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SEPARABLE FUNCTORS IN GROUP CORING

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Abstract

The paper will characterise the separability of the forgetful functor from the category of comodules over a group coring to the category of modules over a suitable algebra. Then the applications of our results to group entwined modules, Doi–Hopf group modules and relative Hopf group modules are considered.

1. Introduction

Turaev [15] introduced the notion of group coalgebras and Hopf group coalgebras in order to generalize the notion of a TQFT and Reshetikhin–Turaev invariant to the case of 3-manifolds endowed with a homotopy classes of maps to $K(\pi, 1)$, where π is any group. Hopf π-coalgebras generalize usual coalgebras and Hopf algebras, in the sense that we recover these notions in the situation where π is the trivial group. A systematic algebraic study of these new structures has been carried out in recent papers by Virelizier [16], Zunino [20, 21], and Wang [17, 18]. Many results from classical Hopf algebra theory can be generalized to Hopf group coalgebras; this has been explained in a paper by Caenepeel and De Lombaerde [3], where it was shown that Hopf group coalgebras are in fact Hopf algebras in a suitable symmetric monoidal category.

The notion of separable functor was introduced by Năstăsescu, Van den Bergh and Van Oystaeyen in [10], where some applications for group-graded rings were done. Every separable functor between abelian categories encodes a Maschke's theorem, which explains the interest concentrated in this notion within the module-theoretical developments in recent years. Separable functors have been investigated in the framework of coalgebras [8], graded homomorphisms of rings [9, 12], Doi–Koppinen modules [4], Doi–Hopf group modules [6], entwined modules [2] or coring [1].

Corings were introduced by Sweedler [13], and were revised by Brzeziński [1]. As the generalization of coring, Caenepeel, Janssen and Wang [5] introduce the group coring and develop Galois theory for group corings. It is natural to ask the following question: How to characterise the separability of the forgetful functor from the category of comodules over a group coring to the category of modules over a suitable algebra under more general assumptions and how to apply our results to some classical cases? This is the motivation of this paper.

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The article is organized as follows.

In Section 2, we recall some basic concepts such as group coalgebras, Hopf group coalgebras, group corings, etc. In Section 3, as the key content, we show that the forgetful functor from the category of comodules over a group coring to the category of modules over a suitable algebra has a right adjoint functor, and discuss under what conditions the adjoint functors are separable. Finally, we apply our main results to some special cases, such as group entwined modules, Doi–Hopf group modules and relative Hopf group modules.

2. Preliminaries

Throughout this paper, we always let π be a group with the unit e and k a field. All coalgebras, algebras, vector spaces and unadorned $\otimes$, Hom, etc., are over k. For terminology concerning coalgebras and Hopf algebras we refer the reader to [14]. Let us recall definitions and basic results related to Hopf group-coalgebras and group corings.

2.1. π-coalgebras. A π-coalgebra is a family of k-vector spaces $C = \{C_\alpha\}_{\alpha\in\pi}$ together with a family of k-linear maps $\Delta = \{\Delta_{\alpha,\beta}\colon C_{\alpha\beta} \to C_\alpha \otimes C_\beta\}_{\alpha,\beta\in\pi}$ (called a *comultiplication*) and a k-linear map $\varepsilon\colon C_e \to k$ (called a *counit*) such that
(C1) For any $\alpha, \beta, \gamma \in \pi$, Δ is coassociative in the sense that

$$(\Delta_{\alpha,\beta} \otimes \mathrm{id}_{C_\gamma}) \circ \Delta_{\alpha\beta,\gamma} = (\mathrm{id}_{C_\alpha} \otimes \Delta_{\beta,\gamma}) \circ \Delta_{\alpha,\beta\gamma}.$$

(C2) For all $\alpha \in \pi$,

$$(\mathrm{id}_{C_\alpha} \otimes \varepsilon) \circ \Delta_{\alpha,e} = \mathrm{id}_{C_\alpha} = (\varepsilon \otimes \mathrm{id}_{C_\alpha}) \circ \Delta_{e,\alpha}.$$

REMARK 2.1. Notice that $(C_e, \Delta_{e,e}, \varepsilon)$ is an ordinary coalgebra in the sense of Sweedler [14].

We use the Sweedler's notation for a comultiplication in the following way: for a π-coalgebra C, for any $\alpha, \beta \in \pi$ and $c \in C_{\alpha\beta}$, we write

$$\Delta_{\alpha,\beta}(c) = c_{(1,\alpha)} \otimes c_{(2,\beta)}.$$

2.2. Hopf π-coalgebras. A Hopf π-coalgebra is a π-coalgebra $H = (\{H_\alpha\}, \Delta, \varepsilon)$ together with a family of k-linear maps $S = \{S_\alpha\colon H_\alpha \to H_{\alpha^{-1}}\}_{\alpha\in\pi}$ (called an *antipode*) such that the following datas hold, for all $\alpha, \beta \in \pi$,
(HC1) Each H_α is an algebra with multiplication m_α and unit $1_\alpha \in H_\alpha$,
(HC2) $\Delta_{\alpha,\beta}$ and $\varepsilon\colon H_e \to k$ are algebra homomorphisms,
(HC3) $m_\alpha \circ (\mathrm{id}_{H_\alpha} \otimes S_{\alpha^{-1}}) \circ \Delta_{\alpha,\alpha^{-1}} = \varepsilon 1_\alpha = m_\alpha \circ (S_{\alpha^{-1}} \otimes \mathrm{id}_{H_\alpha}) \circ \Delta_{\alpha^{-1},\alpha}$.

REMARK 2.2. Note that the notion of a Hopf π-coalgebra is not self-dual and that $(H_e, \Delta_{e,e}, \varepsilon, S_e)$ is an ordinary Hopf algebra.

2.3. π-C-comodules. Let $C = \{C_\alpha\}_{\alpha\in\pi}$ be a π-coalgebra and $V = \{V_\alpha\}_{\alpha\in\pi}$ a family of k-vector spaces. A right π-C-comodule is a couple $(V, \rho^V = \{\rho^V_{\alpha,\beta}\}_{\alpha,\beta\in\pi})$, where for any $\alpha, \beta \in \pi$, $\rho^V_{\alpha,\beta}\colon V_{\alpha\beta} \to V_\alpha \otimes C_\beta$ is a k-linear morphism, which will be called a comodule structure and denoted by $\rho^V_{\alpha,\beta}(v) = v_{[0,\alpha]} \otimes v_{[1,\beta]}$, satisfying the following conditions,
(CC1) ρ^V is coassociative in the sense that, for any $\alpha, \beta, \gamma \in \pi$, we have

$$(\rho^V_{\alpha,\beta} \otimes \mathrm{id}_{C_\gamma}) \circ \rho^V_{\alpha\beta,\gamma} = (\mathrm{id}_{V_\alpha} \otimes \Delta_{\beta,\gamma}) \circ \rho^V_{\alpha,\beta\gamma},$$

(CC2) V is counitary in the sense that, for all $\alpha \in \pi$,

$$(\mathrm{id}_{V_\alpha} \otimes \varepsilon) \circ \rho^V_{\alpha,e} = \mathrm{id}_{V_\alpha}.$$

2.4. π-H-module coalgebras. Let $H = (\{H_\alpha\}_{\alpha\in\pi})$ be a Hopf π-coalgebra and $C = (\{C_\alpha\}_{\alpha\in\pi}, \Delta, \varepsilon)$ a π-coalgebra. C is called a right π-H-module coalgebra, if there is a family of k-linear maps $\phi = \{\phi_\alpha\colon C_\alpha \otimes H_\alpha \to C_\alpha,\ \phi_\alpha(c \otimes h) = c\cdot h\}_{\alpha\in\pi}$ such that the following conditions are satisfied,
(MC1) For any $\alpha, \beta \in \pi$, (C_α, ϕ_α) is a right H_α-module and $c \in C_{\alpha\beta}$, $h \in H_{\alpha\beta}$,

$$\Delta_{\alpha,\beta}(c\cdot h) = c_{(1,\alpha)}\cdot h_{(1,\alpha)} \otimes c_{(2,\beta)}\cdot h_{(2,\beta)}.$$

(MC2) For any $c \in C_e$ and $h \in H_e$, $\varepsilon(c\cdot h) = \varepsilon(c)\varepsilon(h)$.

