in \mathbb{N}$ let $r(\nu) := [\nu(\nu+4)/4]$. Assume that $\nu \leq 7$. Then we have $d_{r(\nu)} = [\nu d/2]$, and for $r(\nu-1) < r < r(\nu)$ we have $d_r = d_{r(\nu)} - (r(\nu) - r)$ $(\nu \geq 2)$. (Furthermore, this remains true for $d = 10, 11, 12$, resp. $d = 13, 14, 15, 16$, resp. $d = 17, 18, 19, 20$ if $\nu \leq 4$, resp. $\nu \leq 5$, resp. $\nu \leq 6$.)*

Proof. By [10], there is a smooth plane curve C of degree d having an automorphism σ of order 2 such that the quotient curve $C/\langle\sigma\rangle$ is isomorphic to X. Let $\pi : C \to X$ be the resulting double covering. Using affine coordinates, σ can be defined by $(x, y) \mapsto (x, -y)$ and π by $(x, y) \mapsto (x, y^2)$ where, by [10], C is defined by the affine equation

$$y^d + a_1(x)y^{d-2} + a_2(x)y^{d-4} + \cdots + a_m(x) = 0$$

if $d = 2m$ is even, resp. by

$$xy^{d-1} + a_1(x)y^{d-3} + a_2(x)y^{d-5} + \cdots + a_m(x) = 0$$

if $d = 2m + 1$ is odd. Here $a_j(x)$ denotes a polynomial of degree at most $2j$ for even d, resp. of degree at most $2j + 1$ for odd d, and $a_m(x)$ is separable of degree d.

σ has d fixed points $P_1, \ldots, P_d$ lying on the line $y = 0$ (thus being defined by the d zeroes of $a_m(x)$), and for odd d there is still another fixed point P_∞ corresponding to $x \neq \infty$, $y = \infty$.

For $\nu \in \mathbb{N}$ let $V^{(\nu)}$ be the vector space of (inhomogeneous) polynomials in x, y of degree at most ν and $V_e^{(\nu)}$ (resp. $V_o^{(\nu)}$) be the subspace of $V^{(\nu)}$ consisting of σ-invariant (resp. σ-anti-invariant) polynomials. Let $e(\nu) := \dim(V_e^{(\nu)})$. Since $V_o^{(\nu)}$ is isomorphic to $V_e^{(\nu-1)}$ we have $e(\nu) + e(\nu - 1) = \dim(V^{(\nu)}) = (\nu + 1)(\nu + 2)/2$ whence $e(\nu) = [((\nu + 2)/2)^2]$. Note that $e(\nu) - 1$ is the number $r(\nu)$ defined in the statement of the proposition, and note that $r(\nu) - [\nu/2] = r(\nu - 1) + 1$ $(\nu \geq 2)$.

Since the σ-invariant part of $|\nu g_d^2| = g_{\nu d}^{\nu(\nu+3)/2}$ can be pushed forward to a $g_{[\nu d/2]}^{r(\nu)}$ on X we have $d_{r(\nu)} \leq [\nu d/2]$, and so $d_{r(\nu)-j} \leq [\nu d/2] - j$ for $j = 0, 1, \ldots, [\nu/2]$. If we can show that we have equality here for $j = [\nu/2]$ (i.e. $d_{r(\nu-1)+1} = d_{r(\nu)-[\nu/2]} = [\nu d/2] - [\nu/2]$) then we have equality for all $j = 0, 1, \ldots, [\nu/2]$. For doing so, let

$$r := r(\nu) - \left[\frac{\nu}{2}\right] = \left[\left(\frac{\nu + 1}{2}\right)^2\right],$$

and we consider the series $|\pi^*(g_{d_r}^r)|$ on C in the following claims.

Claim 1. *Let r be as above. If $d \geq 10$ and $\nu < d/2$ then $|\pi^*(g_{d_r}^r)|$ is very special.*

To see this, observe that $|\pi^*(g_{d_r}^r)|$ is a $g_{2d_r}^{r+\varepsilon}$ on C $(\varepsilon \geq 0)$ of degree $2d_r \leq 2([\nu d/2] - [\nu/2]) \leq \nu d - \nu + 1$ (where $2d_r = \nu d - \nu + 1$ is only possible if d is even and ν is odd). This series has index of speciality $h^1(|\pi^*(g_{d_r}^r)|) = g(C) - 2d_r + r + \varepsilon \geq (d - 1)(d - 2)/2 - 2d_r + r$; plugging in for $2d_r$ and r one easily computes that

$$4h^1(|\pi^*(g_{d_r}^r)|) \geq 2d(d - 3) - 4\nu d + \nu^2 + 6\nu + 1,$$

and since $\nu^2 + 6\nu \geq 12\nu - 9$ we obtain

$$h^1(|\pi^*(g^r_{d_r})|) \geq (d-3)(d-2\nu)/2 - 2.$$

Hence for $2\nu < d$ we see that $h^1(|\pi^*(g^r_{d_r})|) > 1$ if $d \geq 10$.

