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## VARIETIES OF PICARD RANK ONE AS COMPONENTS OF AMPLE DIVISORS

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### Abstract

Let  $\mathcal{V}$  be an integral normal complex projective variety of dimension  $n \geq 3$  and denote by  $\mathcal{L}$  an ample line bundle on  $\mathcal{V}$ . By imposing that the linear system  $|\mathcal{L}|$  contains an element  $A = A_1 + \cdots + A_r$ ,  $r \geq 1$ , where all the  $A_i$ 's are distinct effective Cartier divisors with  $\text{Pic}(A_i) = \mathbb{Z}$ , we show that such a  $\mathcal{V}$  is as special as the components  $A_i$  of  $A \in |\mathcal{L}|$ . After making a list of some consequences about the positivity of the components  $A_i$ , we characterize pairs  $(\mathcal{V}, \mathcal{L})$  as above when either  $A_1 \cong \mathbb{P}^{n-1}$  and  $\text{Pic}(A_j) = \mathbb{Z}$  for  $j = 2, \dots, r$ , or  $\mathcal{V}$  is smooth and each  $A_i$  is a variety of small degree with respect to  $[H_i]_{A_i}$ , where  $[H_i]_{A_i}$  is the restriction to  $A_i$  of a suitable line bundle  $H_i$  on  $\mathcal{V}$ .

### 1. Introduction

Projective manifolds with an irreducible hyperplane section being a special variety have been studied since longtime (see, e.g., [2] and [6]), but the corresponding study for a reducible hyperplane section consisting of a simple normal crossing divisor whose components are special varieties started only recently by Chandler, Howard and Sommese [3]. Therefore, we continue here the study of varieties in terms of a hyperplane section  $A$  which is not irreducible, assuming that  $A$  is a union of distinct irreducible components  $A_1, \dots, A_r$ , with  $r \geq 1$ . More precisely, let  $\mathcal{V}$  be an integral normal complex projective variety of dimension  $n \geq 3$  endowed with an ample line bundle  $\mathcal{L}$ . Assume that

( $\diamond$ )  $|\mathcal{L}|$  contains an element  $A = A_1 + \cdots + A_r$ ,  $r \geq 1$ , where all the components  $A_i$  are distinct and effective Cartier divisors with  $\text{Pic}(A_i)$  of rank one.

Let us observe here that this assumption is a natural generalization of the classical hypothesis  $\text{Pic}(A') \cong \mathbb{Z}$  on a hyperplane section  $A'$  of  $\mathcal{V}$ . Furthermore, if ( $\diamond$ ) holds then every component  $A_i$  of  $A \in |\mathcal{L}|$  does not admit a non-trivial morphism onto a variety  $W_i$  with  $0 < \dim W_i < n - 1$  and in general, also for the smooth case, the main known results on reducible hyperplane sections (see, e.g., [3]) can not be applied on any of the  $A_i$ 's. However, we can show that if a reducible subvariety  $A$  as in ( $\diamond$ ) is contained in  $\mathcal{V}$  as an ample divisor, then it imposes severe restrictions to  $\mathcal{V}$ , that is, the topological and geometric structures of  $\mathcal{V}$  are very closely related to those of each

component of  $A$ . So, in §3 we prove first the following

**Theorem 1.1.** *Let  $\mathcal{V}$  be an integral normal complex projective variety of dimension  $n \geq 3$  and let  $\mathcal{L}$  be an ample line bundle on  $\mathcal{V}$ . Assume that  $(\diamond)$  holds. Then all the  $A_i$ 's are nef and big Cartier divisors on  $\mathcal{V}$  and for any  $i = 1, \dots, r$  there exist proper birational morphisms  $f_i: \mathcal{V} \rightarrow \mathcal{V}_i$  from  $\mathcal{V}$  to a projective normal variety  $\mathcal{V}_i$  given by the map associated to  $|\mathcal{O}_{\mathcal{V}}(m_i A_i)|$  for some  $m_i \gg 0$ . Furthermore, each map  $f_i$  contracts at most a finite number of curves on  $\mathcal{V}$  and it is an isomorphism in a neighborhood of  $A_i$  such that  $f_i(A_i)$  is an effective ample divisor on  $\mathcal{V}_i$ .*

The above result shows that assumption  $(\diamond)$  implies that all the components  $A_i$  of  $A$  are either ample, or at worst big and 1-ample in the sense of [15, (1.3)]. By imposing some restrictions on the singularities of  $\mathcal{V}$ , we are able to deduce that all the  $A_i$ 's are in fact ample Cartier divisors on  $\mathcal{V}$ .

**Theorem 1.2.** *Let  $\mathcal{V}$  be an integral normal complex projective variety of dimension  $n \geq 3$  with at worst Cohen–Macaulay singularities. Let  $\mathcal{L}$  be an ample line bundle on  $\mathcal{V}$  and assume that  $(\diamond)$  holds. Furthermore, suppose that  $\mathcal{V} - F$  is a locally complete intersection for some finite, possibly empty, set  $F \subset \mathcal{V} - \text{Irr}(\mathcal{V})$  with  $\dim \text{Irr}(\mathcal{V}) \leq 0$ , where  $\text{Irr}(\mathcal{V})$  is the set of irrational singularities of  $\mathcal{V}$ . Then all the  $A_i$ 's are ample Cartier divisors on  $\mathcal{V}$  and the maps  $f_i: \mathcal{V} \rightarrow \mathcal{V}_i$  of Theorem 1.1 are all isomorphisms.*

Furthermore, under some additional hypotheses on  $\mathcal{V}$  and on some  $A_i$ , we can finally obtain that  $\text{Pic}(\mathcal{V}) = \mathbb{Z}\langle \Lambda \rangle$  for an ample line bundle  $\Lambda$  on  $\mathcal{V}$  (Corollary 3.1).

All of these results allow us to list in §4 some consequences about the positivity of the  $A_i$ 's and to obtain similar results as in [1, Theorem 1] (see also [14, Proposition VI]) for the case of reducible ample divisors on  $\mathcal{V}$  (Propositions 5.2 and 5.3).

We would like to note that the above results make use of weak hypotheses on  $\mathcal{V}$  and on each  $A_i$ , and that  $(\diamond)$  seems optimal a priori for Theorems 1.1 and 1.2, since easy examples show that these results do not hold assuming that  $\text{Pic}(A_i) \neq \mathbb{Z}\langle \mathcal{H}_i \rangle$  for some  $i = 1, \dots, r$ , also when  $r = 2$  and  $\mathcal{V}$  is smooth (Remark 3.2). Moreover, the techniques we employ in §3 leave out of account any special polarization on each  $A_i$  by a (very) ample line bundle on  $\mathcal{V}$  and they allow us to assume that  $\mathcal{L}$  is simply ample and not necessarily ample and spanned or very ample on  $\mathcal{V}$ .

Finally, as a by-product of §4 and some results obtained by many other authors about smooth complex projective variety  $X$  containing ample divisors of special type (e.g., [1], [2], [6], [7], [9], [10], [11], [14]), in §5.1 we obtain similar results as in [1, Theorem 1] and [14, Proposition VI] (Propositions 5.2 and 5.3), and in §5.2 we classify smooth polarized pairs  $(X, L)$  which admit an ample divisor  $A \in |L|$  such that  $A = A_1 + \dots + A_r$ ,  $r \geq 1$ , and all the components  $A_i$  have small degree with respect to suitable line bundles  $H_i$  on  $X$  for every  $i = 1, \dots, r$ .