2.5. π-corings. Let A be an algebra. A π-group A-coring (or shortly a π-A-coring) $\mathcal{C}$ is a family $\{\mathcal{C}_\alpha\}_{\alpha\in\pi}$ of A-bimodules together with a family of A-bimodule maps

$$\Delta_{\alpha,\beta}\colon \mathcal{C}_{\alpha\beta} \to \mathcal{C}_\alpha \otimes_A \mathcal{C}_\beta, \quad \varepsilon\colon \mathcal{C}_e \to A$$

such that the following conditions hold:
(a) $(\Delta_{\alpha,\beta} \otimes_A \mathrm{id}_{\mathcal{C}_\gamma}) \circ \Delta_{\alpha\beta,\gamma} = (\mathrm{id}_{\mathcal{C}_\alpha} \otimes_A \Delta_{\beta,\gamma}) \circ \Delta_{\alpha,\beta\gamma}$,
(b) $(\mathrm{id}_{\mathcal{C}_\alpha} \otimes_A \varepsilon) \circ \Delta_{\alpha,e} = \mathrm{id}_{\mathcal{C}_\alpha} = (\varepsilon \otimes_A \mathrm{id}_{\mathcal{C}_\alpha}) \circ \Delta_{e,\alpha}$,
for all $\alpha, \beta, \gamma \in \pi$.

For a π-group A-coring C, we also use the following standard notation for the comultiplication maps $\Delta_{\alpha,\beta}$:

$$\Delta_{\alpha,\beta}(c) = c_{(1,\alpha)} \otimes_A c_{(2,\beta)}$$

for all $c \in \mathcal{C}_{\alpha\beta}$.

For a π-A-coring $\mathcal{C}$, a right π-$\mathcal{C}$-comodule M is a family of right A-modules $\{M_\alpha\}_{\alpha\in\pi}$(meaning that each M_α is a right A-module), together with a family of right

A-linear maps $\rho^M = \{\rho^M_{\alpha,\beta}\}_{\alpha,\beta\in\pi}$, where $\rho^M_{\alpha,\beta}\colon M_{\alpha\beta} \to M_\alpha \otimes_A \mathcal{C}_\beta$, such that the following conditions hold:

(i) $(\mathrm{id}_{M_\alpha} \otimes_A \Delta_{\beta,\gamma}) \circ \rho^M_{\alpha,\beta\gamma} = (\rho^M_{\alpha,\beta} \otimes_A \mathrm{id}_{\mathcal{C}_\gamma}) \circ \rho^M_{\alpha\beta,\gamma}$, and

(ii) $(\mathrm{id}_{M_\alpha} \otimes_A \varepsilon) \circ \rho^M_{\alpha,e} = id_{M_\alpha}$

for all $\alpha, \beta, \gamma \in \pi$.

We use the following standard notation:

$$\rho^M_{\alpha,\beta}(m) = m_{[0,\alpha]} \otimes_A m_{[1,\beta]}$$

for $m \in M_{\alpha\beta}$.

A morphism between two right π-$\mathcal{C}$-comodules $M = \{M_\alpha\}_{\alpha\in\pi}$ and $N = \{N_\alpha\}_{\alpha\in\pi}$ is a family of right A-linear maps $f = \{f_\alpha\}_{\alpha\in\pi}$, $f_\alpha\colon M_\alpha \to N_\alpha$ such that

$$(f_\alpha \otimes_A \mathrm{id}_{\mathcal{C}_\beta}) \circ \rho^M_{\alpha,\beta} = \rho^N_{\alpha,\beta} \circ f_{\alpha\beta}.$$

The category of right π-$\mathcal{C}$-comodules will be denoted by $\mathcal{M}^{\pi,\mathcal{C}}$.

REMARK 2.3. If we take $A = k$, π-corings just be π-coalgebra, and then the objects in $\mathcal{M}^{\pi,\mathcal{C}}$ be a comodule over a π-coalgebra.

3. Separable functors in group corings

3.1. Separable functors. We recall now the definitions and some known results of separable functors from [10] and [11].

Separable functors. First, we recall some results relevant to separable functors. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, and $F\colon \mathcal{C} \to \mathcal{D}$ be a covariant functor. Observe that we have two covariant functors

$$\mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet)\colon \mathcal{C}^{op} \times \mathcal{C} \to \underline{\mathrm{Sets}}$$

and

$$\mathrm{Hom}_{\mathcal{D}}(F(\bullet), F(\bullet))\colon \mathcal{C}^{op} \times \mathcal{C} \to \underline{\mathrm{Sets}}$$

and F induces a natural transformation

$$\mathcal{F}\colon \mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet) \to \mathrm{Hom}_{\mathcal{D}}(F(\bullet), F(\bullet)).$$

We say that F is called a *separable functor*, if $\mathcal{F}$ splits, this means that we have a natural transformation

$$\mathcal{P}\colon \mathrm{Hom}_{\mathcal{D}}(F(\bullet), F(\bullet)) \to \mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet)$$

such that

$$\mathcal{P} \circ \mathcal{F} = 1_{\mathrm{Hom}_{\mathcal{C}}(\bullet,\bullet)}$$

the identity natural transformation on $\mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet)$. Separable functors were introduced in [10], and the definition can be found in the following more explicit form: F is separable if and only if for all $A, B \in \mathcal{C}$, we have a map

$$\mathcal{P}_{A,B}\colon \mathrm{Hom}_{\mathcal{D}}(F(A), F(B)) \to \mathrm{Hom}_{\mathcal{C}}(A, B)$$

such that the following conditions hold:
(SF1) For any $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $\mathcal{P}_{A,B}(F(f)) = f$.
(SF2) If we have morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $f_1 \in \mathrm{Hom}_{\mathcal{C}}(A_1, B_1)$ and $g \in \mathrm{Hom}_{\mathcal{D}}(F(A), F(A_1))$, $g' \in \mathrm{Hom}_{\mathcal{D}}(F(B), F(B_1))$ such that the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{g} & F(A_1) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle F(f_1)} \\ F(B) & \xrightarrow{g'} & F(B_1) \end{array}$$

commutes in $\mathcal{D}$, i.e. $F(f_1)g = g'F(f)$, then the diagram

$$\begin{array}{ccc} A & \xrightarrow{\mathcal{P}_{A,A_1}(g)} & A_1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_1} \\ B & \xrightarrow{\mathcal{P}_{B,B_1}(g')} & B_1 \end{array}$$

commutes in $\mathcal{C}$, i.e. $f_1\mathcal{P}_{A,A_1}(g) = \mathcal{P}_{B,B_1}(g')f$.
The terminology comes from the fact that, for a ring homomorphism $R \to S$, the restriction of scalars functor is separable if and only if S/R is separable.

Separable functors are a generalization of the theory of separable field extensions, and of separable algebras. Separability plays a crucial role in several topics in algebra and algebraic geometry (cf. [1, 2, 4, 8, 9, 11, 10] etc.).

Our next result is Rafael's theorem, giving necessary and sufficient conditions for a functor having an adjoint to be separable. Rafael's theorem is a key result in the development of our further theory.

Rafael's theorem [11]. Let $G\colon \mathbb{D} \to \mathbb{C}$ be the right adjoint of $F\colon \mathbb{C} \to \mathbb{D}$ with adjunctions $\eta\colon I_{\mathbb{C}} \to GF$ and $\varsigma\colon FG \to I_{\mathbb{D}}$. Then
(1) F is separable if and only if η splits, i.e., for all object $C \in \mathbb{C}$, there exists a morphism $\nu^C \in \mathrm{Mor}_{\mathbb{C}}(GF(C), C)$ such that $\nu^C \circ \eta^C = C$ and for all $f \in \mathrm{Mor}_{\mathbb{C}}(C, C')$, $\nu^{C'} \circ GF(f) = f \circ \nu^C$.
(2) G is separable if and only if ς cosplits, i.e., for all object $D \in \mathbb{D}$, there exists a morphism $\nu^D \in \mathrm{Mor}_{\mathbb{D}}(D, FG(D))$ such that $\varsigma^D \circ \nu^D = D$ and for all $f \in \mathrm{Mor}_{\mathbb{D}}(D, D')$, $FG(f) \circ \nu^D = \nu^{D'} \circ f$.