Claim 2. *Let r be as above. If $|\pi^*(g^r_{d_r})|$ is a trivial series on C then $d_r = [\nu d/2] - [\nu/2]$ (as wanted).*

To prove the claim, let $|\pi^*(g^r_{d_r})| = |\mu g^2_d - E|$ for an effective divisor E of C and some $\mu \leq d-3$ such that $r+\varepsilon = \dim|\pi^*(g^r_{d_r})| = \mu(\mu+3)/2 - \deg(E)$. The (incomplete) linear subseries $\pi^*(g^r_{d_r}) + E$ of $|\mu g^2_d|$ with base locus E is cut out on C by σ-invariant polynomials passing through E; since its moving part $\pi^*(g^r_{d_r})$ is σ-invariant so is its fixed part E, i.e. $\sigma(E) = E$.

Let $V^{(\mu)}(E)$ denote the subspace of $V^{(\mu)}$ consisting of polynomials of degree at most μ which pass through E, and let $V^{(\mu)}_e(E)$ (resp. $V^{(\mu)}_o(E)$) be the σ-invariant (resp. σ-anti-invariant) subspace of $V^{(\mu)}(E)$. Since $\sigma(E) = E$ we have $V^{(\mu)}(E) = V^{(\mu)}_e(E) \oplus V^{(\mu)}_o(E)$; hence $\dim(V^{(\mu)}_e(E)) = r + 1$ and $\dim(V^{(\mu)}_o(E)) = \varepsilon$. We clearly have $V^{(\mu)}_e(E) \subset V^{(\mu)}_e$, i.e. $[((\nu+1)/2)^2] + 1 = r + 1 \leq e(\mu) = [((\mu+2)/2)^2]$ which implies that $\nu \leq \mu$.

Let Q be a point in E. Then $\sigma(Q) \in E$. Assume that Q is a fixed point of σ. If $Q \in \{P_1, \ldots, P_d\}$, since the tangent line to C at Q is σ-invariant, the intersection multiplicity of it and C at Q is even; so $2Q \subset E$. If d is odd then $Q = P_\infty$ is possible, too. By the equation of C for odd d it is easy to see: If $Q = P_\infty$ and $\deg(E) = \mu d - 2d_r$ is even then $2Q \subset E$, again, and if $Q = P_\infty$ and $\deg(E)$ is odd we even have $2Q \subset E - Q$.

Now, take a point $Q_1 \in E$ (resp. $Q_1 \in E - P_\infty$ iff $\deg(E)$ is odd), and let $Q_2 := \sigma(Q_1)$. Since $|\mu g^2_d - E|$ is trivial so is $|\mu g^2_d - (E - Q_1 - Q_2)|$ which implies that $\dim(|\mu g^2_d - (E - Q_1 - Q_2)|) = r + \varepsilon + 2$. On the other hand, since $\pi(Q_1) = \pi(Q_2)$ and $V^{(\mu)}_o(E)$ is isomorphic to $V^{(\mu-1)}_e(E)$ we have $\dim(V^{(\mu)}_e(E - Q_1 - Q_2)) \leq r + 2$ and $\dim(V^{(\mu)}_o(E - Q_1 - Q_2)) \leq \varepsilon + 1$. Hence we obtain $\dim(V^{(\mu)}_e(E - Q_1 - Q_2)) = r + 2$.

Repeating this process until we have exhausted the points in E (resp. in $E - P_\infty$) we have $e(\mu) = r + 1 + [\deg(E)/2] = e(\nu) - [\nu/2] + [(\mu d - 2d_r)/2]$, i.e.

$$r(\mu) - r(\nu) = \left[\frac{\mu d}{2}\right] - \left[\frac{\nu}{2}\right] - d_r.$$

If $\nu = \mu$ we obtain $d_r = [\nu d/2] - [\nu/2]$, as wanted. So assume that $\nu < \mu$. Note that

$$r(\mu) - r(\nu) = \left[\frac{\mu(\mu+4)}{4}\right] - \left[\frac{\nu(\nu+4)}{4}\right] \leq \frac{\mu^2 - \nu^2 + 4(\mu-\nu) + 1}{4}.$$

