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INVOLUTIONS ON A COMPACT 4-SYMMETRIC SPACE OF EXCEPTIONAL TYPE

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Abstract

Let $(G/H, \sigma)$ be a compact 4-symmetric space of inner and exceptional type. Suppose that the dimension of the center of H is one and H is not a centralizer of a toral subgroup of G. In this paper we shall classify the involution τ of G satisfying $\tau \circ \sigma = \sigma \circ \tau$.

1. Introduction

It is known that k-symmetric spaces is a generalizations of symmetric spaces. The definition is as follows:

Let G be a Lie group and H a closed subgroup of G. A homogeneous space G/H is called a *k-symmetric space* if there exists an automorphism σ on G such that

- $G_o^\sigma \subset H \subset G^\sigma$, where G^σ and G_o^σ are the sets of fixed points of σ and its identity component, respectively,
- $\sigma^k = \mathrm{Id}$ and $\sigma^l \neq \mathrm{Id}$ for any $l < k$.

We denote by $(G/H, \sigma)$ a k-symmetric space with an automorphism σ of order k. Gray [2] classified 3-symmetric spaces (see also Wolf and Gray [14] and [15]). Moreover compact 4-symmetric spaces are classified by Jeménez [4]. The structure of k-symmetric spaces are closely related to the study of finite order automorphisms of Lie groups. Such automorphisms of compact simple Lie groups were classified (cf. Kac [5] and Helgason [3]).

It is known that involutions on k-symmetric spaces are important. For example, the classifications of affine symmetric spaces by Berger [1] are, in essence, the classification of involutions on compact symmetric spaces G/H preserving H. Similarly, such involutions play an important role in the classification of symmetric submanifolds on compact symmetric spaces (cf. Naitoh [9], [10] and [11]).

On a compact 3-symmetric space $(G/H, \sigma)$, an involution τ preserving H satisfies $\tau \circ \sigma = \sigma \circ \tau$ or $\tau \circ \sigma = \sigma^{-1} \circ \tau$. The classification of affine 3-symmetric spaces ([14] and [15]) was made by classifying involutions τ satisfying $\tau \circ \sigma = \sigma \circ \tau$. Moreover, in [12], [13] half-dimensional, totally real and totally geodesic submanifold (with respect to the canonical almost complex structures) of compact Riemannian 3-symmetric spaces

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$(G/H, \langle\ ,\ \rangle, \sigma)$ by using involutions τ on G satisfying $\tau \circ \sigma = \sigma^{-1} \circ \tau$ are classified.

Let G be a compact simple Lie group and $(G/H, \sigma)$ a 4-symmetric space of inner type. Now, we consider the problem of the classification theorem of involutions of G preserving H. In [6], the authors studied involutions of G preserving H for the case where the dimension of the center $Z(H)$ of H is at most one. In particular we classified involutions of G preserving H for the case where $\dim Z(H) = 0$, or $\dim Z(H) = 1$ and H is a centralizer of a toral subgroup of G. In this paper, we treat the case where G/H is exceptional type and H is not a centralizer of a toral subgroup of G. In particular, we classify all involutions τ of G for the case where $\dim Z(H) = 1$ and H is not a centralizer of a toral subgroup of G satisfying $\tau \circ \sigma = \sigma \circ \tau$.

More precisely, let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H, respectively. Then there exists a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ contained in $\mathfrak{h}$ such that $\tau(\mathfrak{t}) = \mathfrak{t}$ for any involution τ preserving $\mathfrak{h}$. We classify involutions $\bar{\tau}$ of the root system of $\mathfrak{h}$ with respect to $\mathfrak{t}$. Then for each involutions $\bar{\tau}$ of the root system of $\mathfrak{h}$, there exists an involution τ_0 preserving $\mathfrak{h}$ such that $\tau_0|_{\mathfrak{t}} = \bar{\tau}$. Each involution τ can be written as $\tau = \tau_0 \circ \mathrm{Ad}(\exp \sqrt{-1}h)$ or $\tau = \mathrm{Ad}(\exp \sqrt{-1}h)$ for some $\sqrt{-1}h \in \mathfrak{t}$, and we classify all τ by considering conjugations within automorphisms preserving $\mathfrak{h}$. For the case where $\bar{\tau} = \mathrm{Id}$, we classify all involutions τ by an argument similar to [6].

Using the result of this paper, the non-compact Riemannian 4-symmetric spaces of exceptional type will be classified in forthcoming paper [7].

The organization of this paper is as follows:

In Section 2, we recall the notions of root systems needed for the remaining part of this paper. Moreover we recall some results on automorphisms of order k ($k \leq 4$).

In Section 3, we remark on some relation between involutions of 4-symmetric space $(G/H, \langle\ ,\ \rangle, \sigma)$ reserving H and root systems of the Lie algebra of G.

In Section 4, by using the results in Section 3, we describe the restrictions of involutions to the root systems for the case where the dimension of the center is one.

In Sections 5 and 6, we enumerate all involutions τ of compact 4-symmetric spaces of exceptional type such that $\tau(H) = H$, the dimension of the center of H is one, $\tau \circ \sigma = \sigma \circ \tau$.

In Section 7, we describe some conjugations between involutions.

In Section 8, by making use of the results in Sections 5 and 6 together with conjugations in Section 7, we give the classification theorem of the conjugation classes of involutions.

2. Preliminaries

2.1. Root systems. Let $\mathfrak{g}$ and $\mathfrak{t}$ be a compact semisimple Lie algebra and a maximal abelian subalgebra of $\mathfrak{g}$, respectively. We denote by $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{t}_{\mathbb{C}}$ the complexifications of $\mathfrak{g}$ and $\mathfrak{t}$, respectively. Let $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ be the root system of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{t}_{\mathbb{C}}$ and $\Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\alpha_1, \dots, \alpha_n\}$ the set of fundamental roots of $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ with respect to a lexicographic order. For $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, put

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g}_{\mathbb{C}}\,;\, [H, X] = \alpha(H)X \text{ for any } H \in \mathfrak{t}_{\mathbb{C}}\}. \tag{2.1}$$

For $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ we define $H_\alpha \in \mathfrak{t}_{\mathbb{C}}$ ($\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$) by $\alpha(H) = B(H_\alpha, H)$, $H \in \mathfrak{t}_{\mathbb{C}}$, where B is the Killing form of $\mathfrak{g}_{\mathbb{C}}$. As in [3], we take the Weyl basis $\{E_\alpha \in \mathfrak{g}_\alpha\,;\, \alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})\}$ of $\mathfrak{g}_{\mathbb{C}}$ so that

$$\begin{aligned}
&[E_\alpha, E_{-\alpha}] = H_\alpha,\\
&[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}, \quad N_{\alpha,\beta} \in \mathbb{R},\\
&N_{\alpha,\beta} = -N_{-\alpha,-\beta},\\
&A_\alpha := E_\alpha - E_{-\alpha}, \quad B_\alpha := \sqrt{-1}(E_\alpha + E_{-\alpha}) \in \mathfrak{g}.
\end{aligned}$$

We denote by $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ the set of positive roots of $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ with respect to the order. Then it follows that

$$\mathfrak{g} = \mathfrak{t} + \sum_{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha), \quad \mathfrak{t} = \sum_{i=1}^n \mathbb{R}\sqrt{-1}H_{\alpha_i}. \tag{2.2}$$

For $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, we define a Lie subalgebra $\mathfrak{su}_\alpha(2)$ of $\mathfrak{g}$ by

$$\mathfrak{su}_\alpha(2) := \mathbb{R}\sqrt{-1}H_\alpha + \mathbb{R}A_\alpha + \mathbb{R}B_\alpha. \tag{2.3}$$

It is obvious that $\mathfrak{su}_\alpha(2) \cong \mathfrak{su}(2)$. We denote by t_α the root reflection along $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Then there exists an extension of t_α to an element of the group $\mathrm{Int}(\mathfrak{g})$ of inner automorphisms of $\mathfrak{g}$, which is denoted by the same symbol as t_α. Since the root reflection of $\mathfrak{su}_\alpha(2)$ along α coincides with the restriction of t_α to $\mathbb{R}\sqrt{-1}H_\alpha$ and t_α is the identical transformation on the orthogonal complement of $\mathbb{R}\sqrt{-1}H_\alpha$ in $\mathfrak{t}$, the following lemma holds.