3.2. Separable functors in group corings. Now we will described the main results of separable functors in group corings in the paper.

Lemma 3.1. *Let A be an algebra and $\mathcal{C}$ be an π-A-coring. Then we have a pair of adjoint functors (F, G) between the categories $\mathcal{M}^{\pi,\mathcal{C}}$ and $\mathcal{M}_A$ (the category of right A-modules), where*

$$F\colon \mathcal{M}^{\pi,\mathcal{C}} \to \mathcal{M}_A, \quad F(M) = M_e, \quad F(f) = f_e$$

for any $M = \{M_\alpha\}_{\alpha\in\pi} \in \mathcal{M}^{\pi,\mathcal{C}}$ and $f = \{f_\alpha\colon M_\alpha \to N_\alpha\}_{\alpha\in\pi}$ in $\mathcal{M}^{\pi,\mathcal{C}}$, and

$$G\colon \mathcal{M}_A \to \mathcal{M}^{\pi,\mathcal{C}}, \quad G(N) = N \otimes_A \mathcal{C} = \{N \otimes_A \mathcal{C}_\alpha\}_{\alpha\in\pi} \quad (N \in \mathcal{M}_A)$$

with the coaction and action maps on $G(N)$ are defined by

$$\rho^{G(N)}_{\alpha,\beta}(n \otimes_A d) = n \otimes_A d_{(1,\alpha)} \otimes_A d_{(2,\beta)}, \quad (n \otimes_A c) \cdot a = n \otimes_A c \cdot a$$

for all $n \in N$, $c \in \mathcal{C}_\alpha$ and $d \in \mathcal{C}_{\alpha\beta}$.

Proof. Standard computations show that $G(N)$ is an object of $\mathcal{M}^{\pi,\mathcal{C}}$. Let us next describe the unit η and the counit ς of the adjunction (F, G). The unit is described as follows: for $M = \{M_\alpha\}_{\alpha\in\pi} \in \mathcal{M}^{\pi,\mathcal{C}}$, we define a family of k-linear maps $\eta^M = \{\eta^M_\alpha\}_{\alpha\in\pi}$,

$$\eta^M_\alpha\colon M_\alpha \to M_e \otimes_A \mathcal{C}_\alpha, \quad \eta^M_\alpha(m) = m_{[0,e]} \otimes_A m_{[1,\alpha]}.$$

It is straightforward to check that η^M is a morphism in $\mathcal{M}^{\pi,\mathcal{C}}$. For any $N \in \mathcal{M}_A$, we define $\varsigma^N\colon N \otimes_A \mathcal{C}_e \to N$ by $\varsigma^N = N \otimes_A \varepsilon$. We can check η and ς are natural transformations, and

$$G(\varsigma^N) \circ \eta^{G(N)} = I_{G(N)}, \quad \varsigma^{F(M)} \circ F(\eta^M) = I_{F(M)}$$

for all $N \in \mathcal{M}_A$ and $M = \{M_\alpha\}_{\alpha\in\pi} \in \mathcal{M}^{\pi,\mathcal{C}}$. □

Theorem 3.2. *Let $\mathcal{C}$ be an π-A-coring. Then the functor G in* Lemma 3.1 *is separable if and only if there exists an invariant $q \in \mathcal{C}^A_e = \{q \in \mathcal{C}_e \mid a \cdot q = q \cdot a \text{ for all } a \in A\}$ such that $\varepsilon(q) = 1_A$.*

Proof. "⇒". Suppose that G is separable, and let ν be split by ς. Since A is a right A-module, by Lemma 3.1 and Rafael's theorem, there exists a

$$\nu^A \in \mathrm{Mor}_A(A, FG(A)) = \mathrm{Mor}_A(A, A \otimes_A \mathcal{C}_e) \cong \mathrm{Mor}_A(A, \mathcal{C}_e).$$

Then we define $q = \nu^A(1_A)$. From ν^A being split by ς^A, it follows that

$$1_A = \varsigma^A \circ \nu^A(1) = \varepsilon \circ \nu^A(1) = \varepsilon(q).$$

Now for any $a \in A$, we define the morphism $f^{\alpha} \in \mathrm{Hom}_A(A, A)$ by $f^a(a') = aa'$. By the naturality of ν, we have the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\nu^A} & FG(A) \\ {\scriptstyle f^a}\big\downarrow & & \big\downarrow{\scriptstyle FG(f^a)} \\ A & \xrightarrow{\nu^A} & FG(A). \end{array}$$

For all $a, a' \in A$, since

$$\begin{aligned} \nu^A(aa') &= \nu^A \circ f^a(a') \\ &= FG(f^a) \circ \nu^A(a') \\ &= (f^a \otimes_A \mathcal{C}_e) \circ \nu^A(a') \\ &= a \cdot \nu^A(a'), \end{aligned}$$

so $\nu^A(a) = a \cdot \nu^A(1_A) = a \cdot q$. Also it follows that

$$\nu^A(a) = \nu^A(1_A) \cdot a = q \cdot a$$

from ν^A being a right A-linear. Hence q is A-central as required.

"$\Leftarrow$". Suppose that there exists $q \in \mathcal{C}_e^A$ such that $\varepsilon(q) = 1_A$. For any $N \in \mathcal{M}_A$, we define

$$\nu^N : N \to N \otimes_A \mathcal{C}_e, \quad \nu^N(n) = n \otimes_A q.$$

It is easily checked that ν^N is a right A-linear from q being A-central. Furthermore

$$\varsigma^N \circ \nu^N(n) = n \cdot \varepsilon(q) = n,$$

i.e., $\varsigma^N \circ \nu^N = N$. ν_N is natural in N. □

REMARK 3.3. Theorem 3.2 is a precise generalization of Theorem 3.3 in [1], which is the case of $\pi = \{e\}$ a trivial group.

Theorem 3.4. *For a π-A-coring $\mathcal{C}$, the forgetful functor $F: \mathcal{M}^{\pi,\mathcal{C}} \to \mathcal{M}_A$ is separable if and only if there exists a family of A-bimodules $\theta = \{\theta^{(\alpha)}: \mathcal{C}_{\alpha^{-1}} \otimes_A \mathcal{C}_\alpha \to A\}_{\alpha\in\pi}$ such that*

$$\theta^{(\alpha)}(c'_{(1,\alpha^{-1})} \otimes_A c'_{(2,\alpha)}) = \varepsilon(c'), \tag{3.1}$$

$$c_{(1,\beta)} \cdot \theta^{(\alpha\beta)}(c_{(2,\beta^{-1}\alpha^{-1})} \otimes_A d) = \theta^{(\alpha)}(c \otimes_A d_{(1,\alpha)}) \cdot d_{(2,\beta)} \tag{3.2}$$

for all $c' \in \mathcal{C}_e$, $c \in \mathcal{C}_{\alpha^{-1}}$, $d \in \mathcal{C}_{\alpha\beta}$.

Proof. Suppose F is separable. Let ν be the splitting of η. For any $\alpha\in\pi$, since $\mathcal{C}^{\alpha^{-1}}=\{\mathcal{C}_{\alpha^{-1}\beta}\}_{\beta\in\pi}$ is an object of $\mathcal{M}^{\pi,\mathcal{C}}$ via $\overline{\Delta}=\{\Delta_{\alpha^{-1}\beta,\gamma}\}_{\beta,\gamma\in\pi}$ and the same A-action on $\mathcal{C}$, there is a morphism $\nu^{\alpha^{-1}}=\{\nu_\beta^{\alpha^{-1}}\}_{\beta\in\pi}$,

$$\nu_\beta^{\alpha^{-1}}\colon \mathcal{C}_{\alpha^{-1}}\otimes_A\mathcal{C}_\beta\to\mathcal{C}_{\alpha^{-1}\beta}.$$

Then we define

$$\theta^{(\alpha)}=\varepsilon\circ\nu_\alpha^{\alpha^{-1}}\colon \mathcal{C}_{\alpha^{-1}}\otimes_A\mathcal{C}_\alpha\to A.$$