**Proposition 1.3.** *Let  $L$  be an ample line bundle on a smooth complex projective variety  $X$  of dimension  $n$  with  $n \geq 5$ . Assume that there is a divisor  $A = A_1 + \dots + A_r \in |L|$ ,  $r \geq 1$ , where each  $A_i$  is an irreducible and reduced normal Gorenstein projective variety with  $\dim \text{Irr}(A_i) \leq 0$ , where  $\text{Irr}(A_i)$  is the set of irrational singularities of  $A_i$ . Suppose that for any  $k = 1, \dots, r$  there exist ample and spanned line bundles  $H_k$  on  $X$  such that  $[H_k]_{A_k}$  is very ample and  $[H_k]_{A_k}^{n-1} \leq 4$ . Then one of the following possibilities holds:*

- (1)  $r \geq 1$ ,  $H_1 = \dots = H_r = H$  and  $(X, H)$  is one of the following pairs:
  - (a)  $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$  and  $A_i \in |\mathcal{O}_{\mathbb{P}^n}(a_i)|$  with  $1 \leq a_i \leq 4$  for every  $i = 1, \dots, r$ ;
  - (b)  $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$  and  $A_i \in |\mathcal{O}_{\mathbb{Q}^n}(a_i)|$  with  $a_i = 1, 2$  for every  $i = 1, \dots, r$ ;
  - (c)  $X \subset \mathbb{P}^{n+1}$  is a hypersurface of degree 3 or 4, and  $A_i = H \in |\mathcal{O}_{\mathbb{P}^{n+1}}(1)_X|$  for every  $i = 1, \dots, r$ ;
  - (d)  $X = \mathbb{Q}_1 \cap \mathbb{Q}_2 \subset \mathbb{P}^{n+2}$  is a complete intersection of two quadric hypersurfaces  $\mathbb{Q}_i \subset \mathbb{P}^{n+2}$  for  $i = 1, 2$ , and  $A_i = H \in |\mathcal{O}_{\mathbb{P}^{n+2}}(1)_X|$  for every  $i = 1, \dots, r$ ;
  - (e)  $\pi: X \rightarrow \mathbb{P}^n$  is a double cover of  $\mathbb{P}^n$  with branch locus  $\Delta \in |\mathcal{O}_{\mathbb{P}^n}(2b)|$ ,  $b = 1, 2$ ,  $H \in |\pi^* \mathcal{O}_{\mathbb{P}^n}(1)|$  and  $A_i \in |\pi^* \mathcal{O}_{\mathbb{P}^n}(a_i)|$ ,  $a_i = 1, 2$ , for every  $i = 1, \dots, r$ ;
  - (f)  $\pi: X \rightarrow \mathbb{Q}^n$  is a double cover of a quadric hypersurface  $\mathbb{Q}^n \subset \mathbb{P}^{n+1}$  with branch locus  $\Delta \in |\mathcal{O}_{\mathbb{Q}^n}(2b')|$ ,  $b' = 1, 2$ , and  $A_i = H \in |\pi^* \mathcal{O}_{\mathbb{Q}^n}(1)|$  for every  $i = 1, \dots, r$ ;
  - (g)  $\pi: X \rightarrow \mathbb{P}^n$  is a  $d$ -cover of  $\mathbb{P}^n$  with  $d = 3, 4$ , and  $A_i = H \in |\pi^* \mathcal{O}_{\mathbb{P}^n}(1)|$  for any  $i = 1, \dots, r$ ;
- (2)  $r \geq 2$ ,  $X \cong \mathbb{P}^1 \times \mathbb{P}^4$  and after renaming  $(A_1, [H_1]_{A_1}) \cong (\mathbb{P}^1 \times \mathbb{P}^3, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1, 1))$ , with  $A_1 \in |\mathcal{O}_X(0, 1)|$  and  $H_1 \in |\mathcal{O}_X(1, 1)|$ ,  $(A_2, [H_2]_{A_2}) \cong (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1))$  with  $A_2 \in |\mathcal{O}_X(1, 0)|$  and  $H_2 \in |\mathcal{O}_X(t, 1)|$  for some integer  $t \geq 1$ , and the remaining polarized pairs  $(A_k, [H_k]_{A_k})$  are of these two types.

**2. Notation**

Let  $\mathcal{V}$  be an integral normal complex projective variety of dimension  $n \geq 3$  endowed with an ample line bundle  $\mathcal{L}$ . All notation and terminology used here are standard in algebraic geometry. We adopt the additive notation for line bundles and the numerical equivalence is denoted by  $\equiv$ , while the linear equivalence by  $\simeq$ . The pull-back  $i^* \mathcal{F}$  of a line bundle  $\mathcal{F}$  on  $\mathcal{V}$  by an embedding  $i: Y \hookrightarrow \mathcal{V}$  is sometimes denoted by  $\mathcal{F}_Y$ . We denote by  $K_{\mathcal{V}}$  the canonical bundle of  $\mathcal{V}$ . For such a polarized variety  $(\mathcal{V}, \mathcal{L})$ , we will use the adjunction theoretic terminology of [2] and we say that  $\mathcal{V}$  is an  $n$ -fold (denoted by  $X$ ) when  $\mathcal{V}$  is smooth.

**3. Proof of Theorems 1.1 and 1.2**

First of all, let us show that  $(\diamond)$  implies that all the components of  $A$  are actually nef and big Cartier divisors on  $\mathcal{V}$ , obtaining the following

Proof of Theorem 1.1. Suppose that  $A_1 \cap A_2 := W$  is nonempty. Define integers  $a_{ij}$  putting  $\mathcal{O}_{\mathcal{V}}(A_i)_{A_j} := \mathcal{O}_{A_j}(a_{ij}\mathcal{H}_j)$ , where  $\mathcal{H}_j$  is the ample generator of  $\text{Pic}(A_j)$  (mod torsion). Then

$$\mathcal{O}_W(a_{11}\mathcal{H}_1) = \mathcal{O}_W(A_1) = \mathcal{O}_W(a_{12}\mathcal{H}_2)$$

is an ample line bundle on  $W$ . Hence  $a_{11}$  is positive. Since  $A$  is a connected divisor on  $\mathcal{V}$  (see [8, III 7.9]), for every  $i = 1, \dots, r$  there exists a component  $A_j$  of  $A \in |\mathcal{L}|$  with  $j \neq i$  such that  $A_i \cap A_j \neq \emptyset$ . This shows that  $\mathcal{O}_{\mathcal{V}}(A_i)_{A_i}$  is ample for any  $i = 1, \dots, r$ , i.e.  $A_i$  is nef and big. From [7, III 4.2] (see also [2, (2.6.5)] or [12, (1.2.30)]), it follows that there exists a proper birational morphism  $f_i: \mathcal{V} \rightarrow \mathcal{V}_i$  from  $\mathcal{V}$  to a projective normal variety  $\mathcal{V}_i$  given by the map associated to  $|\mathcal{O}_{\mathcal{V}}(m_i A_i)|$  for some  $m_i \gg 0$ , which is an isomorphism in a neighborhood of  $A_i$  and such that  $f_i(A_i)$  is an effective ample divisor on  $\mathcal{V}_i$ .

**Claim.** *The morphism  $f_i$  contracts at most a finite number of curves on  $\mathcal{V}$ .*

By [2, (2.5.5)] note that  $A_i \simeq H_i + D_i$  for any  $i = 1, \dots, r$ , where  $H_i$  is  $\mathbb{Q}$ -ample and  $D_i$  is  $\mathbb{Q}$ -effective. This shows that  $A_i \cdot A_j$  is nonzero and so  $A_i \cap A_j$  can not be empty. Let  $W_{ij} := A_i \cap A_j$  and note that  $W_{ij}$  is an ample divisor both in  $A_i$  and in  $A_j$ . Let  $Z$  be an irreducible variety of dimension greater than or equal to two. Since  $\mathcal{L}$  is ample, we have that  $Z \cap A_k \neq \emptyset$  for some  $k = 1, \dots, r$ , and then  $W_{ik} \cap Z \neq \emptyset$  since  $\dim Z \cap A_k \geq 1$ . Put  $R := Z \cap A_i$ . Thus  $R$  is nonempty,  $\dim R \geq 1$  and  $\mathcal{O}(R)_R$  is ample. By [7, III 4.2] (or [12, (1.2.30)]), the linear system  $|\mathcal{O}_{\mathcal{V}}(m_i A_i)|$  restricted to  $Z$  can not contract  $Z$  to a lower dimensional variety.  $\square$