Lemma 2.1. *There exists an element* $\phi \in \mathrm{Int}(\mathfrak{su}_\alpha(2))$ ($\subset \mathrm{Int}(\mathfrak{g})$) *such that* $\phi|_{\mathfrak{t}} = t_\alpha|_{\mathfrak{t}}$.

Define $K_j \in \mathfrak{t}_{\mathbb{C}}$ ($j = 1, \dots, n$) by

$$\alpha_i(K_j) = \delta_{ij}, \quad i, j = 1, \dots, n,$$

and denote the highest root δ of $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ by

$$\delta := \sum_{j=1}^n m_j \alpha_j, \quad m_j \in \mathbb{Z}.$$

In this paper, properly speaking, we denote $\mathrm{Ad}(\exp \pi\sqrt{-1}H)$ ($H \in \mathfrak{t}_{\mathbb{C}}$) by τ_H simply. Then from (2.1) we have

$$\tau_H(E_\alpha) = e^{\pi\sqrt{-1}\alpha(H)}E_\alpha, \quad \alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}). \tag{2.4}$$

Assume that $\mathfrak{g}$ is simple. Then the following is known.

Lemma 2.2 ([8]). *Any inner automorphism of order* 2 *on* $\mathfrak{g}$ *is conjugate within* $\mathrm{Int}(\mathfrak{g})$ *to some* τ_{K_i} *with* $m_i = 1$ *or* 2.

If $h - h' = \sum_{i=1}^{n} a_i K_i$, $a_i \in 2\mathbb{Z}$ for $h, h' \in \mathfrak{t}_{\mathbb{C}}$, we say that h is congruent to h' modulo $2\Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ and it is denoted by $h \equiv h' \pmod{2\Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})}$. It follows from (2.4) that $\tau_h = \tau_{h'}$ if $h \equiv h' \pmod{2\Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})}$.

REMARK 2.3. According to Lemma 2.2, for any inner automorphism τ_H of order 2 on $\mathfrak{g}$, there exists an inner automorphism ν of $\mathfrak{g}$ such that $\nu(H) \equiv K_i \pmod{2\Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})}$, $m_i = 1$ or 2.

We write $h \sim k$ if τ_h is conjugate to τ_k within $\mathrm{Int}(\mathfrak{g})$. Then the following lemmas hold.

Lemma 2.4 ([6]). (A_n) *If* $\mathfrak{g}$ *is of type* A_n, *then* $K_i \sim K_{n+1-i}$.
(D_n) *If* $\mathfrak{g}$ *is of type* D_n, *then* $K_i \sim K_{n-i}$ $(1 \leq i \leq [n/2])$. *In particular, if* n *is odd, then* $K_{n-1} \sim K_n$.
(E_6) *If* $\mathfrak{g}$ *is of type* E_6, *then* $K_1 \sim K_6$, $K_2 \sim K_3 \sim K_5$.

Lemma 2.5 ([4]). *Let* σ *be an inner automorphism of order* 4 *on* $\mathfrak{g}$. *Then* $\sigma \sim \tau_{(1/2)h_a}$ *where either*

$h_0 = K_i, \quad m_i = 4,$

$h_1 = K_i \quad \textit{or} \quad K_j + K_k, \quad m_i = 3, \quad m_j = m_k = 2,$

$h_2 = K_i + K_j, \quad m_i = 1, \quad m_j = 2,$

$h_3 = K_i + K_j + K_k, \quad m_i = m_j = m_k = 1,$

$h_4 = K_i, \quad m_i = 1,$

$h_5 = K_i, \quad K_j + K_k \quad \textit{or} \quad 2K_p + K_q, \quad m_i = 2, \quad m_j = m_k = m_p = m_q = 1.$

REMARK 2.6. (1) If σ is conjugate to $\tau_{(1/2)h_4}$, then a pair $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is symmetric. If σ is conjugate to $\tau_{(1/2)h_5}$, then a pair $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is 3-symmetric. ([6])
(2) Let $\mathfrak{z}$ be the center of $\mathfrak{g}^\sigma$. If $\sigma = \tau_{(1/2)h_a}$ ($a = 0, 1, 2, 3$), then the dimension of $\mathfrak{z}$ is equal to a ([4]).

3. Some remarks on automorphisms of order 4

In this section we use the same notation as in Section 2. Let $(G/H, \sigma)$ be a 4-symmetric space with an inner automorphism σ of order 4. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H, respectively. Note that $\mathfrak{h}$ coincides with the set $\mathfrak{g}^{\sigma}$ of fixed points of σ. Choose a subspace $\mathfrak{m}$ of $\mathfrak{g}$ so that $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is an $\mathrm{Ad}(H)$- and σ-invariant decomposition. Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{g}$ contained in $\mathfrak{h}$, and $\mathfrak{z}$ the center of $\mathfrak{h}$.

Suppose that $\mathfrak{g}$ is a compact simple Lie algebra and $\dim \mathfrak{z} = 1$. Let $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ be the set of automorphisms of $\mathfrak{g}$ preserving $\mathfrak{h}$. Then by Lemma 2.5 σ is conjugate within $\mathrm{Int}(\mathfrak{g})$ to some $\tau_{(1/2)K_i}$, $m_i = 3$ or $\tau_{(1/2)(K_a+K_b)}$, $m_a = m_b = 2$. In the previous paper [6], we classify involutions of $\mathfrak{g}$ preserving $\mathfrak{h}$ such that σ is conjugate to $\tau_{(1/2)K_i}$, $m_i = 3$. Then the following lemma and two remarks hold (see [6]).

Lemma 3.1 ([6]). *Assume $\sigma = \tau_{(1/2)(K_a+K_b)}$, $m_a = m_b = 2$, where $\delta = \sum_{j=1}^{n} m_j \alpha_j$ is the highest root of $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ as in* Section 2. *Then for each $\mu \in \mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$, we have $\mu \circ \sigma \circ \mu^{-1} = \sigma$ or σ^{-1}.*

REMARK 3.2 ([6]). Lemma 3.1 dose not hold in general. If σ is conjugate to $\tau_{(1/2)K_i}$ ($m_i = 3$ or 4), then Lemma 3.1 holds. However in other cases, Lemma 3.1 dose not hold.

REMARK 3.3 ([6]). If σ is an automorphism of order 2 or 3, then we have $\mu \circ \sigma \circ \mu^{-1} = \sigma$ or σ^{-1} for any $\mu \in \mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$.

Now, similarly as in the proof of Lemma 3.2 and Lemma 3.3 in [6], we have the following two lemmas.

Lemma 3.4. *Suppose that $\sigma = \tau_{(1/2)(K_a+K_b)}$ with $m_a = m_b = 2$. Let τ be an involutive automorphism of $\mathfrak{g}$ preserving $\mathfrak{h}$. Then*
(i) *$\tau \circ \sigma = \sigma \circ \tau$ if and only if $\tau(K_a + K_b) \equiv K_a + K_b \pmod{4\Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})}$,*
(ii) *$\tau \circ \sigma = \sigma^{-1} \circ \tau$ if and only if $\tau(K_a + K_b) \equiv -(K_a + K_b) \pmod{4\Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})}$.*

Lemma 3.5. *Suppose that $\sigma = \tau_{(1/2)(K_a+K_b)}$ with $m_a = m_b = 2$.*
(i) *Let τ_1 and τ_2 be involutive automorphisms of $\mathfrak{g}$ preserving $\mathfrak{h}$. If there exists $\mu \in \mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ such that $\mu \circ \tau_1 \circ \mu^{-1} = \tau_2$. Then*

$$\mathfrak{g}^{\tau_1} \cong \mathfrak{g}^{\tau_2}, \quad \mathfrak{h} \cap \mathfrak{g}^{\tau_1} \cong \mathfrak{h} \cap \mathfrak{g}^{\tau_2}. \tag{3.1}$$

(ii) *Put $\tau' := \mu \circ \tau \circ \mu^{-1}$, $\mu \in \mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$. If $\tau \circ \sigma = \sigma^{\pm 1} \circ \tau$, then $\tau' \circ \sigma = \sigma^{\pm 1} \circ \tau'$, respectively.*

In the remaining part of this paper, we suppose that $\sigma = \tau_{(1/2)(K_a+K_b)}$ for some $\alpha_a, \alpha_b \in \Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ with $m_a = m_b = 2$. Then the Dynkin diagram of $\mathfrak{h}$ is isomorphic to the extended Dynkin diagram of $\Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ except α_a and α_b (cf. Theorem 5.15 of Chapter X of [3]). We denote by $\Pi(\mathfrak{h})$ the fundamental root system of $\mathfrak{h}$ corresponding to the Dynkin diagram of $\mathfrak{h}$. Then the following holds.