From ε and $\nu_\beta^{\alpha^{-1}}$ being both right A-linear, it follows that the map $\theta^{(\alpha)}$ is a right A-module morphism. Next for all $a\in A$, we consider a family of k-linear maps $f^{a,\alpha}=\{f_\beta^{a,\alpha}\}_{\beta\in\pi}$,

$$f_\beta^{a,\alpha}\colon \mathcal{C}_{\alpha^{-1}\beta}\to\mathcal{C}_{\alpha^{-1}\beta},\quad f_\beta^{a,\alpha}(c)=a\cdot c.$$

It is checked easily that $f^{a,\alpha}$ is a morphism of $\mathcal{M}^{\pi,\mathcal{C}}$. By the naturality of ν, we have the following commutative diagram

$$\begin{array}{ccc}
\mathcal{C}_{\alpha^{-1}}\otimes_A\mathcal{C}_\beta & \xrightarrow{\nu_\beta^{\alpha^{-1}}} & \mathcal{C}_{\alpha^{-1}\beta}\\
{\scriptstyle f_e^{a,\alpha}\otimes_A\mathcal{C}_\beta}\downarrow & & \downarrow{\scriptstyle f_\beta^{a,\alpha}}\\
\mathcal{C}_{\alpha^{-1}}\otimes_A\mathcal{C}_\beta & \xrightarrow{\nu_\beta^{\alpha^{-1}}} & \mathcal{C}_{\alpha^{-1}\beta}
\end{array}$$

for all $\alpha\in\pi$. Specially, for any $\beta\in\pi$, $c\in\mathcal{C}_{\alpha^{-1}}$, $d\in\mathcal{C}_\beta$, we have

$$\nu_\beta^{\alpha^{-1}}(f_e^{a,\alpha}(c)\otimes_A d)=f_\beta^{a,\alpha}\circ\nu_\beta^{\alpha^{-1}}(c\otimes_A d).$$

It follows from the above equation that, for any $\beta\in\pi$, $\nu_\beta^{\alpha^{-1}}$ is left A-linear. Since $\nu^{\alpha^{-1}}\circ\eta^{\alpha^{-1}}=\mathcal{C}^{\alpha^{-1}}$, it follows that

$$\nu_\beta^{\alpha^{-1}}(c_{(1,\alpha^{-1})}\otimes_A c_{(2,\beta)})=c$$

for all $\beta\in\pi$, $c\in\mathcal{C}_{\alpha^{-1}\beta}$. Then it follows that

$$\theta^{(\alpha)}(c_{(1,\alpha^{-1})}\otimes_A c_{(2,\alpha)})=\varepsilon\circ\nu_\alpha^{\alpha^{-1}}(c_{(1,\alpha^{-1})}\otimes_A c_{(2,\alpha)})=\varepsilon(c).$$

Now, for all $c\in\mathcal{C}_\beta$, we consider a family of k-linear maps

$$l^{(c,\alpha\beta)}=\left\{l_\gamma^{(c,\alpha\beta)}\colon \mathcal{C}_{(\alpha\beta)^{-1}\gamma}\to\mathfrak{R}_\gamma^{(\beta,\alpha)},\ c'\to c\otimes_A c'\right\}_{\gamma\in\pi}$$

and

$$\zeta^{(\alpha,\beta)}=\left\{\zeta_\gamma^{(\alpha,\beta)}\colon \mathcal{C}_{\alpha^{-1}\gamma}\to\mathfrak{R}_\gamma^{(\beta,\alpha)},\quad c\to c_{(1,\beta)}\otimes_A c_{(2,\beta^{-1}\alpha^{-1}\gamma)}\right\}_{\gamma\in\pi},$$

where

$$\mathfrak{R}^{(\beta,\alpha)} = \{\mathfrak{R}^{(\beta,\alpha)}_{\gamma} = \mathcal{C}_\beta \otimes_A \mathcal{C}_{\beta^{-1}\alpha^{-1}\gamma}\}_{\gamma\in\pi}.$$

Standard computation can check $l^{(c,\alpha\beta)}$ and $\zeta^{(\alpha,\beta)}$ is both morphisms in $\mathcal{M}^{\pi,\mathcal{C}}$. By the naturality of ν, we have the following commutative diagrams

$$\begin{array}{ccc}
\mathcal{C}_{(\alpha\beta)^{-1}} \otimes_A \mathcal{C}_\gamma & \xrightarrow{\nu_\gamma^{(\alpha\beta)^{-1}}} & \mathcal{C}_{(\alpha\beta)^{-1}\gamma} \\
\Big\downarrow{\scriptstyle l_e^{(c,\alpha\beta)}\otimes_A \mathcal{C}_\gamma} & & \Big\downarrow{\scriptstyle l_\gamma^{(c,\alpha\beta)}} \\
\mathcal{C}_\beta \otimes_A \mathcal{C}_{(\alpha\beta)^{-1}} \otimes_A \mathcal{C}_\gamma & \xrightarrow{\nu_\gamma^{\mathfrak{R}^{(\beta,\alpha)}}} & \mathcal{C}_\beta \otimes_A \mathcal{C}_{(\alpha\beta)^{-1}\gamma}
\end{array}$$

i.e., for any $c \in \mathcal{C}_\beta$, $c' \in \mathcal{C}_{(\alpha\beta)^{-1}}$ and $d \in \mathcal{C}_\gamma$, we have

$$\nu_\gamma^{\mathfrak{R}^{(\beta,\alpha)}}(c \otimes_A c' \otimes_A d) = c \otimes_A \nu_\gamma^{(\alpha\beta)^{-1}}(c' \otimes_A d) \tag{3.3}$$

and

$$\begin{array}{ccc}
\mathcal{C}_{\alpha^{-1}} \otimes_A \mathcal{C}_\gamma & \xrightarrow{\nu_\gamma^{\alpha^{-1}}} & \mathcal{C}_{\alpha^{-1}\gamma} \\
\Big\downarrow{\scriptstyle \zeta_e^{(\alpha,\beta)}\otimes_A \mathcal{C}_\gamma} & & \Big\downarrow{\scriptstyle \zeta_\gamma^{(\alpha,\beta)}} \\
\mathcal{C}_\beta \otimes_A \mathcal{C}_{\beta^{-1}\alpha^{-1}} \otimes_A \mathcal{C}_\gamma & \xrightarrow{\nu_\gamma^{\mathfrak{R}^{(\beta,\alpha)}}} & \mathcal{C}_\beta \otimes_A \mathcal{C}_{\beta^{-1}\alpha^{-1}\gamma}
\end{array}$$

i.e.,

$$\begin{aligned}
\zeta_\gamma^{(\alpha,\beta)} \circ \nu_\gamma^{\alpha^{-1}}(c \otimes_A d) &= \nu_\gamma^{\mathfrak{R}^{(\beta,\alpha)}}(c_{(1,\beta)} \otimes_A c_{(2,\beta^{-1}\alpha^{-1})} \otimes_A d) \\
&= c_{(1,\beta)} \otimes_A \nu_\gamma^{(\alpha\beta)^{-1}}(c_{(2,\beta^{-1}\alpha^{-1})} \otimes_A d)
\end{aligned} \tag{3.4}$$

for all $c \in \mathcal{C}_{\alpha^{-1}}$ and $d \in \mathcal{C}_\gamma$. Taking $\gamma = \alpha\beta$ and applying $\mathcal{C}_\beta \otimes_A \varepsilon$ to Equation (3.4), we have

$$\nu_{\alpha\beta}^{\alpha^{-1}}(c \otimes_A d) = c_{(1,\beta)} \cdot \theta^{(\alpha\beta)}(c_{(2,\beta^{-1}\alpha^{-1})} \otimes_A d). \tag{3.5}$$

Since $\nu^{\alpha^{-1}}$ is π-$\mathcal{C}$-comodule map, we have

$$\Delta_{e,\beta} \circ \nu_{\alpha\beta}^{\alpha^{-1}}(c \otimes_A d) = \nu_\alpha^{\alpha^{-1}}(c \otimes_A d_{(1,\alpha)}) \otimes_A d_{(2,\beta)} \tag{3.6}$$

for all $d \in \mathcal{C}_{\alpha\beta}$ and $c \in \mathcal{C}_{\alpha^{-1}}$. Applying $\varepsilon \otimes_A \mathcal{C}_\beta$ to Equation (3.6), we have

$$\nu_{\alpha\beta}^{\alpha^{-1}}(c \otimes_A d) = \theta^{(\alpha)}(c \otimes_A d_{(1,\alpha)}) \cdot d_{(2,\beta)}. \tag{3.7}$$

Comparison Equations (3.5) and (3.7), we obtain Equation (3.2) as required.