Proof of Theorem 1.2. Put  $\mathcal{O}_{\mathcal{V}}(A_i)_{A_j} = \mathcal{O}_{A_j}(a_{ij}\mathcal{H}_j)$  and  $l_t := \mathcal{O}_{A_t}(\mathcal{H}_t)^{n-1} > 0$  for any  $i, j, t \in \{1, \dots, r\}$ . From Theorem 1.1 we know that  $a_{ii} > 0$  for every  $i = 1, \dots, r$ . Moreover, since  $A$  is a connected divisor on  $X$  (see [8, III 7.9]), for every  $j = 1, \dots, r$  there exists a component  $A_k$  of  $A \in |L|$  with  $k \neq j$  such that  $A_j \cap A_k \neq \emptyset$ . So we get the following expressions

$$\begin{aligned} A_k^s A_j^{n-s} &= \mathcal{O}_{A_k}(a_{kk}\mathcal{H}_k)^{s-1} \mathcal{O}_{A_k}(a_{jk}\mathcal{H}_k)^{n-s} = a_{kk}^{s-1} a_{jk}^{n-s} l_k, \\ (1) \quad A_k^s A_j^{n-s} &= \mathcal{O}_{A_j}(a_{kj}\mathcal{H}_j)^s \mathcal{O}_{A_j}(a_{jj}\mathcal{H}_j)^{n-s-1} = a_{kj}^s a_{jj}^{n-s-1} l_j, \end{aligned}$$

with  $1 \leq s \leq n - 1$ . Note that  $a_{jj} \neq 0$ ,  $a_{kk} \neq 0$ ,  $a_{kj} > 0$  and  $a_{jk} > 0$ . Moreover, from the equations (1) with  $s = 1, 2$  we deduce that

$$a_{jk}(a_{kj}^2 a_{jj}^{n-3} l_j) = a_{jk}(a_{kk} a_{jk}^{n-2} l_k) = a_{kk}(a_{jk}^{n-1} l_k) = a_{kk}(a_{kj} a_{jj}^{n-2} l_j),$$

that is,

$$(2) \quad a_{jk} a_{kj} = a_{jj} a_{kk}.$$

So we get

$$\begin{aligned} A_s^2 \mathcal{L}^{n-2} &= \sum_{h_1+\dots+h_r=n-2} \frac{(n-2)!}{h_1! \dots h_r!} A_1^{h_1} \dots A_s^{h_s+2} \dots A_r^{h_r} \\ &= \sum_{h_1+\dots+h_r=n-2} \frac{(n-2)!}{h_1! \dots h_r!} a_{1s}^{h_1} \dots a_{ss}^{h_s+1} \dots a_{rs}^{h_r} l_s \\ &= a_{ss}^{n-1} l_s + (\text{non-negative terms}) > 0, \end{aligned}$$

for every  $s = 1, \dots, r$ . Furthermore, for  $j \neq k$  we have also the following equations

$$(3) \quad A_j^2 \mathcal{L}^{n-2} = [A_j]_{A_j} \mathcal{L}_{A_j}^{n-2} = \sum_{k_1+\dots+k_r=n-2} \frac{(n-2)!}{k_1! \dots k_r!} a_{1j}^{k_1} \dots a_{jj}^{k_j+1} \dots a_{rj}^{k_r} l_j,$$

$$(4) \quad A_k^2 \mathcal{L}^{n-2} = [A_k]_{A_k} \mathcal{L}_{A_k}^{n-2} = \sum_{h_1+\dots+h_r=n-2} \frac{(n-2)!}{h_1! \dots h_r!} a_{1k}^{h_1} \dots a_{kk}^{h_k+1} \dots a_{rk}^{h_r} l_k,$$

$$(5) \quad A_j A_k \mathcal{L}^{n-2} = [A_k]_{A_j} \mathcal{L}_{A_j}^{n-2} = \sum_{k_1+\dots+k_r=n-2} \frac{(n-2)!}{k_1! \dots k_r!} a_{1j}^{k_1} \dots a_{jj}^{k_j} \dots a_{kj}^{k_k+1} \dots a_{rj}^{k_r} l_j,$$

$$(6) \quad A_j A_k \mathcal{L}^{n-2} = [A_j]_{A_k} \mathcal{L}_{A_k}^{n-2} = \sum_{h_1+\dots+h_r=n-2} \frac{(n-2)!}{h_1! \dots h_r!} a_{1k}^{h_1} \dots a_{jk}^{h_j+1} \dots a_{kk}^{h_k} \dots a_{rk}^{h_r} l_k.$$

Since by (2) we get

$$\begin{aligned} &(a_{1j}^{k_1} \dots a_{jj}^{k_j+1} \dots a_{rj}^{k_r})(a_{1k}^{h_1} \dots a_{kk}^{h_k+1} \dots a_{rk}^{h_r}) \\ &= a_{jj} a_{kk} (a_{1j}^{k_1} \dots a_{jj}^{k_j} \dots a_{rj}^{k_r})(a_{1k}^{h_1} \dots a_{kk}^{h_k} \dots a_{rk}^{h_r}) \\ &= a_{jk} a_{kj} (a_{1j}^{k_1} \dots a_{jj}^{k_j} \dots a_{rj}^{k_r})(a_{1k}^{h_1} \dots a_{kk}^{h_k} \dots a_{rk}^{h_r}) \\ &= (a_{1j}^{k_1} \dots a_{jj}^{k_j} \dots a_{kj}^{k_k+1} \dots a_{rj}^{k_r})(a_{1k}^{h_1} \dots a_{jk}^{h_j+1} \dots a_{kk}^{h_k} \dots a_{rk}^{h_r}), \end{aligned}$$

from (3), (4), (5) and (6), we obtain that

$$(A_k A_j \mathcal{L}^{n-2})^2 = (A_k^2 \mathcal{L}^{n-2})(A_j^2 \mathcal{L}^{n-2}) > 0$$

for every  $k$  and  $j$  such that  $A_j \cap A_k \neq \emptyset$ .

Thus, by [2, (2.5.4)] we see that there exists a rational number  $\lambda_{jk}$  such that  $A_j$  is numerically equivalent to  $\lambda_{jk} A_k$ . Since from (2) it follows that  $A_k$  meets all the other components of  $A$ , by an inductive argument we get

$$(7) \quad \mathcal{L} \simeq A_1 + \dots + A_r \equiv (\lambda_{1k} + \dots + \widehat{\lambda_{kk}} + \dots + \lambda_{rk} + 1) A_k = \mu_k A_k,$$

where  $\mu_k := \lambda_{1k} + \dots + \widehat{\lambda_{kk}} + \dots + \lambda_{rk} + 1$  and the symbol  $\widehat{\phantom{x}}$  denotes suppression.

Moreover, since

$$0 < \mathcal{L}_{A_j}^{n-1} = A_j \mathcal{L}^{n-1} = \lambda_{jk} A_k \mathcal{L}^{n-1} = \lambda_{jk} \mathcal{L}_{A_k}^{n-1},$$

we have that  $\lambda_{jk} > 0$  for every  $j \neq k$  and from (7) it follows that  $\mu_k \geq 1$ , i.e.  $A_k$  is an ample Cartier divisor on  $\mathcal{V}$  for any  $k = 1, \dots, r$ . By combining this with Theorem 1.1, we obtain that every  $f_i$  is an isomorphism.  $\square$

**Corollary 3.1.** *Let  $\mathcal{V}$  be an integral normal complex projective variety of dimension  $n \geq 3$  with at worst Cohen–Macaulay singularities. Let  $\mathcal{L}$  be an ample line bundle on  $\mathcal{V}$  and assume that  $(\diamond)$  holds. Furthermore, suppose that one of the following conditions holds:*

- (a)  $\mathcal{V}$  is a locally complete intersection;
- (b)  $n = 3$ ,  $\mathcal{V} - A_k$  and  $\mathcal{V} - F$  are locally complete intersections for some  $k = 1, \dots, r$  and some finite set  $F \subset \mathcal{V} - \text{Irr}(\mathcal{V})$ .