Lemma 3.6 ([6]). *For any involutive automorphism τ of $\mathfrak{g}$ preserving $\mathfrak{h}$, there exists $\mu \in \mathrm{Int}(\mathfrak{h})$ such that $\mu \circ \tau \circ \mu^{-1}(\Pi(\mathfrak{h})) = \Pi(\mathfrak{h})$.*

From Remark 3.3 and Lemma 3.1 if $\dim \mathfrak{z} = 1$, then we have the following two cases:

$$\tau \circ \sigma = \sigma \circ \tau \quad \text{or} \quad \tau \circ \sigma = \sigma^{-1} \circ \tau.$$

In the following sections, we shall classify the conjugation classes of involutive automorphisms τ within $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ such that $\dim \mathfrak{z} = 1$ and $\tau \circ \sigma = \sigma \circ \tau$.

4. The restriction of τ to $\mathfrak{t}$

In the remaining part of this paper we use the same notation as in Sections 2 and 3. Let $(G/H, \sigma)$ be a compact 4-symmetric space such that G is compact simple Lie group and σ is inner automorphisms of order 4. As before, let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{g}$ contained in $\mathfrak{h}$. We suppose that $\sigma = \tau_{(1/2)(K_a+K_b)}$ for some $\alpha_a, \alpha_b \in \Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ with $m_a = m_b = 2$. Then $\mathfrak{z} = \mathbb{R}\sqrt{-1}(K_a - K_b)$. Let τ be an involution of $\mathfrak{g}$ satisfying $\tau \circ \sigma = \sigma \circ \tau$. By Lemma 3.6, we may assume that $\tau(\mathfrak{t}) = \mathfrak{t}$ and $\tau(\Pi(\mathfrak{h})) = \Pi(\mathfrak{h})$. Then it follows from Lemma 3.4 that
(C1) $\tau(K_a - K_b) = K_a - K_b$, $\tau(K_a + K_b) \equiv K_a + K_b \pmod{4\Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})}$,
(C2) $\tau(K_a - K_b) = -(K_a - K_b)$, $\tau(K_a + K_b) \equiv K_a + K_b \pmod{4\Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})}$.

Now we denote by $\langle m, n\rangle$ the set of $\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ such that the coefficients of α_a and α_b in α are m and n, respectively. Then we have

$$\begin{aligned}
&\mathfrak{g}^{\tau_{K_a}} = \{\pm\langle 0,0\rangle, \pm\langle 0,1\rangle, \pm\langle 0,2\rangle, \pm\langle 2,0\rangle, \pm\langle 2,1\rangle, \pm\langle 2,2\rangle\},\\
&(\mathfrak{g}^{\tau_{K_a}})^{\perp} = \{\pm\langle 1,0\rangle, \pm\langle 1,1\rangle, \pm\langle 1,2\rangle\},\\
&\mathfrak{g}^{\tau_{K_b}} = \{\pm\langle 0,0\rangle, \pm\langle 1,0\rangle, \pm\langle 0,2\rangle, \pm\langle 2,0\rangle, \pm\langle 1,2\rangle, \pm\langle 2,2\rangle\},\\
&(\mathfrak{g}^{\tau_{K_b}})^{\perp} = \{\pm\langle 0,1\rangle, \pm\langle 1,1\rangle, \pm\langle 2,1\rangle\}.
\end{aligned} \tag{4.1}$$

The right hand sides are the sums of the root spaces of the specified roots.

CASE (C1) We have $\tau(K_s) \equiv K_s \pmod{4\Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})}$, $s = a, b$ and hence $\tau \circ \tau_{K_s} \circ \tau^{-1} = \tau_{K_s}$, $s = a, b$. Thus we have

$$\tau(\mathfrak{g}^{\tau_{K_s}}) = \mathfrak{g}^{\tau_{K_s}}, \quad \tau((\mathfrak{g}^{\tau_{K_s}})^{\perp}) = (\mathfrak{g}^{\tau_{K_s}})^{\perp} \quad (s = a, b)$$

and

$$\tau(\mathfrak{g}^{\tau_{K_a}} \cap \mathfrak{g}^{\tau_{K_b}}) = \mathfrak{g}^{\tau_{K_a}} \cap \mathfrak{g}^{\tau_{K_b}}, \quad \tau((\mathfrak{g}^{\tau_{K_a}})^{\perp} \cap (\mathfrak{g}^{\tau_{K_b}})^{\perp}) = (\mathfrak{g}^{\tau_{K_a}})^{\perp} \cap (\mathfrak{g}^{\tau_{K_b}})^{\perp}.$$

Therefore by (4.1) the involution τ satisfies following relations.

$$
\begin{aligned}
&\tau(\{\pm\langle 1,1\rangle\}) = \{\pm\langle 1,1\rangle\},\\
&\tau(\{\pm\langle 1,0\rangle, \pm\langle 1,2\rangle\}) = \{\pm\langle 1,0\rangle, \pm\langle 1,2\rangle\},\\
&\tau(\{\pm\langle 0,1\rangle, \pm\langle 2,1\rangle\}) = \{\pm\langle 0,1\rangle, \pm\langle 2,1\rangle\},\\
&\tau(\{\pm\langle 2,0\rangle, \pm\langle 0,2\rangle\}) = \{\pm\langle 2,0\rangle, \pm\langle 0,2\rangle\}.
\end{aligned}
\tag{4.2}
$$

CASE (C2) We have $\tau \circ \tau_{K_a} \circ \tau^{-1} = \tau_{K_b}$ and hence

$$\tau(\mathfrak{g}^{\tau_{K_a}}) = \mathfrak{g}^{\tau_{K_b}}, \quad \tau((\mathfrak{g}^{\tau_{K_a}})^{\perp}) = (\mathfrak{g}^{\tau_{K_b}})^{\perp}.$$

Therefore the involution τ satisfies following relations.

$$
\begin{aligned}
&\tau(\{\pm\langle 0,0\rangle, \pm\langle 2,2\rangle\}) = \{\pm\langle 0,0\rangle, \pm\langle 2,2\rangle\},\\
&\tau(\{\pm\langle 0,1\rangle, \pm\langle 0,2\rangle, \pm\langle 2,0\rangle, \pm\langle 2,1\rangle\}) = \{\pm\langle 1,0\rangle, \pm\langle 0,2\rangle, \pm\langle 2,0\rangle, \pm\langle 1,2\rangle\},\\
&\tau(\{\pm\langle 1,0\rangle, \pm\langle 1,1\rangle, \pm\langle 1,2\rangle\}) = \{\pm\langle 0,1\rangle, \pm\langle 1,1\rangle, \pm\langle 2,1\rangle\}.
\end{aligned}
$$

Similarly as in the case (C1), since

$$\tau(\mathfrak{g}^{\tau_{K_a}} \cap \mathfrak{g}^{\tau_{K_b}}) = \mathfrak{g}^{\tau_{K_a}} \cap \mathfrak{g}^{\tau_{K_b}}, \quad \tau((\mathfrak{g}^{\tau_{K_a}})^{\perp} \cap (\mathfrak{g}^{\tau_{K_b}})^{\perp}) = (\mathfrak{g}^{\tau_{K_a}})^{\perp} \cap (\mathfrak{g}^{\tau_{K_b}})^{\perp},$$

it follows from (4.1) that

$$
\begin{aligned}
&\tau(\{\pm\langle 1,1\rangle\}) = \{\pm\langle 1,1\rangle\},\\
&\tau(\{\pm\langle 1,0\rangle, \pm\langle 1,2\rangle\}) = \{\pm\langle 0,1\rangle, \pm\langle 2,1\rangle\},\\
&\tau(\{\pm\langle 2,0\rangle, \pm\langle 0,2\rangle\}) = \{\pm\langle 2,0\rangle, \pm\langle 0,2\rangle\}.
\end{aligned}
\tag{4.3}
$$

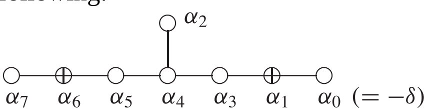
Next, we investigate the possibilities of $\tau|_{\mathfrak{t}}$ for the case for $\mathfrak{g}$ is a simple Lie algebra of exceptional type and $\sigma = \tau_{(1/2)(K_a+K_b)}$, $m_a = m_b = 2$.