"$\Leftarrow$". Suppose that there exists θ as in the theorem. Then for all $M \in \mathcal{M}^{\pi,\mathcal{C}}$, we define a family of k-linear maps $\nu^M = \{\nu_\alpha^M\}_{\alpha\in\pi}$,

$$\nu_\alpha^M : M_e \otimes_A \mathcal{C}_\alpha \to M_\alpha, \quad m \otimes_A c \to m_{[0,\alpha]} \cdot \theta^{(\alpha)}(m_{[1,\alpha^{-1}]} \otimes_A c).$$

Notice first that ν^M is right A-linear. Next, we show that ν^M is right π-$\mathcal{C}$-colinear, i.e., the following diagram is commutative

$$\begin{array}{ccc} M_e \otimes_A \mathcal{C}_{\alpha\beta} & \xrightarrow{\nu_{\alpha\beta}^M} & M_{\alpha\beta} \\ {\scriptstyle M_e\otimes_A \Delta_{\alpha,\beta}}\downarrow & & \downarrow{\scriptstyle \rho_{\alpha,\beta}^M} \\ M_e \otimes_A \mathcal{C}_\alpha \otimes_A \mathcal{C}_\beta & \xrightarrow{\nu_\alpha^M \otimes_A \mathcal{C}_\beta} & M_\alpha \otimes_A \mathcal{C}_\beta. \end{array}$$

In fact, for any $m \in M_e$, $c \in \mathcal{C}_{\alpha\beta}$, we have

$$\begin{aligned} &(\nu_\alpha^M \otimes_A \mathcal{C}_\beta) \circ (M_e \otimes_A \Delta_{\alpha,\beta})(m \otimes_A c) \\ &= m_{[0,\alpha]} \cdot \theta^{(\alpha)}(m_{[1,\alpha^{-1}]} \otimes_A c_{(1,\alpha)}) \otimes_A c_{(2,\beta)} \\ &= m_{[0,\alpha]} \otimes_A \theta^{(\alpha)}(m_{[1,\alpha^{-1}]} \otimes_A c_{(1,\alpha)}) \cdot c_{(2,\beta)} \\ &= m_{[0,\alpha]} \otimes_A m_{[1,\alpha^{-1}](1,\beta)} \cdot \theta^{(\alpha\beta)}(m_{[1,\alpha^{-1}](2,\beta^{-1}\alpha^{-1})} \otimes_A c) \\ &= m_{[0,\alpha\beta][0,\alpha]} \otimes_A m_{[0,\alpha\beta][1,\beta]} \cdot \theta^{(\alpha\beta)}(m_{[1,\beta^{-1}\alpha^{-1}]} \otimes_A c) \\ &= \rho_{\alpha,\beta}^M \circ \nu_{\alpha\beta}^M(m \otimes_A c). \end{aligned}$$

Next, we shall show that $\nu^M \circ \eta^M = M$. Indeed, we take any $m \in M_\alpha$ and compute

$$\begin{aligned} \nu_\alpha^M \circ \eta_\alpha^M(m) &= \nu_\alpha^M(m_{[0,e]} \otimes_A m_{[1,\alpha]}) \\ &= m_{[0,e][0,\alpha]} \cdot \theta^{(\alpha)}(m_{[0,e][1,\alpha^{-1}]} \otimes_A m_{[1,\alpha]}) \\ &= m_{[0,\alpha]} \cdot \theta^{(\alpha)}(m_{[1,e](1,\alpha^{-1})} \otimes_A m_{[1,e](2,\alpha)}) \\ &= m_{[0,\alpha]} \cdot \varepsilon(m_{[1,e]}) = m. \end{aligned}$$

This shows that ν^M is the required splitting of η^M. It is evidently natural in M. □

REMARK 3.5. Theorem 3.4 is a precise generalization of Theorem 3.5 in [1], which is the case of $\pi = \{e\}$ a trivial group.

4. Applications

In this section, we shall apply Theorem 3.4 to some classical modules.

4.1. π-entwining modules. Recall first from [18] the π-entwining structure. Let $C = (\{C_\alpha\}_{\alpha\in\pi}, \Delta, \varepsilon)$ be a π-coalgebra and A be an algebra over k. Let ψ be a family of k-linear maps

$$\psi = \{\psi_\alpha : C_\alpha \otimes A \to A \otimes C_\alpha\}_{\alpha\in\pi}$$

(denoted by $\psi_\alpha(c\otimes a) = a_{\psi_\alpha} \otimes c^{\psi_\alpha} = a_{\Psi_\alpha} \otimes c^{\Psi_\alpha} = \cdots$) such that the following conditions are satisfied:

(ET1) $(ab)_{\psi_\alpha} \otimes c^{\psi_\alpha} = a_{\Psi_\alpha} b_{\psi_\alpha} \otimes c^{\Psi_\alpha \psi_\alpha}$ for all $a, b \in A$, $c \in C_\alpha$,

(ET2) $1_{A\psi_\alpha} \otimes c^{\psi_\alpha} = 1_A \otimes c$ for all $c \in C_\alpha$,

(ET3) $a_{\psi_{\alpha\beta}} \otimes d_{(1,\alpha)}^{\psi_{\alpha\beta}} \otimes d_{(2,\beta)}^{\psi_{\alpha\beta}} = a_{\psi_\beta \Psi_\alpha} \otimes d_{(1,\alpha)}^{\Psi_\alpha} \otimes d_{(2,\beta)}^{\psi_\beta}$ for all $d \in C_{\alpha\beta}$,

(ET4) $\varepsilon_C(c^{\psi_e}) a_{\psi_e} = \varepsilon_C(c) a$ for all $c \in C_e$, $a \in A$.

The triple (A, C, ψ) is called a (right-right) π*-entwining structure* and is denoted by $(A, C)_{\pi\text{-}\psi}$. The map ψ is called a π*-entwining map*.

Given a (right-right) π-entwining structure $(A, C)_{\pi\text{-}\psi}$. Then one can form the category $\mathcal{M}_A^{\pi\text{-}C}(\psi)$ of right $(A, C)_{\pi\text{-}\psi}$-modules. The objects of $\mathcal{M}_A^{\pi\text{-}C}(\psi)$ are right π-C-comodules $M = \{M_\alpha\}_{\alpha\in\pi}$ together with a family of k-linear maps

$$\phi = \{\phi_\alpha : M_\alpha \otimes A \to M_\alpha,\ \phi_\alpha(m \otimes a) = m \cdot a\}$$

such that the following conditions hold:

(1) For each $\alpha \in \pi$, (M_α, ϕ_α) is a right A-module.

(2) For all $\alpha, \beta \in \pi$, $m \in M_{\alpha\beta}$ and $a \in A$, $\rho_{\alpha,\beta}^M(m \cdot a) = m_{[0,\alpha]} \cdot a_{\psi_\beta} \otimes m_{[1,\beta]}^{\psi_\beta}$.

The morphisms of $\mathcal{M}_A^{\pi\text{-}C}(\psi)$ are not only right A-linear but also right C-colinear.

EXAMPLE 4.1. Let $(A,C)_{\pi\text{-}\psi}$ be a π-entwining structure. Then $\mathcal{C} = \{A\otimes C_\alpha\}_{\alpha\in\pi}$ can be endowed with A-bimodule structure via

$$b' \cdot (a \otimes c) \cdot b = b'ab_{\psi_\alpha} \otimes c^{\psi_\alpha}$$

for all $b', b \in A$ and $c \in C_\alpha$, and we have a π-A-coring $\mathcal{C} = \{A \otimes C_\alpha\}_{\alpha\in\pi}$ with the comultiplication and the counit given by

$$\overline{\Delta}_{\alpha,\beta} = A \otimes \Delta_{\alpha,\beta}, \quad \overline{\varepsilon} = A \otimes \varepsilon.$$

The following lemma reveals the relation between the category of the modules over the π-A-coring $\mathcal{C} = \{A \otimes C_\alpha\}_{\alpha\in\pi}$ and the category of the modules over π-entwining structure $(A, C)_{\pi\text{-}\psi}$.