*Then  $\text{Pic}(\mathcal{V}) = \mathbb{Z}\langle \Lambda \rangle$ , where  $\Lambda$  is an ample line bundle on  $\mathcal{V}$ . In particular, all the  $A_i$ 's are ample Cartier divisors on  $\mathcal{V}$  and the maps  $f_i: \mathcal{V} \rightarrow \mathcal{V}_i$  of Theorem 1.1 are all isomorphisms.*

*Proof.* In both cases (a) and (b), we simply use Theorem 1.2 and [2, (2.3.4)].  $\square$

**REMARK 3.2.** Let  $\mathcal{V} \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-d))$  with  $0 < d < n$  and denote by  $\pi: \mathcal{V} \rightarrow \mathbb{P}^{n-1}$  the projection map. Note that there exists a smooth divisor  $E$  on  $\mathcal{V}$  which is a section of  $\pi$  such that  $E \cong \mathbb{P}^{n-1}$  and  $E_E \in |\mathcal{O}_{\mathbb{P}^{n-1}}(-d)|$ . Put  $\mathcal{L} := E + \pi^* \mathcal{O}_{\mathbb{P}^{n-1}}(a)$ . Then, for a suitable integer  $a \gg 0$ , we have that  $\mathcal{L}$  is ample on  $\mathcal{V}$  and that there exists a divisor  $A \in |\mathcal{L}|$  such that  $A = A_1 + A_2$ , where  $A_1 = E$  and  $A_2 \in |\pi^* \mathcal{O}_{\mathbb{P}^{n-1}}(a)|$  is a smooth divisor on  $\mathcal{V}$ . This example shows that if  $r \geq 2$  then the above results can not be improved by assuming in  $(\diamond)$  that  $\text{Pic}(A_i) \neq \mathbb{Z}$  for some  $i = 1, \dots, r$ , also when  $r = 2$  and  $\mathcal{V}$  is a Fano  $n$ -fold with  $n \geq 3$ .

#### 4. Some immediate consequences

Let us collect here some results due to the different positivity of the components  $A_i$  of  $A \in |\mathcal{L}|$  under the hypothesis  $(\diamond)$ .

**4.0.1. Nefness and bigness of the  $A_i$ 's.** From Theorem 1.1, we obtain the following

**Corollary 4.1.** *Let  $\mathcal{V}$  be an integral normal complex projective variety of dimension  $n \geq 3$  and let  $\mathcal{L}$  be an ample line bundle on  $\mathcal{V}$ . Assume that  $(\diamond)$  holds. Set  $D = \sum A_{i_h}$ , where all the  $i_h \in \{1, \dots, r\}$  are not necessarily distinct indexes. Moreover, let  $\text{Irr}(\mathcal{V})$  be the set of irrational singularities of  $\mathcal{V}$ . Then we have the following properties:*

- (Vanishing type theorems).

(1) *Let  $\varphi: \mathcal{V} \rightarrow Y$  be a morphism from  $\mathcal{V}$  to a projective variety  $Y$ . Then*

$$\varphi_{(i)}(K_{\mathcal{V}} + D) = 0 \quad \text{for } i \geq \max_{y \in \varphi(\text{Irr}(\mathcal{V}))} \dim(\varphi^{-1}(y) \cap \text{Irr}(\mathcal{V})) + 1,$$

where  $\varphi_{(i)}(\mathcal{T})$  is the  $i$ -th higher derived functor of the direct image  $\varphi_* (\mathcal{T})$  of a sheaf  $\mathcal{T}$  on  $\mathcal{V}$ ;

(2)  $H^1(\mathcal{V}, -D) = 0$ ;

(3) assuming that  $\text{Irr}(\mathcal{V})$  is finite and nonempty, we have that

$$\dim H^0(\mathcal{V}, K_{\mathcal{V}} + D) \geq \#\text{Irr}(\mathcal{V}) > 0,$$

where  $\#\text{Irr}(\mathcal{V})$  is the number of points in  $\text{Irr}(\mathcal{V})$ ; in particular, if  $\mathcal{A} \in |D|$  lies in the set of Cohen–Macaulay points of  $\mathcal{V}$ , then

$$\dim H^0(\mathcal{V}, K_{\mathcal{V}}) + \dim H^0(\mathcal{A}, K_{\mathcal{A}}) \geq \#\text{Irr}(\mathcal{V}) > 0.$$

- (Lefschetz type theorems).

(4) Let  $\mathcal{A} \in |D|$  be a divisor such that  $\mathcal{V} - \mathcal{A}$  is a local complete intersection. Then under the restriction map it follows that

$$H^j(\mathcal{V}, \mathbb{Z}) \cong H^j(\mathcal{A}, \mathbb{Z}) \quad \text{for } j \leq n - 3,$$

and

$$H^j(\mathcal{V}, \mathbb{Z}) \rightarrow H^j(\mathcal{A}, \mathbb{Z}) \quad \text{for } j = n - 2,$$

is injective with torsion free cokernel; moreover, we have that  $\text{Pic}(\mathcal{V}) \cong \text{Pic}(\mathcal{A})$  for  $n \geq 5$ , and the restriction mapping  $\text{Pic}(\mathcal{V}) \rightarrow \text{Pic}(\mathcal{A})$  is injective with torsion free cokernel for  $n = 4$ ; in particular, if  $\mathcal{V} - A_i$  is a local complete intersection for some  $i = 1, \dots, r$ , then  $\text{Pic}(\mathcal{V}) = \mathbb{Z}\langle \Lambda \rangle$  for  $n \geq 4$  and some ample line bundle  $\Lambda$  on  $\mathcal{V}$ ;

- (The Albanese mapping).

(5) Assume that  $\mathcal{A} \in |D|$  is normal. Moreover, suppose that  $\mathcal{A}$  and  $\mathcal{V}$  have at worst rational singularities. Then the map  $\text{Alb}(\mathcal{A}) \rightarrow \text{Alb}(\mathcal{V})$  induced by inclusion is an isomorphism.

- (Hodge index type theorems).

(6) For  $n_{i_1} + \dots + n_{i_k} = n - 1$  and  $n_{i_1} \geq 1$ , we have

$$(A_{i_0} \cdot A_{i_1}^{n_{i_1}} \cdots A_{i_k}^{n_{i_k}})^2 \geq (A_{i_0}^2 \cdot A_{i_1}^{n_{i_1}-1} \cdots A_{i_k}^{n_{i_k}})(A_{i_1}^{n_{i_1}+1} \cdots A_{i_k}^{n_{i_k}}),$$

and for  $n_{i_1} + \dots + n_{i_k} = n$ , we get also

$$(A_{i_1}^{n_{i_1}} \cdots A_{i_k}^{n_{i_k}})^n \geq (A_{i_1}^n)^{n_{i_1}} \cdots (A_{i_k}^n)^{n_{i_k}} > 0,$$

where  $i_h \in \{1, \dots, r\}$ ;

(7) we have the following inequality:  $(A_i^{n-1} \cdot A_j)(A_i \cdot A_j^{n-1}) \geq A_i^n A_j^n > 0$ ;

(8) for  $t \geq 1$  and any nef and big line bundle  $H_i$  on  $\mathcal{V}$  with  $1 \leq i \leq t$ , it follows that  $\mathcal{O}_{\mathcal{V}}(A_j)^{n-t} \cdot \prod_{i=1}^t H_i > 0$ , where  $j \in \{1, \dots, r\}$ ;



(9) let  $H$  be a line bundle such that  $\dim H^0(\mathcal{V}, NH) \geq 2$  for some  $N \geq 1$ . Then  $H \cdot \mathcal{O}_{\mathcal{V}}(A_{i_1}) \cdots \mathcal{O}_{\mathcal{V}}(A_{i_{n-1}}) > 0$ , where  $i_h \in \{1, \dots, r\}$ .