From [4], a pair $(\mathfrak{g}, \mathfrak{h})$ is one of the following:

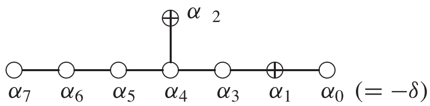
$$
\begin{aligned}
&(\mathfrak{e}_6, \mathfrak{so}(6) \oplus \mathfrak{so}(4) \oplus \mathbb{R}), \quad (\mathfrak{e}_7, \mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R}), \quad (\mathfrak{e}_7, \mathfrak{so}(8) \oplus \mathfrak{so}(4) \oplus \mathbb{R}),\\
&(\mathfrak{e}_8, \mathfrak{so}(12) \oplus \mathfrak{su}(2) \oplus \mathbb{R}), \quad (\mathfrak{f}_4, \mathfrak{sp}(2) \oplus \mathfrak{sp}(1) \oplus \mathbb{R}).
\end{aligned}
\tag{4.4}
$$

Suppose that $\mathfrak{g}$ is of type $\mathfrak{e}_7$. According to [4, Theorem 3.2] we may assume the Dynkin diagram of $\mathfrak{h}$ coincides with the extended Dynkin diagram of $\mathfrak{e}_7$ except $\oplus$ in the following:

(i)



(ii)



CASE (i) In this case, $\sigma = \tau_{(1/2)(K_1+K_6)}$. Put $\gamma := \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$. Then we have $\langle 0, 2\rangle = \{\gamma\}$ and $\langle 2, 0\rangle = \emptyset$. Therefore by (4.2) and (4.3) we have

$$\tau(\gamma) = \gamma \quad \text{or} \quad -\gamma. \tag{4.5}$$

From the Dynkin diagram (i) the involution τ satisfies one of following relations:
(i1) $\tau(\alpha_i) = \alpha_i$ $(i = 0, 2, 3, 4, 5, 7)$,
(i2) $\tau(\alpha_0) = \alpha_7,\ \tau(\alpha_i) = \alpha_i$ $(i = 2, 3, 4, 5)$,
(i3) $\tau(\alpha_3) = \alpha_5,\ \tau(\alpha_i) = \alpha_i$ $(i = 0, 2, 4, 7)$,
(i4) $\tau(\alpha_0) = \alpha_7,\ \tau(\alpha_3) = \alpha_5,\ \tau(\alpha_i) = \alpha_i$ $(i = 2, 4)$,
(i5) $\tau(\alpha_2) = \alpha_3,\ \tau(\alpha_i) = \alpha_i$ $(i = 0, 4, 5, 7)$,
(i6) $\tau(\alpha_0) = \alpha_7,\ \tau(\alpha_2) = \alpha_3,\ \tau(\alpha_i) = \alpha_i$ $(i = 4, 5)$,
(i7) $\tau(\alpha_2) = \alpha_5,\ \tau(\alpha_i) = \alpha_i$ $(i = 0, 3, 4, 7)$,
(i8) $\tau(\alpha_0) = \alpha_7,\ \tau(\alpha_2) = \alpha_5,\ \tau(\alpha_i) = \alpha_i$ $(i = 3, 4)$.
For each case (i1)–(i8) we calculate $\tau(\alpha_1)$ and $\tau(\alpha_6)$.

(i1): It follows from (4.5) that $\tau|_{\mathfrak{t}} = \mathrm{Id}_{\mathfrak{t}}$ or $\tau(\alpha_i) = \alpha_i$, $i = 0, 2, 3, 4, 5, 7$, $\tau(\alpha_1) = \alpha_1 + \gamma$ and $\tau(\alpha_6) = -\gamma + \alpha_6$. This is of Type I in Table I below.

(i2): Since $\tau(\alpha_0) = \alpha_7$ and $\tau(\alpha_i) = \alpha_i$, $i = 2, 3, 4, 5$, we have $\tau(\alpha_1) + \tau(\alpha_6) = \alpha_1 + \alpha_6$. Then from (4.5) we have $\tau(\alpha_6) \notin \Delta$. Thus this case dose not occur.

(i3): As in (i2) we have $\tau(\alpha_6) \notin \Delta$, and this case dose not occur.

(i4): Similarly as in the case (i2) if $\tau(\gamma) = \gamma$, then we have $\tau(\alpha_6) = \alpha_1 + \gamma$ and $\tau(\alpha_1) = -\gamma + \alpha_6$. This is of Type VI in Table I below. If $\tau(\gamma) = -\gamma$, then we have $\tau(\alpha_6) = \alpha_1$ and $\tau(\alpha_1) = \alpha_6$. This is of Type IV in Table I below.

(i5): We have $2\tau(\alpha_1) + 2\tau(\alpha_6) = 2\alpha_1 - \alpha_2 + \alpha_3 + 2\alpha_6$ and this case dose not occur.

(i6)–(i8): Similarly as in the case (i5) these cases do not occur.

CASE (ii) In this case, $\sigma = \tau_{(1/2)(K_1+K_2)}$, and an involution τ satisfies one of the following:
(ii1) $\tau(\alpha_i) = \alpha_i$ $(i = 0, 3, 4, 5, 6, 7)$,
(ii2) $\tau(\alpha_0) = \alpha_0,\ \tau(\alpha_3) = \alpha_7,\ \tau(\alpha_4) = \alpha_6,\ \tau(\alpha_5) = \alpha_5$.
Note that

$$\langle 0, 1\rangle = \left\{\begin{matrix} \alpha_2,\ \alpha_2+\alpha_4,\ \alpha_2+\alpha_3+\alpha_4,\ \alpha_2+\alpha_4+\alpha_5, \\ \alpha_2+\alpha_3+\alpha_4+\alpha_5,\ \alpha_2+\alpha_4+\alpha_5+\alpha_6,\ \alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \\ \alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_7,\ \alpha_2+\alpha_3+2\alpha_4+\alpha_5, \\ \alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7,\ \alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6, \\ \alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7,\ \alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6, \\ \alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7,\ \alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7 \end{matrix}\right\}. \tag{4.6}$$

(ii1): In the case (C1) since $\alpha_2 + \alpha_i \notin \Delta$ and $\alpha_2 - \alpha_j \notin \Delta$, $1 \le i \le 7$, $i \ne 4$, $1 \le j \le 7$ we have $\tau(\alpha_2 + \alpha_i) \notin \Delta$ $(1 \le i \le 7,\ i \ne 4)$ and $\tau(\alpha_1 - \alpha_i) \notin \Delta$ $(1 \le i \le 7)$. Thus we have

$$\tau(\alpha_2) + \alpha_i \notin \Delta, \quad \tau(\alpha_2) - \alpha_j \notin \Delta, \quad i = 3, 5, 6, 7,\ j = 3, 4, 5, 6, 7. \tag{4.7}$$

From (4.2) and the fact that $\{\pm\langle 2, 1\rangle\} = \emptyset$, we have $\tau(\alpha_2) \in \{\pm\langle 0, 1\rangle\}$ and therefore it follows from (4.6) and (4.7) that $\tau(\alpha_2) = \alpha_2$ and $\tau(\alpha_1) = \alpha_1$. Thus we have $\tau|_{\mathfrak{t}} = \mathrm{Id}_{\mathfrak{t}}$. In the case (C2) since $\alpha_1 + \alpha_i \notin \Delta$, $\alpha_1 - \alpha_j \notin \Delta$, $1 \le i \le 7$, $i \ne 3$, $1 \le j \le 7$, we have $\tau(\alpha_1 + \alpha_i) \notin \Delta$ $(1 \le i \le 7,\ i \ne 3)$ and $\tau(\alpha_1 - \alpha_j) \notin \Delta$ $(1 \le j \le 7)$. Therefore we have

$$\tau(\alpha_1) + \alpha_i \notin \Delta, \quad \tau(\alpha_1) - \alpha_j \notin \Delta, \quad i = 4, 5, 6, 7,\ j = 3, 4, 5, 6, 7. \tag{4.8}$$

Since $\{\pm\langle 2, 1\rangle\} = \emptyset$, it follows from (4.3) that $\tau(\alpha_1) \in \{\pm\langle 0, 1\rangle\}$. Hence by (4.6) and (4.8) this case dose not occur.