Lemma 4.2. *Let $(A, C)_{\pi\text{-}\psi}$ be a π-entwining structure. Then*

$$\mathcal{M}^{\pi,\mathcal{C}} \cong \mathcal{M}_A^{\pi\text{-}C}(\psi).$$

From Theorem 3.4 and Lemma 4.2, we have the following conclusion.

Theorem 4.3. *For a π-entwined structure $(A, C)_{\pi\text{-}\psi}$, the following statements are equivalent*:
(1) *The forgetful functor $F: \mathcal{M}_A^{\pi\text{-}C}(\psi) \to \mathcal{M}_A$ is separable.*
(2) *There exists a family of A-bimodules $\theta = \{\theta^{(\alpha)}: A \otimes C_{\alpha^{-1}} \otimes C_\alpha \to A\}_{\alpha\in\pi}$ such that, for all $c' \in C_e$, $d \in C_{\alpha\beta}$,*

$$\theta^{(\alpha)}(a \otimes c'_{(1,\alpha^{-1})} \otimes c'_{(2,\alpha)}) = a\varepsilon_C(c'), \tag{4.1}$$

$$\theta^{(\alpha\beta)}(1_A \otimes c_{(2,\beta^{-1}\alpha^{-1})} \otimes d)_{\psi_\beta} \otimes c_{(1,\beta)}{}^{\psi_\beta} = \theta^{(\alpha)}(1_A \otimes c \otimes d_{(1,\alpha)}) \otimes d_{(2,\beta)}. \tag{4.2}$$

Here, the induced A-bimodule structure on $\{A \otimes C_{\alpha^{-1}} \otimes C_\alpha\}_{\alpha\in\pi}$ is

$$b \cdot (a \otimes c \otimes d) \cdot b' = bab'_{\psi_\alpha \psi_{\alpha^{-1}}} \otimes c^{\psi_{\alpha^{-1}}} \otimes d^{\psi_\alpha}$$

for all $a, a', b \in A$, $c \in C_{\alpha^{-1}}$ and $d \in C_\alpha$.
(3) *There exists a family of k-linear maps $\vartheta = \{\vartheta^{(\alpha)}: C_{\alpha^{-1}} \otimes C_\alpha \to A\}_{\alpha\in\pi}$ such that*

$$\vartheta^{(\alpha)}(c'_{(1,\alpha^{-1})} \otimes c'_{(2,\alpha)}) = \varepsilon_C(c'), \tag{4.3}$$

$$\vartheta^{(\alpha\beta)}(c_{(2,\beta^{-1}\alpha^{-1})} \otimes d')_{\psi_\beta} \otimes c_{(1,\beta)}{}^{\psi_\beta} = \vartheta^{(\alpha)}(c \otimes d'_{(1,\alpha)}) \otimes d'_{(2,\beta)}, \tag{4.4}$$

$$\vartheta^{(\alpha)}(c \otimes d)a = a_{\psi_\alpha \psi_{\alpha^{-1}}} \vartheta^{(\alpha)}(c^{\psi^{\alpha^{-1}}} \otimes d^{\psi^\alpha}) \tag{4.5}$$

for all $c' \in C_e$, $c \in C_{\alpha^{-1}}$, $d' \in C_{\alpha\beta}$ and $d \in C_\alpha$.

Proof. We apply Theorem 3.4 to the special A-π-coring $\mathcal{C} = \{A \otimes C_\alpha\}_{\alpha\in\pi}$, Equation (3.1) and (3.2) directly corresponds to Equation (4.1) and

$$\begin{aligned} &\theta^{(\alpha\beta)}(b_{\psi_{\beta^{-1}\alpha^{-1}}} \otimes c_{(2,\beta^{-1}\alpha^{-1})}{}^{\psi_{\beta^{-1}\alpha^{-1}}} \otimes d)_{\psi_\beta} \otimes c_{(1,\beta)}{}^{\psi_\beta} \\ &= \theta^{(\alpha)}(b_{\psi_{\alpha^{-1}}} \otimes c^{\psi_{\alpha^{-1}}} \otimes d_{(1,\alpha)}) \otimes d_{(2,\beta)} \end{aligned} \tag{4.6}$$

for all $\alpha, \beta \in \pi$, $b \in A$, $c \in C_{\alpha^{-1}}$ and $d \in C_{\alpha\beta}$. The equation (4.6) is equivalent to the equation

$$\theta^{(\alpha\beta)}(1_A \otimes c_{(2,\beta^{-1}\alpha^{-1})} \otimes d)_{\psi_\beta} \otimes c_{(1,\beta)}{}^{\psi_\beta} = \theta^{(\alpha)}(1_A \otimes c \otimes d_{(1,\alpha)}) \otimes d_{(2,\beta)} \tag{4.7}$$

for all $\alpha, \beta \in \pi$, $c \in C_{\alpha^{-1}}$ and $d \in C_{\alpha\beta}$. In fact, if we take $b = 1_A$ in (4.6), then it follows Equation (4.7). Assume that (4.7) holds, by using (ET3) and (ET1), we have

$$\begin{aligned} &b_{\psi_{\alpha^{-1}}} \theta^{(\alpha)}(1_A \otimes c^{\psi_{\alpha^{-1}}} \otimes d_{(1,\alpha)}) \otimes d_{(2,\beta)} \\ &= b_{\psi_{\alpha^{-1}}} \theta^{(\alpha\beta)}(1_A \otimes (c^{\psi_{\alpha^{-1}}})_{(2,\beta^{-1}\alpha^{-1})} \otimes d)_{\psi_\beta} \otimes (c^{\psi_{\alpha^{-1}}})_{(1,\beta)}{}^{\psi_\beta} \\ &\overset{(ET3)}{=} b_{\psi_{\beta^{-1}\alpha^{-1}}\Psi_\beta} \theta^{(\alpha\beta)}\big(1_A \otimes c_{(2,\beta^{-1}\alpha^{-1})}{}^{\psi_{\beta^{-1}\alpha^{-1}}} \otimes d\big)_{\psi_\beta} \otimes c_{(1,\beta)}{}^{\Psi_\beta \psi_\beta} \\ &\overset{(ET1)}{=} \big(b_{\psi_{\beta^{-1}\alpha^{-1}}} \theta^{(\alpha\beta)}\big(1_A \otimes c_{(2,\beta^{-1}\alpha^{-1})}{}^{\psi_{\beta^{-1}\alpha^{-1}}} \otimes d\big)\big)_{\psi_\beta} \otimes c_{(1,\beta)}{}^{\psi_\beta}. \end{aligned}$$

Since $\theta^{(\alpha\beta)}$ and $\theta^{(\alpha)}$ are left A-linear, it follows that Equation (4.6) holds. So we show that (1) $\Leftrightarrow$ (2).

Suppose that there exists a family of A-bimodules $\theta = \{\theta^{(\alpha)}\colon A \otimes C_{\alpha^{-1}} \otimes C_\alpha \to A\}_{\alpha\in\pi}$ such that (4.1) and (4.2) hold. Then we define a family of k-linear maps $\vartheta = \{\vartheta^{(\alpha)}\}_{\alpha\in\pi}$,

$$\vartheta^{(\alpha)}\colon C_{\alpha^{-1}} \otimes C_\alpha \to A, \quad c \otimes d \mapsto \theta^{(\alpha)}(1_A \otimes c \otimes d)$$

for all $c \in C_{\alpha^{-1}}$ and $d \in C_\alpha$. By the properties of θ, we have

$$\begin{aligned}
&\vartheta^{(\alpha)}\big(c'_{(1,\alpha^{-1})} \otimes c'_{(2,\alpha)}\big) = \varepsilon_C(c'),\\
&\vartheta^{(\alpha\beta)}\big(c_{(2,\beta^{-1}\alpha^{-1})} \otimes d'\big)_{\psi_\beta} \otimes c^{\psi_\beta}_{(1,\beta)} = \vartheta^{(\alpha)}(c \otimes d'_{(1,\alpha)}) \otimes d'_{(2,\beta)},\\
&\vartheta^{(\alpha)}(c \otimes d)a = \theta^{(\alpha)}(1_A \otimes c \otimes d)a\\
&\qquad = \theta^{(\alpha)}((1_A \otimes c \otimes d) \cdot a)\\
&\qquad = \theta^{(\alpha)}\big(a_{\psi_\alpha\psi_{\alpha^{-1}}} \otimes c^{\psi^{\alpha^{-1}}} \otimes d^{\psi^\alpha}\big)\\
&\qquad = a_{\psi_\alpha\psi_{\alpha^{-1}}}\vartheta^{(\alpha)}\big(c^{\psi^{\alpha^{-1}}} \otimes d^{\psi^\alpha}\big)
\end{aligned}$$

for all $c' \in C_e$, $c \in C_{\alpha^{-1}}$, $d' \in C_{\alpha\beta}$ and $d \in C_\alpha$, that is, Equations (4.3)–(4.5) hold.