Proof. First of all, from Theorem 1.1 we deduce that the effective divisor  $D = \sum A_{i_h}$  is nef and big on  $\mathcal{V}$ . Thus (1), (2) and (3) of the statement follow from [2, (2.2.5), (2.2.7), (2.2.8)]. Finally, cases (4) to (9) follow from [2, (2.3.3), (2.3.4), (2.4.4), (2.5.1), (2.5.3), (2.5.2), (2.5.8) and (2.5.9)].  $\square$

By applying Theorem 1.1 also to the zero locus of special sections of  $k$ -ample vector bundles on an  $n$ -fold  $X$  for  $k \geq 0$  (see [15, §1] and [2, §2.1]), we obtain the following

**Corollary 4.2** (Lefschetz–Sommese type theorem). *Let  $\mathcal{E}$  be a  $k$ -ample vector bundle of rank  $r \geq 1$  on an  $n$ -fold  $X$  with  $n \geq 4$ . Assume that there exists a section  $s \in \Gamma(\mathcal{E})$  whose zero locus  $Z = (s)_0$  is an integral normal complex variety such that  $\dim Z \geq 3$ . Suppose that there exists a divisor  $A = A_1 + \cdots + A_s, s \geq 1$ , on  $Z$  which satisfies  $(\diamond)$ . If  $Z - A_i$  is a local complete intersection for some  $i = 1, \dots, s$ , then*

$$H^i(X, \mathbb{Z}) \cong H^i(A_i, \mathbb{Z}) \quad \text{for } i \leq \min\{\dim Z - 3, n - r - k - 1\},$$

and the restriction maps  $H^i(X, \mathbb{Z}) \rightarrow H^i(A_i, \mathbb{Z})$  are injective with torsion free cokernel for  $i = \min\{\dim Z - 2, n - r - k\}$ .

Proof. From Theorem 1.1, we know that each  $A_k$  is at worst 1-ample on  $Z$ . Thus by Corollary 4.1 (4), we have that

$$H^j(Z, \mathbb{Z}) \cong H^j(A_i, \mathbb{Z}) \quad \text{for } j \leq \dim Z - 3,$$

and the restriction maps  $H^j(Z, \mathbb{Z}) \rightarrow H^j(A_i, \mathbb{Z})$  are injective with torsion free cokernel for  $j = \dim Z - 2$ . Moreover, from the Lefschetz–Sommese’s theorem for  $k$ -ample vector bundles on an  $n$ -fold  $X$  (see [15, (1.16)] and [13, (7.1.1), (7.1.9)]), we deduce that

$$H^j(Z, \mathbb{Z}) \cong H^j(X, \mathbb{Z}) \quad \text{for } j \leq n - r - k - 1,$$

and that the restriction maps  $H^j(X, \mathbb{Z}) \rightarrow H^j(Z, \mathbb{Z})$  are injective with torsion free cokernel for  $j = n - r - k$ .  $\square$

**4.0.2. Ampleness of the  $A_i$ ’s.** Under the same assumption of Theorem 1.2, we first deduce the following

**Corollary 4.3.** *Let  $\mathcal{V}$  be an integral normal complex projective variety of dimension  $n \geq 3$  with at worst Cohen–Macaulay singularities. Let  $\mathcal{L}$  be an ample line bundle on  $\mathcal{V}$  and assume that  $(\diamond)$  holds. Moreover, suppose that  $\mathcal{V} - F$  is a local complete intersection for some finite, possibly empty, set  $F \subset \mathcal{V} - \text{Irr}(\mathcal{V})$  with  $\dim \text{Irr}(\mathcal{V}) \leq 0$ ,*

where  $\text{Irr}(\mathcal{V})$  is the set of irrational singularities of  $\mathcal{V}$ . Set  $D = \sum A_{i_h}$ , where all the  $i_h \in \{1, \dots, r\}$  are not necessarily distinct indexes. Then we have the following properties:

(I) (Fujita’s vanishing type theorem). Given any coherent sheaf  $\mathcal{F}$  on  $\mathcal{V}$ , there exists an integer  $m(\mathcal{F}, D)$  such that

$$H^i(\mathcal{V}, \mathcal{F} \otimes \mathcal{O}_{\mathcal{V}}(mD + N)) = 0 \quad \text{for } i > 0, m \geq m(\mathcal{F}, D),$$

where  $N$  is any nef divisor on  $\mathcal{V}$ ;

(II)  $\dim H^0(\mathcal{V}, D) \leq D^n + n$ , with equality if and only if  $\mathcal{V}$  is one of the following:

- (a)  $\mathbb{P}^n$ ;
- (b) a quadric hypersurface  $\mathbb{Q}^n \subset \mathbb{P}^{n+1}$ ;
- (c) a  $\mathbb{P}^{n-1}$ -bundle over  $\mathbb{P}^1$ ;
- (d) a generalized cone over a smooth submanifold  $V \subset \mathcal{V}$  as in (a), (b), (c);

(III) (Lefschetz type theorems). Let  $\mathcal{A} \in |D|$  be a divisor such that  $\mathcal{V} - \mathcal{A}$  is a local complete intersection. Then under the restriction map it follows that

$$H^j(\mathcal{V}, \mathbb{Z}) \cong H^j(\mathcal{A}, \mathbb{Z}) \quad \text{for } j \leq n - 2,$$

and

$$H^j(\mathcal{V}, \mathbb{Z}) \rightarrow H^j(\mathcal{A}, \mathbb{Z}) \quad \text{for } j = n - 1,$$

is injective with torsion free cokernel; moreover, we have that  $\text{Pic}(\mathcal{V}) \cong \text{Pic}(\mathcal{A})$  for  $n \geq 4$ , and the restriction mapping  $\text{Pic}(\mathcal{V}) \rightarrow \text{Pic}(\mathcal{A})$  is injective with torsion free cokernel for  $n = 3$ ; in particular, if  $\mathcal{V} - A_i$  is a local complete intersection for some  $i = 1, \dots, r$ , then  $\text{Pic}(\mathcal{V}) = \mathbb{Z}\langle \Lambda \rangle$  for  $n \geq 3$  and some ample line bundle  $\Lambda$  on  $\mathcal{V}$ .

*Proof.* By Nakai–Moishezon–Kleiman criterion and Theorem 1.2, we see that  $D = \sum A_{i_h}$  is ample on  $\mathcal{V}$ . Thus case (I) of the statement follows from [5, §1, Theorem 1] (see also [12, (1.4.35)]). Finally, we obtain case (II) by [6, I (4.2), (5.10), (5.15)], while (III) follows from [2, (2.3.3)] and [2, (2.3.4)] respectively. □

Finally, let us deduce also the following result for ample vector bundles on a smooth variety.

**Corollary 4.4.** *Let  $\mathcal{E}$  be an ample vector bundle of rank  $r \geq 1$  on an  $n$ -fold  $X$  with  $n \geq 4$ . Assume that there exists a section  $s \in \Gamma(\mathcal{E})$  whose zero locus  $Z = (s)_0$  is a smooth submanifold of  $X$ . Suppose that there exists a divisor  $A = A_1 + \dots + A_s, s \geq 1$ , on  $Z$  which satisfies  $(\diamond)$ . If  $r < n - 2$  then both  $\text{Pic}(X)$  and  $\text{Pic}(Z)$  have rank one.*

*Proof.* It follows easily from Corollary 4.3 (III) and [15, (1.16)], or [13, (7.1.5) (ii)]. □

## 5. Some applications

Here are two applications.

**5.1. All the  $A_i$ 's are Fano varieties of Picard rank one.** First of all, let us prove the following

**Lemma 5.1.** *Let  $\mathcal{L}, \mathcal{V}, A_i, \mathcal{V}_i, f_i$  be as in Theorem 1.1. If  $\mathcal{V}_k$  is  $\mathbb{Q}$ -factorial for some  $k = 1, \dots, r$ , then  $A_k$  is ample on  $\mathcal{V}$  and  $f_k: \mathcal{V} \rightarrow \mathcal{V}_k$  is in fact an isomorphism.*