(ii2): We note that

$$\tau(\alpha_1) + \tau(\alpha_2) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - \alpha_6 - \alpha_7. \tag{4.9}$$

Similarly as in the case (ii1) of (C1) we have

$$\tau(\alpha_2) + \alpha_i \notin \Delta, \quad \tau(\alpha_2) - \alpha_j \notin \Delta, \quad i = 3, 5, 6, 7,\ j = 3, 4, 5, 6, 7 \tag{4.10}$$

and $\tau(\alpha_2) \in \{\pm\langle 0, 1\rangle\}$ which, together with (4.6) implies that $\tau(\alpha_2) = -\alpha_2 - \alpha_3 - 2\alpha_4 - 2\alpha_5 - 2\alpha_6 - \alpha_7$. From this and (4.9) we have $\tau(\alpha_1) = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$. This is of Type VII in Table I below.

By a similar argument as above, we obtain the possibilities of $\tau|_{\mathfrak{t}} \ne \mathrm{Id}$ which are listed in Table I below.

REMARK 4.1. According to [4, Corollary 3.5], $K_2 + K_3 \sim K_2 + K_5 \sim K_3 + K_5$ for the case where $\mathfrak{g} = \mathfrak{e}_6$.

REMARK 4.2. For Type VIII in Table I, it is easy to see $\tau(K_1 + K_8) = -K_1 + 3K_8 \equiv -(K_1 + K_8) \pmod{4\Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})}$. Hence we have $\tau \circ \sigma = \sigma^{-1} \circ \tau$. Similarly, for Type I and Type IV we have $\tau \circ \sigma = \sigma \circ \tau$ and for the other types, we have $\tau \circ \sigma = \sigma^{-1} \circ \tau$.

By the above argument together with Remark 4.1 we obtain following proposition.

Proposition 4.3. *Suppose that $(G/H, \sigma)$ is compact 4-symmetric space of exceptional type and $\sigma = \tau_{(1/2)(K_a + K_b)}$, $m_a = m_b = 2$. Let τ be an involution of $\mathfrak{g}$ such that $\tau \circ \sigma = \sigma \circ \tau$ and $\tau|_{\mathfrak{t}} \ne \mathrm{Id}_{\mathfrak{t}}$. Then τ is conjugate within one of two involutions of* Type I *and* Type IV *in* Table I.

Table I. The possibilities of $\tau|_{\mathfrak{t}}$ such that $\tau|_{\mathfrak{t}} \neq \mathrm{Id}$ ($\sigma = \tau_{(1/2)h}$).

Type	$\mathfrak{g}$	h	$\tau\|_{\mathfrak{t}}$
I	$\mathfrak{e}_6$	$K_3 + K_5$	$\alpha_i \mapsto \alpha_i$ $(i = 0, 2, 4)$, $\alpha_1 \mapsto \alpha_6$, $\alpha_3 \mapsto \alpha_5$
II	$\mathfrak{e}_6$	$K_3 + K_5$	$\alpha_i \mapsto \alpha_i$ $(i = 2, 6)$, $\alpha_0 \mapsto \alpha_4$ $\alpha_3 \mapsto \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$ $\alpha_5 \mapsto \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5$
III	$\mathfrak{e}_6$	$K_3 + K_5$	$\alpha_2 \mapsto \alpha_2$, $\alpha_0 \mapsto \alpha_4$ $\alpha_3 \mapsto \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5$ $\alpha_5 \mapsto \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$
IV	$\mathfrak{e}_7$	$K_1 + K_6$	$\alpha_i \mapsto \alpha_i$ $(i = 2, 4)$, $\alpha_1 \mapsto \alpha_6$, $\alpha_3 \mapsto \alpha_5$, $\alpha_7 \mapsto \alpha_0$
V	$\mathfrak{e}_7$	$K_1 + K_6$	$\alpha_i \mapsto \alpha_i$ $(i = 0, 2, 3, 4, 5, 7)$ $\alpha_1 \mapsto \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$ $\alpha_6 \mapsto -\alpha_2 - \alpha_3 - 2\alpha_4 - 2\alpha_5 - \alpha_6 - \alpha_7$
VI	$\mathfrak{e}_7$	$K_1 + K_6$	$\alpha_i \mapsto \alpha_i$ $(i = 2, 4)$, $\alpha_3 \mapsto \alpha_5$, $\alpha_7 \mapsto \alpha_0$ $\alpha_1 \mapsto -\alpha_2 - \alpha_3 - 2\alpha_4 - 2\alpha_5 - \alpha_6 - \alpha_7$ $\alpha_6 \mapsto \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$
VII	$\mathfrak{e}_7$	$K_1 + K_2$	$\alpha_0 \mapsto \alpha_0$, $\alpha_3 \mapsto \alpha_7$, $\alpha_4 \mapsto \alpha_6$, $\alpha_5 \mapsto \alpha_5$ $\alpha_1 \mapsto \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ $\alpha_2 \mapsto -\alpha_2 - \alpha_3 - 2\alpha_4 - 2\alpha_5 - 2\alpha_6 - \alpha_7$
VIII	$\mathfrak{e}_8$	$K_1 + K_8$	$\alpha_i \mapsto \alpha_i$ $(i = 0, 2, 3, 4, 5, 6, 7)$ $\alpha_8 \mapsto 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8$ $\alpha_1 \mapsto -\alpha_1 - 2\alpha_2 - 3\alpha_3 - 4\alpha_4 - 3\alpha_5 - 2\alpha_6 - \alpha_7$
IX	$\mathfrak{f}_4$	$K_1 + K_4$	$\alpha_i \mapsto \alpha_i$ $(i = 0, 2, 3)$, $\alpha_1 \mapsto \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4$ $\alpha_4 \mapsto -\alpha_2 - 2\alpha_3 - \alpha_4$

Finally, in the previous paper [6], we proved the following lemma.

Lemma 4.4 ([6]). *Let* $\mathfrak{t}_+$ *be the* $(+1)$*-eigenspace of* $\tau|_{\mathfrak{t}}$*. Then*

$$\begin{aligned}\dim \mathfrak{g}^\tau = \dim \mathfrak{t}_+ + \#\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) + 2\#\{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}});\ \tau(E_\alpha) = E_\alpha\} \\ - \#\{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}});\ \tau(\alpha) = \alpha\}.\end{aligned}$$ □

5. The involution for the case where $\tau|_{\mathfrak{t}} \neq \mathrm{Id}$

As before, let $(G/H, \sigma)$ be a compact 4-symmetric space such that G is simple and σ is inner. Suppose that $\dim \mathfrak{z} = 1$ and $\tau_{(1/2)(K_a+K_b)}$, $m_a = m_b = 2$. In this section we investigate involutions τ of $\mathfrak{g}$ such that $\tau \circ \sigma = \sigma \circ \tau$ and $\tau|_{\mathfrak{t}} \neq \mathrm{Id}$.

From Proposition 4.3 we investigate the Type I and Type IV in Table I.

Type I: In this case, $\mathfrak{g} = \mathfrak{e}_6$ and $\mathfrak{h} = \mathfrak{so}(6) \oplus \mathfrak{so}(4) \oplus \mathbb{R}$. Let $\mathfrak{t}_\pm$ be the (± 1)-eigenspaces of $\tau|_\mathfrak{t}$, respectively. Since $\alpha_i(\tau(K_j)) = \tau(\alpha_i)(K_j)$, we have

$$\sqrt{-1}\mathfrak{t}_+ = \mathrm{span}\{K_1 + K_6, K_2, K_3 + K_5, K_4\}, \quad \sqrt{-1}\mathfrak{t}_- = \mathrm{span}\{K_1 - K_6, K_3 - K_5\}.$$

For each $h_- \in \sqrt{-1}\mathfrak{t}_-$, we have $\tau \circ \tau_{h_-} \circ \tau = \tau_{\tau(h_-)} = \tau_{-h_-}$. Thus we have

$$(\tau_{h_-})^{-1} \circ \tau \circ \tau_{h_-} = \tau \circ \tau_{2h_-}. \tag{5.1}$$

From (5.1) for each $h \in \mathfrak{t}_-$, an involution $\tau \circ \tau_h$ is conjugate within $\mathrm{Int}(\mathfrak{h})$ to τ. Then using $h_- := t(K_1 - K_6) \in \sqrt{-1}\mathfrak{t}_-$, we may assume $\tau(E_{\alpha_1}) = E_{\alpha_6}$. Indeed, if $\tau(E_{\alpha_1}) = aE_{\alpha_6}$ ($a \in \mathbb{C}$, $|a| = 1$), then it follows from (2.4) and (5.1) that

$$(\tau_{h_-})^{-1} \circ \tau \circ \tau_{h_-}(E_{\alpha_1}) = ae^{2t\pi\sqrt{-1}}E_{\alpha_6}.$$