Conversely, suppose that $\vartheta = \{\vartheta^{(\alpha)}\colon C_{\alpha^{-1}} \otimes C_\alpha \to A\}_{\alpha\in\pi}$ such that Equations (4.3)–(4.5) hold. Then we define a family of k-linear maps $\theta = \{\theta^{(\alpha)}\}_{\alpha\in\pi}$,

$$\theta^{(\alpha)}\colon A \otimes C_{\alpha^{-1}} \otimes C_\alpha \to A, \quad a \otimes c \otimes d \mapsto a\vartheta^{\alpha}(c \otimes d)$$

for all $a \in A$, $c \in C_{\alpha^{-1}}$ and $d \in C_\alpha$. It is straightforward to check that θ is a A-bimodule map and satisfies Equations (4.1) and (4.2). Thus, we prove that (2) $\Leftrightarrow$ (3). □

4.2. Doi–Hopf π-modules. Given a Hopf π-coalgebra $H = (\{H_\alpha\}_{\alpha\in\pi})$. Recall that a (right) π-H-comodule algebra is an algebra A over k together with a family of k-linear maps $\rho^A = \{\rho^A_\alpha\colon A \to A \otimes H_\alpha\}_{\alpha\in\pi}$ such that,
(a) For any $\alpha, \beta \in \pi$,

$$(\mathrm{id}_A \otimes \Delta_{\alpha,\beta}) \circ \rho^A_{\alpha\beta} = (\rho^A_\alpha \otimes \mathrm{id}_{H_\beta}) \circ \rho^A_\beta, \quad (\mathrm{id}_A \otimes \varepsilon) \circ \rho^A_e = \mathrm{id}_A,$$

(b) For any $\alpha \in \pi$, ρ_α is algebra homomorphism.
We shall adopt the standard notation, for any $a \in A$, $\rho^A_\alpha(a) = a_{[0,\alpha]} \otimes a_{[1,\alpha]}$.

A *Doi–Hopf π-datum* is a triple (H, A, C), where A is a right π-H-comodule algebra and C a right π-H-module coalgebra (see Section 2.4). A *Doi–Hopf π-module* $M = \{M_\alpha\}_{\alpha\in\pi}$ is a right A-module(meaning that each M_α is right A-module) which is also a right π-C-comodule with the coaction structure

$$\rho^M = \{\rho^M_{\alpha,\beta}\colon M_{\alpha\beta} \to M_\alpha \otimes C_\beta\}_{\alpha,\beta\in\pi}$$

such that the following compatible condition holds:

$$\rho^M_{\alpha,\beta}(m \cdot a) = m_{[0,\alpha]} \cdot a_{[0,\beta]} \otimes m_{[1,\beta]} \cdot a_{[1,\beta]}$$

for all $\alpha, \beta \in \pi$ and $m \in M_{\alpha\beta}$, $a \in A$.

The set of Doi–Hopf π-modules together with both a right A-module maps and a right π-C-comodule maps will form a category of Doi–Hopf π-modules and will be denoted by $\mathcal{M}_A^{\pi\text{-}C}$ (called a *Doi–Hopf π-modules category*).

Given a Doi–Hopf π-datum (H, A, C), we define a family of k-linear maps

$$\psi = \{\psi_\alpha : C_\alpha \otimes A \to A \otimes C_\alpha,\ c \otimes a \to a_{[0,\alpha]} \otimes c \cdot a_{[1,\alpha]}\}_{\alpha\in\pi}.$$

Then we have a special π-entwining structure $(A, C)_{\pi\text{-}\psi}$ associated to Doi–Hopf π-datum (H, A, C).

Lemma 4.4. *Let (H, A, C) be a Doi–Hopf π-datum. Then*

$$\mathcal{M}_A^{\pi\text{-}C} \cong \mathcal{M}_A^{\pi\text{-}C}(\psi).$$

Proof. Straightforward. □

From Theorem 4.3 and Lemma 4.4, we have the following theorems:

Theorem 4.5. *For a Doi–Hopf π-datum (H, A, C), the following statements are equivalent*:
(1) *The forgetful functor $F : \mathcal{M}_A^{\pi\text{-}C} \to \mathcal{M}_A$ is separable.*
(2) *There exists a family of k-linear maps $\vartheta = \{\vartheta^{(\alpha)} : C_{\alpha^{-1}} \otimes C_\alpha \to A\}_{\alpha\in\pi}$ such that the following equations are satisfied,*

$$\vartheta^{(\alpha)}\big(c'_{(1,\alpha^{-1})} \otimes c'_{(2,\alpha)}\big) = \varepsilon_C(c'),$$
$$\vartheta^{(\alpha\beta)}(c_{(2,\beta^{-1}\alpha^{-1})} \otimes d')_{[0,\beta]} \otimes c_{(1,\beta)} \cdot \vartheta^{(\alpha\beta)}(c_{(2,\beta^{-1}\alpha^{-1})} \otimes d')_{[1,\beta]} = \vartheta^{(\alpha)}(c \otimes d'_{(1,\alpha)}) \otimes d'_{(2,\beta)},$$
$$\vartheta^{(\alpha)}(c \otimes d)a = a_{[0,\alpha][0,\alpha^{-1}]}\vartheta^{(\alpha)}(c \cdot a_{[0,\alpha][1,\alpha^{-1}]} \otimes d \cdot a_{[1,\alpha]})$$

for all $c' \in C_e$, $c \in C_{\alpha^{-1}}$, $d' \in C_{\alpha\beta}$ and $d \in C_\alpha$.

Let A be a right π-H-comodule algebra. Then we have a special Doi–Hopf π-datum (H, A, H). The corresponding Doi–Hopf π-modules category $\mathcal{M}_A^{\pi\text{-}H}$ is called a relative Hopf π-modules category. From Theorem 4.5, we have

Theorem 4.6. *Let A be a right π-H-comodule algebra. Then the following statements are equivalent*:
(1) *The forgetful functor $F : \mathcal{M}_A^{\pi\text{-}H} \to \mathcal{M}_A$ is separable.*

(2) *There exists a family of k-linear maps* $\vartheta = \{\vartheta^{(\alpha)}: H_{\alpha^{-1}} \otimes H_\alpha \to A\}_{\alpha\in\pi}$ *such that the following equations are satisfied,*

$$\vartheta^{(\alpha)}(h'_{(1,\alpha^{-1})} \otimes h'_{(2,\alpha)}) = \varepsilon(h'), \tag{4.8}$$

$$\begin{aligned}&\vartheta^{(\alpha\beta)}(h_{(2,\beta^{-1}\alpha^{-1})} \otimes d')_{[0,\beta]} \otimes h_{(1,\beta)}\vartheta^{(\alpha\beta)}(h_{(2,\beta^{-1}\alpha^{-1})} \otimes d')_{[1,\beta]}\\ &= \vartheta^{(\alpha)}(h \otimes d'_{(1,\alpha)}) \otimes d'_{(2,\beta)},\end{aligned} \tag{4.9}$$

$$\vartheta^{(\alpha)}(h \otimes d)a = a_{[0,\alpha][0,\alpha^{-1}]}\vartheta^{(\alpha)}(ha_{[0,\alpha][1,\alpha^{-1}]} \otimes da_{[1,\alpha]}) \tag{4.10}$$

for all $h' \in H_e$, $h \in H_{\alpha^{-1}}$, $d' \in H_{\alpha\beta}$ *and* $d \in H_\alpha$.