*Proof.* Take any line bundle  $\mathcal{D}$  on  $\mathcal{V}$  and consider the following commutative diagram (\*):

$$\begin{array}{ccc} U & \xrightarrow{f_k|_U} & U' \\ j \downarrow & & \downarrow j_k \\ \mathcal{V} & \xrightarrow{f_k} & \mathcal{V}_k \end{array}$$

where  $j: U \rightarrow \mathcal{V}$  and  $j_k: U' \rightarrow \mathcal{V}_k$  are the inclusion maps and  $f_k|_U: U \rightarrow U'$  is the isomorphism induced by  $f_k: \mathcal{V} \rightarrow \mathcal{V}_k$ . Since  $\mathcal{V}_k$  is  $\mathbb{Q}$ -factorial, we see that  $f_{k*}(N\mathcal{D}) = \mathcal{L}'$  is a line bundle on  $\mathcal{V}_k$  for some positive integer  $N$ . Write  $\mathcal{L}' = \sum_h a_h \mathcal{L}'_h$ , where  $\mathcal{L}'_h$  are the generators of  $\text{Pic}(\mathcal{V}_k)$ . Then by (\*) we get

$$\begin{aligned} N\mathcal{D}|_U &= j^*(N\mathcal{D}) = f_k|_{U*} j^*(N\mathcal{D}) = j_k^* f_{k*}(N\mathcal{D}) \\ &= \sum_h a_h j_k^* \mathcal{L}'_h = \sum_h a_h (f_k|_U)^* j_k^* \mathcal{L}'_h = \sum_h a_h j^* f_k^* \mathcal{L}'_h \\ &= j^* \left( \sum_h a_h f_k^* \mathcal{L}'_h \right) = \left( \sum_h a_h f_k^* \mathcal{L}'_h \right) \Big|_U. \end{aligned}$$

By Hartogs' lemma (see, e.g., [4, (11.4)]), this gives  $N\mathcal{D} = \sum_h a_h f_k^*(\mathcal{L}'_h)$ , i.e.  $\mathcal{D} = \sum_h (a_h/N) f_k^*(\mathcal{L}'_h)$ . Therefore, if  $A_k$  is not ample, then we deduce that there exists an irreducible curve  $\Gamma \subset \mathcal{V}$  such that  $m_k A_k \cdot \Gamma = 0$  for any positive integer  $m_k$ , i.e. the map  $f_k$  contracts the curve  $\Gamma$ . Hence  $f_k^*(\mathcal{L}'_h) \cdot \Gamma = 0$  for any  $h$ , i.e.  $\mathcal{D} \cdot \Gamma = 0$ , but this leads to a contradiction by taking  $\mathcal{D} = \mathcal{L}$ .  $\square$

Similar results as in [1, Theorem 1] (see also [14, Proposition VI]) for the case of reducible ample divisors on  $\mathcal{V}$  can be now proved.

**Proposition 5.2.** *Let  $\mathcal{V}$  be an integral normal complex projective variety of dimension  $n \geq 3$  and let  $\mathcal{L}$  be an ample line bundle on  $\mathcal{V}$ . Assume that  $(\diamond)$  holds. If, up to renaming,  $A_1 \cong \mathbb{P}^{n-1}$ , then  $\mathcal{V}$  is the cone  $\mathcal{C}(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(s))$  on  $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(s))$  with  $A_1 = v_s(\mathbb{P}^{n-1})$  and  $\mathcal{N}_{A_1/\mathcal{V}} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(s)$  for a suitable integer  $s > 0$ , where  $v_s$  is the  $s^{\text{th}}$  Veronese embedding of  $\mathbb{P}^{n-1}$ .*

Proof. Since  $(\diamond)$  holds and  $A_1 \cong \mathbb{P}^{n-1}$ , by Theorem 1.1 we have that  $f_1(A_1) \cong \mathbb{P}^{n-1}$  is ample on  $\mathcal{V}_1$ . By [1, Theorem 1] we see that  $\mathcal{V}_1$  is the cone  $\mathcal{C}(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(s))$  over  $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(s))$ , where  $s$  is a positive integer such that  $\mathcal{N}_{f_1(A_1)/\mathcal{V}_1} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(s)$ .

Since  $\mathcal{V}_1$  is  $\mathbb{Q}$ -factorial and  $\text{Pic}(\mathcal{V}_1) = \mathbb{Z}$ , by Lemma 5.1 we deduce that  $f_1$  is an isomorphism and  $\text{Pic}(\mathcal{V}) = \mathbb{Z}$ . Therefore,  $A_1$  is an ample divisor on  $\mathcal{V}$  and by applying now [1, Theorem 1] to the pair  $(\mathcal{V}, A_1)$ , we get that  $\mathcal{V} \cong \mathcal{C}(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(s))$ ,  $A_1 \cong v_s(\mathbb{P}^{n-1})$  and  $\mathcal{N}_{A_1/\mathcal{V}} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(s)$  for a suitable integer  $s > 0$ .  $\square$

**Proposition 5.3.** *Let  $\mathcal{V}$  be an integral normal Gorenstein projective variety of dimension  $n \geq 3$  and let  $\mathcal{L}$  be an ample line bundle on  $\mathcal{V}$ . Suppose that  $(\diamond)$  holds and that  $\dim \text{Irr}(\mathcal{V}) \leq 0$ , where  $\text{Irr}(\mathcal{V})$  is the set of irrational singularities of  $\mathcal{V}$ . Assume that each  $A_i$  is a normal Gorenstein variety such that  $K_{A_i} + \tau_i \mathcal{H}_i \simeq \mathcal{O}_{A_i}$  for some integer  $\tau_i$ . If  $\mathcal{V} - A_k$  is a local complete intersection for some  $k = 1, \dots, r$  and either  $n \geq 4$ , or  $n = 3$  and  $\mathcal{V} - F$  is a local complete intersection for some finite, possibly empty set  $F \subset \mathcal{V} - \text{Irr}(\mathcal{V})$ , then  $\text{Pic}(\mathcal{V}) = \mathbb{Z}\langle \Lambda \rangle$ , where  $\Lambda$  is an ample line bundle on  $\mathcal{V}$ ,  $K_{\mathcal{V}} = \rho \Lambda$ ,  $A_i = a_i \Lambda$  and  $\Lambda_{A_i} = h_i \mathcal{H}_i$  with  $\tau_i = -h_i(\rho + a_i)$ , where  $\rho, a_i > 0$  and  $h_i > 0$  are integers. In particular, for  $n \geq 5$  we have  $h_i = 1$  for every  $i = 1, \dots, r$ .*

Proof. Assume that  $n \geq 4$ . From case (4) of Corollary 4.1 it follows that either (a)  $n \geq 5$  and  $\text{Pic}(\mathcal{V}) \cong \text{Pic}(A_i) = \mathbb{Z}$  for any  $i = 1, \dots, r$ , or (b)  $n = 4$  and  $\text{Pic}(\mathcal{V})$  restricts injectively into  $\text{Pic}(A_i) = \mathbb{Z}$ . In both situations, we see that  $\text{Pic}(\mathcal{V}) = \mathbb{Z}\langle \Lambda \rangle$  for some ample line bundle  $\Lambda$  on  $\mathcal{V}$ . Moreover, in (a) we have that  $\Lambda_{A_i} \simeq \mathcal{H}_i$ , while in (b) we have that  $\Lambda_{A_i} \simeq h_i \mathcal{H}_i$  for some positive integer  $h_i$ . By adjunction formula, we obtain that

$$\mathcal{O}_{A_i} \simeq K_{A_i} + \tau_i \mathcal{H}_i \simeq \begin{cases} (K_{\mathcal{V}} + A_i + \tau_i \Lambda)_{A_i} & \text{in case (a),} \\ \left( K_{\mathcal{V}} + A_i + \frac{\tau_i}{h_i} \Lambda \right)_{A_i} & \text{in case (b).} \end{cases}$$

i.e.  $K_{\mathcal{V}} + A_i + (\tau_i/h_i)\Lambda \simeq \mathcal{O}_{\mathcal{V}}$  for some  $h_i \geq 1$ . If we put  $K_{\mathcal{V}} = \rho \Lambda$  and  $A_i = a_i \Lambda$  for some integers  $\rho$  and  $a_i \geq 1$ , then we see that  $\tau_i = -h_i(\rho + a_i)$ , with  $h_i = 1$  when  $n \geq 5$ .

As to the case  $n = 3$ , from (III) of Corollary 4.3 it follows that  $\text{Pic}(\mathcal{V})$  restricts injectively into  $\text{Pic}(A_i) = \mathbb{Z}$ . Then by arguing as above, we conclude that also in this situation  $\text{Pic}(\mathcal{V}) = \mathbb{Z}\langle \Lambda \rangle$  for some ample line bundle  $\Lambda$  on  $\mathcal{V}$  and, with the same notation as above,  $\tau_i = -h_i(\rho + a_i)$ , where  $h_i$  and  $a_i$  are positive integers.  $\square$

REMARK 5.4. When  $n \geq 6$ , Proposition 5.3 generalizes [16, Theorem 1].