Taking t so that $a = e^{-2t\pi\sqrt{-1}}$, we may assume $\tau(E_{\alpha_1}) = E_{\alpha_6}$. Similarly, using $h_- = t(K_3 - K_5)$, we may assume $\tau(E_{\alpha_3}) = E_{\alpha_5}$. Therefore we have

$$\tau(E_{\alpha_1}) = E_{\alpha_6}, \quad \tau(E_{\alpha_2}) = \pm E_{\alpha_2}, \quad \tau(E_{\alpha_3}) = E_{\alpha_5}, \quad \tau(E_{\alpha_4}) = \pm E_{\alpha_4}. \tag{5.2}$$

On the other hand, it is known that there exists an involutive automorphism ϕ of outer type satisfying

$$\phi(E_{\alpha_1}) = E_{\alpha_6}, \quad \phi(E_{\alpha_2}) = E_{\alpha_2}, \quad \phi(E_{\alpha_3}) = E_{\alpha_5}, \quad \phi(E_{\alpha_4}) = E_{\alpha_4}. \tag{5.3}$$

By (5.3) it is obvious that $\phi|_\mathfrak{t} = \tau|_\mathfrak{t}$. Then, by Proposition 5.3 of Chapter IX of [3], there exists $h \in \sqrt{-1}\mathfrak{t}$ such that $\tau = \phi \circ \tau_h$. Put

$$\begin{aligned} &h := h_+ + h_-, \quad h_+ \in \sqrt{-1}\mathfrak{t}_+,\ h_- \in \sqrt{-1}\mathfrak{t}_-, \\ &h_+ := k_1(K_1 + K_6) + k_2K_2 + k_3(K_3 + K_5) + k_4K_4, \\ &h_- := k_5(K_1 - K_6) + k_6(K_3 - K_5), \end{aligned}$$

where $k_1, \dots, k_6 \in \mathbb{R}$. Since $\tau^2 = \mathrm{Id}$ and $\phi(h) = h_+ - h_-$, we have $\tau_{2h_+} = \mathrm{Id}$ and hence $2h_+ \equiv 0 \pmod{2\Pi(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})}$. Therefore we have $k_1, k_2, k_3, k_4 \in \mathbb{Z}$. Considering (5.2) and (5.3) together with (2.4), we have

$$\alpha_1(h) \equiv \alpha_3(h) \equiv \alpha_5(h) \equiv \alpha_6(h) \equiv 0 \pmod 2,$$

and therefore

$$h \equiv k_2K_2 + k_4K_4 \pmod{2\Pi(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})}.$$

Hence τ is conjugate within $\mathrm{Aut}_\mathfrak{h}(\mathfrak{g})$ to one of the following involutions:

$$\phi, \quad \phi \circ \tau_{K_2}, \quad \phi \circ \tau_{K_4}, \quad \phi \circ \tau_{K_2 + K_4}.$$

We shall check conjugations between the above involutions. Since $\mathfrak{su}_{\alpha_2}(2)$, $\mathfrak{su}_{\alpha_0+\alpha_2}(2) \subset \mathfrak{g}^{\phi}$, $t_{\alpha_2}(K_2) = -K_2 + K_4$ and $t_{\alpha_0+\alpha_2}(K_2) = 2K_2 - K_4$, we have $\phi \circ t_{\alpha_2} = t_{\alpha_2} \circ \phi$, $\phi \circ t_{\alpha_0+\alpha_2} = t_{\alpha_0+\alpha_2} \circ \phi$ and

$$\phi \circ \tau_{K_2+K_4} = \phi \circ \tau_{t_{\alpha_2}(K_2)} = \phi \circ t_{\alpha_2} \circ \tau_{K_2} \circ t_{\alpha_2}^{-1} = t_{\alpha_2} \circ (\phi \circ \tau_{K_2}) \circ t_{\alpha_2}^{-1},$$
$$\phi \circ \tau_{K_4} = \phi \circ \tau_{t_{\alpha_0+\alpha_2}(K_2)} = \phi \circ t_{\alpha_0+\alpha_2} \circ \tau_{K_2} \circ t_{\alpha_0+\alpha_2}^{-1} = t_{\alpha_0+\alpha_2} \circ (\phi \circ \tau_{K_2}) \circ t_{\alpha_o+\alpha_2}^{-1}.$$

Note that $t_{\alpha_2}, t_{\alpha_0+\alpha_2} \in \mathrm{Int}(\mathfrak{h})$ since $\mathfrak{su}_{\alpha_2}(2), \mathfrak{su}_{\alpha_0+\alpha_2}(2) \subset \mathfrak{h}$. Hence we have $\phi \circ \tau_{K_2+K_4} \approx \phi \circ \tau_{K_2} \approx \phi \circ \tau_{K_4}$, where we write $\tau_H \approx \tau_{H'}$ if τ_H is conjugate to $\tau_{H'}$ within $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$. Consequently we obtain $\tau \approx \phi$ or $\tau \approx \phi \circ \tau_{K_2}$.

Put $\nu := \tau|_{\mathfrak{t}}$. It is easy to see that the set Δ_ν^+ of positive roots α satisfying $\nu(\alpha) = \alpha$ coincides with

$$\Delta_\nu^+ = \left\{\begin{array}{l} \alpha_2,\ \alpha_4,\ \alpha_2+\alpha_4,\ \alpha_3+\alpha_4+\alpha_5, \\ \alpha_2+\alpha_3+\alpha_4+\alpha_5,\ \alpha_2+\alpha_3+2\alpha_4+\alpha_5, \\ \alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6,\ \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \\ \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6,\ \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6, \\ \alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6,\ -\alpha_0 \end{array}\right\}.$$

Hence it follows from Lemma 4.4 and (5.3) that $\dim \mathfrak{g}^{\phi} = 52$. By using the classification of symmetric spaces, we have $\mathfrak{g}^{\phi} \cong F_4$.

The subset $\{\alpha, \beta\}$ of $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ such that $\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, $\phi(\alpha) = \beta$, $\alpha \neq \pm\beta$ and $\alpha(K_3 + K_5) \equiv 0 \pmod 4$ is only $\{\alpha_1, \alpha_6\}$. Furthermore $\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ such that $\phi(\alpha) = \alpha$ and $\alpha(K_3 + K_5) \equiv 0 \pmod 4$ are $-\alpha_0$, α_2, α_4, $\alpha_2 + \alpha_4$, $-(\alpha_0 + \alpha_2)$ and $-(\alpha_0 + \alpha_2 + \alpha_4)$. Since $\dim \mathfrak{t}_+ = 4$, we have

$$\dim(\mathfrak{h} \cap \mathfrak{g}^{\phi}) = 4 + ((1 + 6) \times 2) = 18.$$

Since $(\mathfrak{h}, \mathfrak{h} \cap \mathfrak{g}^{\phi})$ is symmetric pair we can see $\mathfrak{h} \cap \mathfrak{g}^{\phi} \cong D_3 \oplus A_1$.

Similarly as above we obtain $\mathfrak{g}^{\phi\circ\tau_{K_2}} = C_4$ and $\mathfrak{h} \cap \mathfrak{g}^{\phi\circ\tau_{K_2}} = B_1 \oplus B_1 \oplus A_1 \oplus \mathbb{R}$.