So we have $d_r \geq [\mu d/2] - [\nu/2] - ((\mu - \nu)(\mu + \nu + 4) + 1)/4$. It suffices to show that the right hand side of this inequality is at least $[\nu d/2] - [\nu/2]$, i.e. to show that

$[\mu d/2]-[\nu d/2] \geq ((\mu-\nu)(\mu+\nu+4)+1)/4$. But $[\mu d/2]-[\nu d/2] \geq ((\mu-\nu)d-1)/2$, and we have $((\mu-\nu)d-1)/2 \geq ((\mu-\nu)(\mu+\nu+4)+1)/4$ since this latter inequality just means that $(\mu-\nu)(2d-4-\nu-\mu) \geq 3$ which is true for $\nu < \mu \leq d-3$. This proves the claim.

The next claim proves the proposition.

Claim 3. *Let r be as above, and assume that $\nu \leq 4$ for $10 \leq d \leq 12$, $\nu \leq 5$ for $13 \leq d \leq 16$, $\nu \leq 6$ for $17 \leq d \leq 20$, and $\nu \leq 7$ for $d \geq 21$. Then $d_r = [\nu d/2]-[\nu/2]$.*

In fact, our assumptions on ν imply that $\nu < d/2$, and so Claim 1 implies that $|\pi^*(g^r_{d_r})|$ is very special. Write our $r = [((\nu+1)/2)^2]$ in the form $r = (x+1)(x+2)/2 - \beta$ with non-negative integers x and $\beta \leq x$. If we then have $\deg(\pi^*(g^r_{d_r})) = 2d_r < d(r) := (x+3)(d-3)-\beta$ then, according to the main result in [5], the series $|\pi^*(g^r_{d_r})|$ is even trivial.

Now assume that $d_r < [\nu d/2]-[\nu/2]$, i.e. $2d_r \leq \nu d - \nu - 1$ if d is even and ν is odd, and $2d_r \leq \nu d - \nu - 2$ for the other parities of d and ν. Then it turns out (by our hypotheses on d and ν) that $2d_r < d(r)$. (In fact, we note that $r = 1, 2, 4, 6, 9, 12, 16$ for $\nu = 1, \ldots, 7$, and one computes $d(1) = 3d-9$, $d(2) = 4d-13$, $d(4) = 5d-17$, $d(6) = 5d-15$, $d(9) = 6d-19$, $d(12) = 7d-24$, $d(16) = 8d-29$.) By [5], we consequently see that $|\pi^*(g^r_{d_r})|$ is trivial ($r \leq 16$). But then Claim 2 implies that $d_r = [\nu d/2]-[\nu/2]$, a contradiction. □

Recall that $r(\nu) = [\nu(\nu+4)/4]$ and that, by Lemma 4.1, $\dim|\nu g^3_d| = \nu(\nu+2) = r(2\nu)$ for $\nu \leq (d-2)/2$. Here we add

Lemma 4.3. $\dim|\nu g^3_d + g^1_{d_1}| = r(2\nu+1)$ *if $\nu \leq (d-2)/2$. In particular, $d_{r(\mu)} \leq ((\mu-1)/2)d + d_1 = [\mu d/2]$ for odd $\mu \leq d-1$.*

Proof. If $\nu = [(d-2)/2]$ then $|\nu g^3_d + g^1_{d_1}|$ is non-special, and the claim follows from the Riemann–Roch theorem. So let $\nu < [(d-2)/2]$. First, let X lie on a smooth quadric surface. Since this surface has two rulings X has a pencil of degree d_1 different from our chosen $g^1_{d_1}$ resp. a base point free pencil $g^1_{d_1+1}$ if $d = 2d_1$ resp. $d = 2d_1+1$. Call this pencil L; we then have $g^3_d = |g^1_{d_1} + L|$. By the base point free pencil trick ([1], III. ex. B-4),

$$2\dim|\nu g^3_d + g^1_{d_1}| \leq \dim|(\nu g^3_d + g^1_{d_1}) + L| + \dim|(\nu g^3_d + g^1_{d_1}) - L|,$$

and

$$2\dim|\nu g^3_d - g^1_{d_1}| \leq \dim|(\nu g^3_d - g^1_{d_1}) + L| + \dim|(\nu g^3_d - g^1_{d_1}) - L|.$$