DEFINITION 4.7. Let A be a π-H-comodule algebra. A family of k-linear maps $\varphi = \{\varphi_\beta: H_\beta \to A\}_{\beta\in\pi}$ is called a total integral for A, if φ satisfies

$$\rho^A_\beta \circ \varphi_{\alpha\beta} = (\varphi_\alpha \otimes \mathrm{id}_{H_\beta}) \circ \Delta_{\alpha,\beta}, \quad \varphi_\beta(1_\beta) = 1_A$$

for all $\alpha, \beta \in \pi$.

REMARK 4.8. If $\pi = \{e\}$ is a trivial group, the total integrals in Definition 4.7 are reduced to the total integrals in ordinary Hopf algebras [7].

Let $\varphi = \{\varphi_\alpha: H_\alpha \to A\}_{\alpha\in\pi}$ be a total integral. Now we define

$$\vartheta^{(\beta)}: H_{\beta^{-1}} \otimes H_\beta \to A, \quad \vartheta^{(\beta)}(g \otimes h) = \varphi_\beta(S_{\beta^{-1}}(g)h)$$

for all $\beta \in \pi$, $h \in H_\beta$ and $g \in H_{\beta^{-1}}$. Standard computation can check that conditions (4.8) and (4.9) hold (also see [6] in detail). Also that the equality (4.10) holds is equivalent to that φ_β with $\beta \in \pi$ satisfies

$$a_{[0,e]}\varphi_\beta(S_{\beta^{-1}}(a_{[1,e](1,\beta^{-1})})S_{\beta^{-1}}(d)ba_{[1,e](2,\beta)}) = \varphi_\beta(S_{\beta^{-1}}(d)b)a$$

for all $a \in A$ and $d \in H_{\beta^{-1}}$, $b \in H_\beta$.

Assume that there exists a family of k-linear maps

$$\vartheta = \{\vartheta^{(\beta)}: H_{\beta^{-1}} \otimes H_\beta \to A\}_{\beta\in\pi}$$

such that the conditions (4.8)–(4.10) hold. Then we define a family of k-linear maps $\varphi = \{\varphi_\beta: H_\beta \to A\}_{\beta\in\pi}$, where

$$\varphi_\beta(h) = \vartheta^{(\beta)}(1_{\beta^{-1}} \otimes h)$$

for all $h \in H_\beta$. Using conditions (4.8) and (4.9), we can easily get

Proposition 4.9. *Under the assumptions above,* $\varphi = \{\varphi_\beta\}_{\beta\in\pi}$ *is a total integral.*

4.3. Hopf π-coalgebras. Let $H = (\{H_\alpha\}_{\alpha\in\pi})$ be a Hopf π-coalgebra with the invertible antipode $S = \{S_\alpha\}_{\alpha\in\pi}$, i.e, for each $\alpha \in \pi$, S_α is bijective. A *left* (*resp. right*) *π-integral* for H is a family of k-linear forms

$$\lambda = \{\lambda\}_{\alpha\in\pi} \in \prod_{\alpha\in\pi} H_\alpha^*$$

such that, for all $\alpha, \beta \in \pi$,

$$(\mathrm{id}_{H_\alpha} \otimes \lambda_\beta) \circ \Delta_{\alpha,\beta} = \lambda_{\alpha\beta} 1_\alpha, \quad (\text{resp. } (\lambda_\alpha \otimes \mathrm{id}_{H_\beta}) \circ \Delta_{\alpha,\beta} = \lambda_{\alpha\beta} 1_\beta).$$

Note that λ_e is a usual left (resp. right) integral for the Hopf algebra H_e^*. Virelizier [16] showed that a Hopf π-coalgebra H is cosemisimple if and only if there exists a right π-integral $\lambda = \{\lambda_\alpha\}_{\alpha\in\pi}$ such that $\lambda_\alpha(1_\alpha) = 1$, for all $\alpha \in \pi$.

Lemma 4.10. *Suppose that S_β is bijective for all $\beta \in \pi$, then for a right π-integral $\lambda = \{\lambda_\alpha\}_{\alpha\in\pi}$ for H, we have*

$$(\lambda_\alpha, g h_{(1,\alpha)}) h_{(2,\beta)} = (\lambda_{\alpha\beta}, g_{(1,\alpha\beta)} h) S_\beta^{-1}(g_{(2,\beta^{-1})})$$

for all $g \in H_\alpha$ and $h \in H_{\alpha\beta}$.

Proof. For any $\alpha, \beta \in \pi$, $g \in H_\alpha$, $h \in H_{\alpha\beta}$, we have

$$\begin{aligned}
(\lambda_\alpha, g h_{(1,\alpha)}) h_{(2,\beta)} &= \varepsilon_H(g_{(2,e)})(\lambda_\alpha, g_{(1,\alpha)} h_{(1,\alpha)}) h_{(2,\beta)} \\
&= S_\beta^{-1}(g_{(2,e)(2,\beta^{-1})}) g_{(2,e)(1,\beta)} (\lambda_\alpha, g_{(1,\alpha)} h_{(1,\alpha)}) h_{(2,\beta)} \\
&= S_\beta^{-1}(g_{(1,\beta^{-1})}) g_{(1,\alpha\beta)(2,\beta)} (\lambda_\alpha, g_{(1,\alpha\beta)(1,\alpha)} h_{(1,\alpha)}) h_{(2,\beta)} \\
&= S_\beta^{-1}(g_{(1,\beta^{-1})})(\lambda_{\alpha\beta}, g_{(1,\alpha\beta)} h).
\end{aligned}$$

This ends the proof. □

As we know, k can be viewed as a π-H-comodule algebra with the comodule structure $\rho^k = \{\rho_\alpha^k : k \to k \otimes H_\alpha, a \mapsto a \otimes 1_\alpha\}_{\alpha\in\pi}$. Then φ in Definition 4.7 has the following form

$$\varphi_{\alpha\beta} = (\varphi_\alpha \otimes \mathrm{id}_{H_\beta}) \circ \Delta_{\alpha,\beta}$$

for all $\alpha, \beta \in \pi$, which is just a right π-integral. Meanwhile, we get a special relative Hopf π-modules category $\mathcal{M}^{\pi\text{-}H}$ (called the category of comodules over Hopf π-coalgebra H).

Theorem 4.11. *Let $H = (\{H_\alpha\}_{\alpha\in\pi})$ be a Hopf π-coalgebra. Then the following statements are equivalent*:

(1) *The forgetful functor* $F\colon \mathcal{M}^{\pi\text{-}H} \to \mathcal{M}_k$ (*the category of all vector spaces over* k) *is separable, where* $F(M) = M_e$ *for* $M = \{M_\alpha\}_{\alpha\in\pi} \in \mathcal{M}^{\pi\text{-}H}$.
(2) H *is cosemisimple.*

Proof. Assume that (2) holds, we have a right π-integral $\varphi = \{\varphi_\alpha\}_{\alpha\in\pi}$ such that $\lambda_\alpha(1_\alpha) = 1$, for all $\alpha \in \pi$. Using the right π-integral $\varphi = \{\varphi_\alpha\}_{\alpha\in\pi}$, we define a family of k-linear maps $\vartheta = \{\vartheta_\beta\}_{\beta\in\pi}$, where

$$\vartheta^{(\beta)}\colon H_{\beta^{-1}} \otimes H_\beta \to k, \quad \vartheta^{(\beta)}(g \otimes h) = \varphi_\beta(S_{\beta^{-1}}(g)h)$$

for all $\beta \in \pi$, $h \in H_\beta$ and $g \in H_{\beta^{-1}}$. It is checked straightforward that ϑ satisfies the conditions (4.8)–(4.10). From Theorem 4.6, we can get the desired result.

Conversely, assume that the forgetful functor F is separable. Then, by Theorem 4.6, there exists a family of k-linear maps

$$\theta = \{\theta^{(\alpha)}\colon H_{\alpha^{-1}} \otimes H_\alpha \to k\}_{\alpha\in\pi}$$

satisfying Equations (4.8) and (4.9). Let $\lambda_\alpha \in H_\alpha^*$ be the k-linear functional defined by

$$\lambda_\alpha(h) = \theta^{(\alpha)}(1_{\alpha^{-1}} \otimes h)$$

for all $h \in H_\alpha$. By Proposition 4.9, the family $\lambda = \{\lambda_\alpha\}_{\alpha\in\pi}$ is a total integral for the right π-H-comodule algebra k. By Theorem 5.4 in [16], one can conclude that H is cosemisimple. □

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