Type IV: In this case, $\mathfrak{g} = \mathfrak{e}_7$ and $\mathfrak{h} = \mathfrak{so}(8) \oplus \mathfrak{so}(4) \oplus \mathbb{R}$. According to Section 5 in [6], there exists an involution ψ on $\mathfrak{g}$ such that

$$\psi(E_{\alpha_1}) = E_{\alpha_6}, \quad \psi(E_{\alpha_2}) = E_{\alpha_2}, \quad \psi(E_{\alpha_3}) = E_{\alpha_5}, \quad \psi(E_{\alpha_4}) = E_{\alpha_4},$$
$$\psi(E_{\alpha_5}) = E_{\alpha_3}, \quad \psi(E_{\alpha_6}) = E_{\alpha_1}, \quad \psi(E_{\alpha_7}) = E_{\alpha_0}, \quad \psi(E_{\alpha_0}) = E_{\alpha_7}.$$

Let $\mathfrak{t}_\pm$ be the (± 1)-eigenspaces of ψ. Then we have

$$\mathfrak{t}_+ = \mathrm{span}\{K_1 + K_6 - 2K_7,\ K_2 - K_7,\ K_3 + K_5 - 3K_7,\ K_4 - 2K_7\},$$
$$\mathfrak{t}_- = \mathrm{span}\{-K_1 + K_6,\ -K_3 + K_5,\ K_7\}.$$

By an argument similar to the case of Type I, we can prove that τ is conjugate within $\mathrm{Int}(\mathfrak{h})$ to one of the following involutions:

$$\psi, \quad \psi \circ \tau_{K_2}, \quad \psi \circ \tau_{K_4}, \quad \psi \circ \tau_{K_2} \circ \tau_{K_4}.$$

We shall check conjugations between the above involutions. Since $t_{\alpha_2} \circ \psi = \psi \circ t_{\alpha_2}$, we have

$$t_{\alpha_2} \circ \psi \circ \tau_{K_2} \circ t_{\alpha_2}^{-1} = \psi \circ \tau_{K_2+K_4},$$

which implies that $\psi \circ \tau_{K_2} \approx \psi \circ \tau_{K_2} \circ \tau_{K_4}$. Moreover since $t_{\alpha_4} \circ \psi = \psi \circ t_{\alpha_4}$ and above equation, we have

$$\begin{aligned} t_{\alpha_4} \circ (t_{\alpha_2} \circ \psi \circ \tau_{K_2} \circ t_{\alpha_2}^{-1}) \circ t_{\alpha_4}^{-1} &= t_{\alpha_4} \circ (\psi \circ \tau_{K_2+K_4}) \circ t_{\alpha_4}^{-1} \\ &= \psi \circ t_{\alpha_4} \circ \tau_{K_2+K_4} \circ t_{\alpha_4}^{-1} \\ &= \psi \circ \tau_{t_{\alpha_4}(K_2+K_4)}. \end{aligned} \tag{5.4}$$

On the other hand, it is easy to see that

$$t_{\alpha_4}(K_2 + K_4) = 2K_2 + K_3 - K_4 + K_5 \equiv K_4 + (K_3 + K_5) \pmod{2\Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})}, \tag{5.5}$$

$$\psi(K_3 - K_5) = -(K_3 - K_5). \tag{5.6}$$

Therefore from (5.4), (5.5) and (5.6) we have

$$\begin{aligned} t_{\alpha_4} \circ (t_{\alpha_2} \circ \psi \circ \tau_{K_2} \circ t_{\alpha_2}^{-1}) \circ t_{\alpha_4}^{-1} &= \psi \circ \tau_{K_4} \circ \tau_{(K_3-K_5)} \\ &= \tau_{-(1/2)(K_3-K_5)} \circ (\psi \circ \tau_{K_4}) \circ \tau_{(1/2)(K_3-K_5)}. \end{aligned}$$

Hence using t_{α_2}, t_{α_4} and $\tau_{(1/2)(K_3-K_5)} \in \mathrm{Int}(\mathfrak{h})$, we have $\psi \circ \tau_{K_2} \approx \psi \circ \tau_{K_4}$.

By an argument similar to the case of Type I, we obtain $\mathfrak{g}^{\psi} = E_6 \oplus \mathbb{R}$, $\mathfrak{h} \cap \mathfrak{g}^{\psi} = B_3 \oplus A_1$, $\mathfrak{g}^{\psi \circ \tau_{K_2}} = A_7$ and $\mathfrak{h} \cap \mathfrak{g}^{\psi \circ \tau_{K_2}} = B_2 \oplus B_1 \oplus A_1$.

6. The involution for the case where $\tau|_{\mathfrak{t}} = \mathrm{Id}$

First, we suppose that $\mathfrak{g}$ is of type $\mathfrak{e}_7$ and $\sigma = \tau_{(1/2)(K_1+K_2)}$. Then by Dynkin diagram (ii) in Section 4, we have

$$\mathfrak{h} \cong A_1 \oplus A_5 \oplus \mathbb{R}\sqrt{-1}(K_1 - K_2).$$

Moreover a maximal abelian subalgebra $\mathfrak{t}$ is decomposed into $\mathfrak{t} = (A_1 \cap \mathfrak{t}) \oplus (A_5 \cap \mathfrak{t}) \oplus \mathbb{R}\sqrt{-1}(K_1 - K_2)$. Hence we can write

$$\tau = \tau_{T_1} \circ \tau_{T_2} \circ \tau_{m(K_1-K_2)}, \quad \sqrt{-1}T_1 \in A_1 \cap \mathfrak{t},\ \sqrt{-1}T_2 \in A_5 \cap \mathfrak{t}.$$

We define $v_0 \in \sqrt{-1}(A_1 \cap \mathfrak{t})$ and $v_i \in \sqrt{-1}(A_5 \cap \mathfrak{t})$, $i \in \Lambda := \{3,4,5,6,7\}$ by $\alpha_i(v_j) = \delta_{ij}$, $i, j \in \{0\} \cup \Lambda$. Since $(\tau_{T_1}|_{A_1})^2 = \mathrm{Id}_{A_1}$ and $(\tau_{T_2}|_{A_5})^2 = \mathrm{Id}_{A_5}$, it follows from Lemma 2.2 and Remark 2.3 that there exist $\mu_1 \in \mathrm{Int}(A_1)$ and $\mu_2 \in \mathrm{Int}(A_5)$ such that

$$\mu_1(T_1) \equiv \begin{cases} 0 & \mod 2\Pi_{A_1}, \\ v_0 & \mod 2\Pi_{A_1}, \end{cases} \quad \mu_2(T_2) \equiv \begin{cases} 0 & \mod 2\Pi_{A_5}, \\ v_i & \mod 2\Pi_{A_5} \quad (i \in \Lambda), \end{cases} \tag{6.1}$$

where Π_{A_l} denotes the fundamental root system of Type A_l. Therefore considering Lemma 2.4 we may assume

$$T_1 = \begin{cases} 2m_0v_0, \\ v_0 + 2m_0v_0, \end{cases} \qquad T_2 = \begin{cases} 2m_3v_3 + \cdots + 2m_7v_7, \\ v_i + 2m_3v_3 + \cdots + 2m_7v_7, \end{cases} \tag{6.2}$$

where $i = 5, 6, 7$ and $m_0, m_3, \ldots, m_7 \in \mathbb{Z}$. Consequently τ is conjugate within $\mathrm{Int}(\mathfrak{h})$ to one of the following automorphisms:

$$\begin{cases} \tau_{2m_0v_0+2m_3v_3+\cdots+2m_7v_7+m(K_1-K_2)}, \\ \tau_{v_i+2m_0v_0+2m_3v_3+\cdots+2m_7v_7+m(K_1-K_2)}, \\ \tau_{v_0+v_j+2m_0v_0+2m_3v_3+\cdots+2m_7v_7+m(K_1-K_2)}, \end{cases} \tag{6.3}$$

where $i = 0, 5, 6, 7$, $j = 5, 6, 7$ and $m_0, m_3, m_4, \ldots, m_7 \in \mathbb{Z}$, $m \in \mathbb{R}$.

Now we shall write v_i $(i = 0, 3, 4, \ldots, 7)$ by the linear combination of $K_1, \ldots, K_7$. Put $v_i = \sum_{j=1}^7 a_j^i K_j$, $a_j^i \in \mathbb{R}$. First we compute v_0. Since $A_1 \cap \mathfrak{t} = \mathbb{R}\sqrt{-1}H_{\alpha_0}$ and

$$A_1 \cap \mathfrak{t} = \{\sqrt{-1}H \in \mathfrak{t};\ \alpha_j(H) = 0,\ j = 2, \ldots, 7\},$$

we have $a_2^0 = \cdots = a_7^0 = 0$. Hence we have $v_0 = a_1^0 K_1$. Because $\alpha_0(v_0) = 1$, we have

$$v_0 = -\frac{1}{2}K_1. \tag{6.4}$$

Next, we compute v_i $(i = 3, 4, \ldots, 7)$. Then computing simultaneous equations $\alpha_i(v_j) = \delta_{ij}$, $i, j \in \Lambda$, we have

$$\begin{aligned} &v_3 = a_1^3K_1 - \frac{1}{2}(2a_1^3 + 3)K_2 + K_3, \quad v_4 = a_1^4K_1 - \frac{1}{2}(2a_1^4 + 4)K_2 + K_4, \\ &v_5 = a_1^5K_1 - \frac{1}{2}(2a_1^5 + 3)K_2 + K_5, \quad v_6 = a_1^6K_1 - \frac{1}{2}(2a_1^6 + 2)K_2 + K_6, \\ &v_7 = a_1^7K_1 - \frac{1}{2}(2a_1^7 + 1)K_2 + K_7. \end{aligned} \tag{6.5}$$