**5.2. All the  $A_i$ 's have small degrees.** Denote now by  $X$  an  $n$ -fold with  $n \geq 3$  and by  $L$  an ample line bundle on  $X$ . In this subsection we prove the last result stated in the Introduction.

Proof of Proposition 1.3. Since  $[H_i]_{A_i}^{n-1} \leq 4$ , by [2, (8.10.1)] we see that  $(A_i, [H_i]_{A_i})$  satisfies one of the following two conditions:

- (a)  $\text{Pic}(A_i) = \mathbb{Z}\langle [H_i]_{A_i} \rangle$ ;
- (b)  $(A_i, [H_i]_{A_i}) \cong (\mathbb{P}^1 \times \mathbb{P}^3, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1, 1))$ .

STEP (I). First of all, assume that  $r = 1$ . In Case (a), by Lefschetz's theorem we see that  $\text{Pic}(X) = \mathbb{Z}\langle H_1 \rangle$ . Write  $A_1 = a_1 H_1$  for a positive integer  $a_1$ . Since  $a_1 H_1^n = A_1 H_1^{n-1} = [H_1]_{A_1}^{n-1} \leq 4$ , we deduce that  $a_1 H_1^n \leq 4$ . Consider the map  $\varphi: X \rightarrow \mathbb{P}^N$  associated to  $|H_1|$ . Since  $|H_1|$  is ample and spanned, the morphism  $\varphi$  is finite and such that

$$[H_1]^n = \deg \varphi \cdot \deg \varphi(X) \leq 4.$$

This gives the following three possibilities:

- (1)  $\deg \varphi = 1$ ,  $\deg \varphi(X) \leq 4$ ;
- (2)  $\deg \varphi = 2$ ,  $\deg \varphi(X) \leq 2$ ;
- (3)  $\deg \varphi = 3, 4$  and  $\deg \varphi(X) = 1$ .

In (1), since  $n \geq 5$  and  $\text{Pic}(X) = \mathbb{Z}\langle H_1 \rangle$ , by [2, (8.10.1)] (see also [6], [9], [11]) we deduce that  $(X, H_1)$  is one of the following pairs:

- $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ , where  $A_1 \in |\mathcal{O}_{\mathbb{P}^n}(a_1)|$  with  $0 < a_1 \leq 4$ ;
- $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$ , where  $A_1 \in |\mathcal{O}_{\mathbb{Q}^n}(a_1)|$  with  $0 < a_1 \leq 2$ ;
- $(V_d, \mathcal{O}_{\mathbb{P}^{n+1}}(1)_{V_d})$  with  $A_1 \in |H_1|$ , where  $V_d \subset \mathbb{P}^{n+1}$  is a smooth hypersurface of degree  $d = 3, 4$ ;
- $(W, \mathcal{O}_{\mathbb{P}^{n+2}}(1)_W)$  with  $A_1 \in |H_1|$ , where  $W = \mathbb{Q}_1 \cap \mathbb{Q}_2 \subset \mathbb{P}^{n+2}$  is a complete intersection of two quadric hypersurfaces  $\mathbb{Q}_i \subset \mathbb{P}^{n+2}$  for  $i = 1, 2$ .

In (2), the map  $\varphi$  is a double cover of either (i)  $\mathbb{P}^n$ , or (ii)  $\mathbb{Q}^n \subset \mathbb{P}^{n+1}$ . In case (i), we see that  $H_1 = \varphi^* \mathcal{O}_{\mathbb{P}^n}(1)$  and  $A_1 \in |a_1 H_1|$  with  $a_1 = 1, 2$ . In case (ii), we get  $H_1 = \varphi^* \mathcal{O}_{\mathbb{Q}^n}(1)$  and  $A_1 \in |H_1|$ . Finally, in (3) the morphism  $\varphi$  is a  $d$ -cover of  $\mathbb{P}^n$  with  $d = 3, 4$ , and  $A_1 = H_1 = \varphi^* \mathcal{O}_{\mathbb{P}^n}(1)$ .

Finally, consider Case (b). Note that  $\text{Pic}(X) \neq \mathbb{Z}$ . Moreover, since  $A_1$  is ample, for any ample line bundle  $H$  on  $\mathbb{P}^1 \times \mathbb{P}^3$ , we have

$$\begin{aligned} 0 &= \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1, 0) \cdot \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(2, 0) \cdot H^2 = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1, 0) \cdot (K_{A_1} + 4H_{1A_1}) \cdot H^2 \\ &= \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1, 0) \cdot (K_X + 4H_1 + A_1)_{A_1} \cdot H^2 \geq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1, 0) \cdot (K_X + 5H_1)_{A_1} \cdot H^2, \end{aligned}$$

i.e.  $K_X + 5H_1$  can not be ample on  $X$ . So by [10], [6] and [2, §7.2], we see that  $(X, H_1)$  is a scroll over  $\mathbb{P}^1$  of dimension five, i.e.  $X \cong \mathbb{P}(\mathcal{E})$ , where  $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_5)$  with  $1 \leq a_1 \leq \cdots \leq a_5$ , and  $H_1$  is the tautological line bundle  $\xi$  of  $\mathbb{P}(\mathcal{E})$ . Put  $\xi' = \xi - a_1 F$ . Since  $A_1$  is ample on  $X$ , write  $A_1 = a\xi' + bF$  for some positive integers  $a$  and  $b$  (see [2, (3.2.4)]). Then

$$[H_1]_{A_1}^4 = (a\xi' + bF)(\xi)^4 = a(\xi - a_1 F)(\xi)^4 + b = a(a_2 + \cdots + a_5) + b \geq 5,$$

but this is absurd. Thus Case (b) can not occur for  $r = 1$ .

STEP (II). Suppose now that  $r \geq 2$ . First of all, let us prove the following

**Claim.** *If  $(A_1, H_{1A_1}) \cong (\mathbb{P}^1 \times \mathbb{P}^3, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1, 1))$ , then  $K_X + 4H_i$  is not nef for some  $i = 1, \dots, r$ .*

Proof. Suppose that  $K_X + 4H_k$  is nef for every  $k = 1, \dots, r$ . Let  $\Phi_1: A_1 \rightarrow \mathbb{P}^1$  be the nefvalue morphism of  $(A_1, H_{1A_1})$ . Up to renaming, assume that  $A_2, \dots, A_s$  are the only components of  $A$  such that  $A_1 \cap A_h \neq \emptyset$  for  $h = 2, \dots, s$  with  $s \leq r$ . Put  $h_k := [A_k]_{A_1}$  for  $k = 2, \dots, r$  and note that  $h_t$  is trivial for  $t = s + 1, \dots, r$ .

First of all, if  $\Phi_1(h_k)$  is a union of points of  $\mathbb{P}^1$  for every  $k = 2, \dots, s$ , then for a general fiber  $F \cong \mathbb{P}^3$  of  $\Phi_1$  we have that

$$(L_{A_1})_F = ([A_1]_{A_1})_F + [h_2]_F + \dots + [h_s]_F = ([A_1]_{A_1})_F = \mathcal{O}_F(a),$$

for a suitable integer  $a \geq 1$ . Moreover, note that  $K_{A_1} + 4H_{1A_1} \simeq 2F$ .

Thus by adjunction we obtain that

$$\begin{aligned} [K_X + 4H_1]_F H_{1F}^2 &= ([K_X + 4H_1]_{A_1})_F (H_{1A_1})_F^2 = (2F - [A_1]_{A_1})_F (H_{1A_1})_F^2 \\ &= -([A_1]_{A_1})_F (H_{1A_1})_F^2 = -\mathcal{O}_F(a)\mathcal{O}_F(1)^2 < 0, \end{aligned}$$

but this is absurd, since  $K_X + 4H_1$  is nef and  $H_1$  is ample on  $X$ .