Observe that $|\nu g^3_d + g^1_{d_1} + L| = |(\nu+1)g^3_d|$, $|\nu g^3_d - g^1_{d_1} - L| = |(\nu-1)g^3_d|$ and $\deg|\nu g^3_d + g^1_{d_1} - L| \leq \nu d$. But $|\nu g^3_d - g^1_{d_1} + L|$ has degree νd resp. $\nu d + 1$ if d is even resp. odd. Let $g^r_{\nu d+1} := |\nu g^3_d - g^1_{d_1} + L|$ for odd d. By Lemma 4.1 we know that $r \le r(2\nu)+1$. Assume that $r = r(2\nu)+1$. Then, for some $P \in X$, $|g^r_{\nu d+1} - P| = g^{r(\nu)}_{\nu d} = |\nu g^3_d|$, by Lemma 4.1, which implies that $|K_X - \nu g^3_d|$ has the base point P; but we have $|K_X - ((d-5)/2)g^3_d| = g^1_{d_1}$, and so the series $|K_X - \nu g^3_d| = |(((d-5)/2)g^3_d + g^1_{d_1}) - \nu g^3_d| = |((d-5)/2 - \nu)g^3_d + g^1_{d_1}|$ is base point free. Hence we have $r \le r(2\nu)$.

Now Lemma 4.1 gives us

$$\begin{aligned} 2\dim|\nu g^3_d + g^1_{d_1}| &\le r(2\nu+2) + (r(2\nu) - 1) \\ &= (\nu+1)(\nu+3) + \nu(\nu+2) - 1 = 2\nu^2 + 6\nu + 2 = 2r(2\nu+1), \\ 2\dim|\nu g^3_d - g^1_{d_1}| &\le r(2\nu) + r(2\nu-2) \\ &= \nu(\nu+2) + (\nu-1)(\nu+1) = 2\nu^2 + 2\nu - 1 = 2r(2\nu-1) + 1, \end{aligned}$$

i.e. $\dim|\nu g^3_d + g^1_{d_1}| \le r(2\nu+1)$ and $\dim|\nu g^3_d - g^1_{d_1}| \le r(2\nu-1)$.

On the other hand, it follows that $\dim|\nu g^3_d + g^1_{d_1}| \ge 2\dim|\nu g^3_d| - \dim|\nu g^3_d - g^1_{d_1}| \ge 2r(2\nu) - r(2\nu-1) = 2\nu(\nu+2) - (\nu^2+\nu-1) = \nu^2 + 3\nu + 1 = r(2\nu+1)$, and this proves our claim.

Let X lie on a quadric cone. Then X has a unique $g^1_{d_1}$, and $|2g^1_{d_1}| = g^3_d$ resp. $|2g^1_{d_1} + P| = g^3_d$ for some point $P \in X$ if $d = 2d_1$ resp. $d = 2d_1 + 1$. Since X is a specialization of an extremal space curve of degree d on a smooth quadric surface we have $\dim|\nu g^3_d + g^1_{d_1}| \ge r(2\nu+1)$, by semi-continuity. On the other hand,

$$2\dim|\nu g^3_d + g^1_{d_1}| \le \dim|(\nu g^3_d + g^1_{d_1}) + g^1_{d_1}| + \dim|(\nu g^3_d + g^1_{d_1}) - g^1_{d_1}|,$$

and $|\nu g^3_d + g^1_{d_1} + g^1_{d_1}|$ is $|(\nu+1)g^3_d|$ resp. $|(\nu+1)g^3_d - P|$ if d is even resp. odd. Hence we have

$$2\dim|\nu g^3_d + g^1_{d_1}| \le (\nu+1)(\nu+3) + \nu(\nu+2) = 2r(2\nu+1) + 1,$$

i.e.

$$\dim|\nu g^3_d + g^1_{d_1}| \le r(2\nu+1). \qquad \square$$

An extremal space curve on a smooth quadric surface is a generization of an extremal space curve of the same degree on a quadric cone, and the numbers in the gonality sequence can only grow by generization ([12], 3.4). Hence our previous results in this section imply the

Theorem 4.4. *The claims of* Proposition 4.2 *hold for any extremal space curve X of degree d.*

Corollary 4.5. *Let $d \ge 10$. Then $d_3/3 < d_4/4$ and $d_8/8 < d_9/9$, and if $d > 10$ we also have $d_5/5 < d_6/6$.*

In [3], Ballico already observed, in a broader context, that $d_3/3 < d_4/4$ for $d \gg 0$.

QUESTIONS. How far does the pattern in the gonality sequence of X (observed in its first part) continue to hold? And can Corollary 4.5 be generalized to an extremal curve of degree $d \gg 0$ in $\mathbb{P}^r$; in particular, do we have $d_r/r < d_{r+1}/(r+1)$ $(r \geq 4)$?

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