Assume now that  $\Phi_1(h_t) = \mathbb{P}^1$  for some  $t = 2, \dots, s$ . If  $\text{Pic}(A_t) = \mathbb{Z}$ , then by [3, (4.2)] we get a contradiction by taking  $B = A_t$  and  $A = A_1$ . Thus we can assume that  $(A_t, H_{tA_t}) \cong (\mathbb{P}^1 \times \mathbb{P}^3, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1, 1))$ . Note that there exists a general fiber  $F$  of  $\Phi_1$  such that  $F \not\subseteq h_t$  and  $[h_t]_F$  is effective and not trivial. Moreover, we have that  $\text{Pic}(F) \cong \mathbb{Z}$  and  $F \not\subseteq A_t$ , since otherwise  $F \subseteq A_1 \cap A_t = h_t$ . Therefore  $F \cap A_t \neq \emptyset$  and  $F \cap A_t \neq F$ . Since

$$\dim F + \dim A_t - \dim \Phi_1(A_1) = 3 + 4 - 1 = 6 > 5 = \dim X,$$

by the same argument as in the proof of [3, (4.2)] with  $B = F$  and  $A = A_t$ , we can conclude that  $F \subseteq A_t$ , but this gives a contradiction.  $\square$

By the above Claim, if one of the  $A_k$ , say  $A_1$ , is such that

$$(A_1, H_{1A_1}) \cong (\mathbb{P}^1 \times \mathbb{P}^3, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1, 1)),$$

then  $n = 5$  and  $K_X + 4H_i$  is not nef for some  $i = 1, \dots, r$ . Since  $\text{Pic}(A_1) \neq \mathbb{Z}$ , by [10], [6] and [2, §7] we deduce that  $(X, H_i)$  is a  $\mathbb{P}^4$ -bundle over a smooth curve  $C$ . Moreover,  $A_1$  dominates  $C$ . So  $C \cong \mathbb{P}^1$  and  $X \cong \mathbb{P}(\mathcal{E})$  for some vector bundle  $\mathcal{E}$  of rank-5 on  $\mathbb{P}^1$  such that  $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(\bar{a}_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^1}(\bar{a}_4)$  with  $0 \leq \bar{a}_1 \leq \dots \leq \bar{a}_4$ . Let  $\xi$  be the tautological line bundle of  $\mathbb{P}(\mathcal{E})$ . Write  $A_1 = a_1\xi + b_1F$  and  $H_1 = \alpha_1\xi + \beta_1F$  with  $b_1 \geq 0$  and  $a_1, \alpha_1, \beta_1$  positive integers (see [2, (3.2.4)]). Furthermore, put  $F_{A_1} \simeq$

$\mathcal{O}_{A_1}(a, 0)$  and  $\xi_{A_1} \simeq \mathcal{O}_{A_1}(b, c)$  with  $a > 0$  and  $b, c \geq 0$ . By the adjunction formula we have

$$\begin{aligned} \mathcal{O}_{A_1}(-2, -4) &\simeq K_{A_1} \simeq (K_X + A_1)_{A_1} \simeq [-5\xi + (\deg \mathcal{E} - 2)F + a_1\xi + b_1F]_{A_1} \\ &= (a_1 - 5)\xi_{A_1} + (b_1 + \deg \mathcal{E} - 2)F_{A_1} \\ &\simeq \mathcal{O}_{A_1}(b(a_1 - 5) + a(b_1 + \deg \mathcal{E} - 2), c(a_1 - 5)). \end{aligned}$$

This gives the following two equations:

$$\begin{aligned} (b) \quad &4 = c(5 - a_1), \\ (\natural) \quad &2 = b(5 - a_1) - a(b_1 + \deg \mathcal{E} - 2). \end{aligned}$$

Note that from (b) it follows that  $1 \leq a_1 \leq 4$  and then

$$4 \geq a_1 = [A_1]_F \cdot [\xi]_F^3 = F_{A_1} \cdot \xi_{A_1}^3 = \mathcal{O}_{A_1}(a, 0) \cdot \mathcal{O}_{A_1}(b, c)^3 = ac^3.$$

Thus we deduce that  $c = 1$  and  $a_1 = a$ . By (b) we have also that  $a = a_1 = 1$  and the equation ( $\natural$ ) gives  $4b = b_1 + \deg \mathcal{E}$ . Since

$$\mathcal{O}_{A_1}(1, 1) \simeq [H_1]_{A_1} = [\alpha_1\xi + \beta_1F]_{A_1} = \mathcal{O}_{A_1}(\alpha_1b + \beta_1, \alpha_1),$$

we see that  $\alpha_1 = 1$  and  $1 = b + \beta_1 \geq b + 1 \geq 1$ , i.e.  $b = 0$  and  $\beta_1 = 1$ . Therefore, by ( $\natural$ ) we get  $0 = 4b = b_1 + \deg \mathcal{E} \geq \deg \mathcal{E}$ , i.e.  $\bar{a}_1 = \dots = \bar{a}_4 = 0$  and  $b_1 = 0$ . This shows that  $X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 5}) \cong \mathbb{P}^1 \times \mathbb{P}^4$ ,  $A_1 \in |\mathcal{O}_X(0, 1)|$  and  $H_1 \in |\mathcal{O}_X(1, 1)|$ . Consider  $A_i \in |\mathcal{O}_X(d, e)|$  and  $H_i \in |\mathcal{O}_X(t, s)|$  for some  $i = 2, \dots, r$ , where  $d, e \geq 0$  and  $t, s > 0$ . Then we have

$$4 \geq [H_i]_{A_i}^4 = \mathcal{O}_X(d, e) \cdot \mathcal{O}_X(t, s)^4 = s^3(ds + 4et) \geq s^3.$$

This gives  $s = 1$  and  $0 \leq e \leq 1$ . If  $e = 1$ , then we get  $d = 0, t = s = e = 1$  and  $(A_i, [H_i]_{A_i})$  is of the same type as  $(A_1, [H_1]_{A_1})$ . If  $e = 0$ , then we see that  $1 \leq d \leq 4, t \geq 1$  and  $s = 1$ . This shows that  $A_i \in |\mathcal{O}_X(d, 0)|$  and  $H_i \in |\mathcal{O}_X(t, 1)|$  with  $t \geq 1$  and  $1 \leq d \leq 4$ , i.e.  $A_i \rightarrow \mathbb{P}^4$  is a  $d$ -cover of  $\mathbb{P}^4$  with  $1 \leq d \leq 4$ . Note that in this case we have

$$|\mathcal{O}_X(d, 0)| = \underbrace{|\mathcal{O}_X(1, 0)| + \dots + |\mathcal{O}_X(1, 0)|}_{d\text{-times}}.$$

Thus, since  $A_i \in |\mathcal{O}_X(d, 0)|$  is irreducible and reduced, we conclude that  $d = 1$ .

Finally, since the above argument works independently from the choice of the component  $A_1$  of  $A \in |L|$ , we can assume, without loss of generality, that every component  $A_i$  of  $A$  is such that  $\text{Pic}(A_i) = \mathbb{Z}\langle [H_i]_{A_i} \rangle$ . Then by Proposition 5.3 and Corollary 4.3 (III), we conclude that  $\text{Pic}(X) = \mathbb{Z}\langle \Lambda \rangle$  for some ample line bundle  $\Lambda$  on  $X$ . Write

$A_i = a_i \Lambda$  and  $H_i = b_i \Lambda$  for suitable positive integers  $a_i$  and  $b_i$ . Thus we obtain that

$$a_i b_i^{n-1} \Lambda^n = (a_i \Lambda)(b_i \Lambda)^{n-1} = [H_i]_{A_i}^{n-1} \leq 4,$$

i.e.  $H_1 = \cdots = H_r = \Lambda$  and  $a_i \Lambda^n \leq 4$ . Consider the map  $\varphi: X \rightarrow \mathbb{P}^N$  associated to  $|\Lambda|$ . Since  $\Lambda$  is ample and spanned, we have  $\Lambda^n = \deg \varphi \cdot \deg \varphi(X) \leq 4$  and by arguing as in case  $r = 1$ , we obtain the statement.  $\square$

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