Now, in order to determine a_1^i $(i \in \Lambda)$, we write $K_1 - K_2$ by the linear combination of $H_{\alpha_1}, \dots, H_{\alpha_7}$. Put $K_i = \sum_{j=1}^{7} c_j^i H_{\alpha_j}$, $c_j^i \in \mathbb{R}$, $i = 1, 2$. Then since $\delta_{ij} = \alpha_i(K_j)$, we have

$$\delta_{i1} = \alpha_i(K_1) = \sum_{j=1}^{7} c_j^1 \alpha_i(H_{\alpha_j}), \quad i = 1, 2, \dots, 7 \tag{6.6}$$

and therefore

$$\left(c_1^1 - \frac{c_3^1}{2}\right)\alpha_1(H_{\alpha_1}) = 1, \quad c_2^1 - \frac{c_4^1}{2} = 0, \quad -\frac{c_1^1}{2} + c_3^1 - \frac{c_4^1}{2} = 0, \quad -\frac{c_2^1}{2} - \frac{c_3^1}{2} + c_4^1 - \frac{c_5^1}{2} = 0,$$
$$-\frac{c_4^1}{2} + c_5^1 - \frac{c_6^1}{2} = 0, \quad -\frac{c_5^1}{2} + c_6^1 - \frac{c_7^1}{2} = 0, \quad -\frac{c_6^1}{2} + c_7^1 = 0.$$

Indeed, considering the α_1-series containing α_j, we have $\alpha_j(H_{\alpha_1}) = 0$ for $j \neq 1, 3$ and $2\alpha_3(H_{\alpha_1})/\alpha_1(H_{\alpha_1}) = -1$. Thus if $i = 1$ in (6.6), then we have

$$\begin{aligned} 1 = \alpha_1(K_1) &= \sum_{j=1}^{7} c_j^1 \alpha_1(H_{\alpha_j}) = c_1^1 \alpha_1(H_{\alpha_1}) + c_3^1 \alpha_1(H_{\alpha_3}) \\ &= c_1^1 \alpha_1(H_{\alpha_1}) + c_3^1\left(-\frac{1}{2}\alpha_1(H_{\alpha_1})\right) = \left(c_1^1 - \frac{c_3^1}{2}\right)\alpha_1(H_{\alpha_1}). \end{aligned}$$

We can get the other equations by a similar computation as above.

Computing these simultaneous equations we obtain

$$K_1 = \frac{2}{\alpha_1(H_{\alpha_1})}(2H_{\alpha_1} + 2H_{\alpha_2} + 3H_{\alpha_3} + 4H_{\alpha_4} + 3H_{\alpha_5} + 2H_{\alpha_6} + H_{\alpha_7}).$$

By an argument similar as above, we obtain

$$K_2 = \frac{1}{\alpha_2(H_{\alpha_2})}(4H_{\alpha_1} + 7H_{\alpha_2} + 8H_{\alpha_3} + 12H_{\alpha_4} + 9H_{\alpha_5} + 6H_{\alpha_6} + 3H_{\alpha_7}).$$

From the Dynkin diagram (ii) in Section 4, we can put $k := \alpha_1(K_1) = \alpha_2(K_2)$ and therefore

$$K_1 - K_2 = \frac{1}{k}(-3H_{\alpha_2} - 2H_{\alpha_3} - 4H_{\alpha_4} - 3H_{\alpha_5} - 2H_{\alpha_6} - H_{\alpha_7}). \tag{6.7}$$

Since $v_i \perp (K_1 - K_2)$ $(i \in \Lambda)$, it follows from (6.5) that $a_1^3 = -5/6$, $a_1^4 = -2/3$, $a_1^5 = -1/2$, $a_1^6 = -1/3$ and $a_1^7 = -1/6$, which implies that

$$\begin{aligned} &v_3 = -\frac{5}{6}K_1 - \frac{2}{3}K_2 + K_3, \quad v_4 = -\frac{2}{3}K_1 - \frac{4}{3}K_2 + K_4, \\ &v_5 = -\frac{1}{2}K_1 - K_2 + K_5, \quad v_6 = -\frac{1}{3}K_1 - \frac{2}{3}K_2 + K_6, \quad v_7 = -\frac{1}{6}K_1 - \frac{1}{3}K_2 + K_7. \end{aligned} \tag{6.8}$$

It follows from (6.3), (6.4) and (6.8) that τ is conjugate within $\mathrm{Int}(\mathfrak{h})$ to one of the following:

$$\tau_{aK_1+bK_2}, \quad \tau_{v_i+aK_1+bK_2}, \quad \tau_{v_0+v_j+aK_1+bK_2}, \quad i=0,5,6,7,\ j=5,6,7 \tag{6.9}$$

where $a=-m_0-5m_3/3-4m_4/3-m_5-2m_6/3-m_7/3+m$ and $b=-4m_3/3-8m_4/3-2m_5-4m_6/3-2m_7/3-m$. Moreover, since $\tau^2=\mathrm{Id}$, it follows from (6.4), (6.8) and (6.9) that τ is conjugate within $\mathrm{Int}(\mathfrak{h})$ to some τ_h where h is one of the following:

$$K_i, \quad K_1+K_2, \quad K_1+K_j, \quad K_2+K_j, \quad K_1+K_2+K_j$$

where $i=1,2,5,6,7$ and $j=5,6,7$.

If $h=K_5$, then from Lemma 4.4 we have $\dim \mathfrak{g}^{\tau_{K_5}}=63$ and $\mathfrak{g}^{\tau_{K_5}}\cong A_7$. Furthermore we have

$$\mathfrak{h}\cap\mathfrak{g}^{\tau_{K_5}}=\mathfrak{t}\oplus\sum_{\substack{\alpha\in\Delta^+(\mathfrak{g}_{\mathbb{C}},\mathfrak{t}_{\mathbb{C}})\\ \alpha(K_1)+\alpha(K_2)\equiv 0 \bmod 4\\ \alpha(K_5)\equiv 0 \bmod 2}}(\mathbb{R}A_\alpha+\mathbb{R}B_\alpha)\subset\mathfrak{h}(\cong\mathfrak{su}(2)\oplus\mathfrak{su}(6)\oplus\mathbb{R}).$$

In this case, $A_5^{\tau_{K_5}}\cong A_2\oplus A_2\oplus\mathbb{R}$ and $A_1^{\tau_{K_5}}\cong\mathbb{R}$, and hence

$$\mathfrak{h}\cap\mathfrak{g}^{\tau_{K_5}}\cong A_2\oplus A_2\oplus\mathbb{R}^3.$$

Similarly as above, we can get $(\mathfrak{g}^\tau,\mathfrak{h}\cap\mathfrak{g}^\tau)$ for each $\tau=\tau_h$.

Note that $\mathfrak{su}_{\alpha_0}(2)\subset\mathfrak{h}$ and $t_{\alpha_0}\in\mathrm{Int}(\mathfrak{h})$. It is easy to check that t_{α_0} maps $K_1+K_5\mapsto -4K_1+K_5$, $K_1+K_7\mapsto -2K_1+K_7$, $K_1+K_2+K_5\mapsto -6K_1+K_2+K_5$ and $K_1+K_2+K_7\mapsto -4K_1+K_2+K_7$. Therefore we have $\tau_{K_1+K_5}\approx\tau_{K_5}$, $\tau_{K_1+K_7}\approx\tau_{K_7}$, $\tau_{K_1+K_2+K_5}\approx\tau_{K_2+K_5}$ and $\tau_{K_1+K_2+K_7}\approx\tau_{K_2+K_7}$.

REMARK 6.1. From Lemma 2.4, we can see that $\tau_{v_3}|_{A_5}$ is conjugate within $\mathrm{Int}(A_5)$ ($\subset\mathrm{Int}(\mathfrak{h})$) to $\tau_{v_7}|_{A_5}$. Therefore by the above argument, τ_{K_3} is conjugate within $\mathrm{Int}(\mathfrak{h})$ to τ_{K_7}, $\tau_{K_1+K_7}$, $\tau_{K_2+K_7}$ or $\tau_{K_1+K_2+K_7}$. However, $\mathfrak{g}^{\tau_{K_3}}\ncong\mathfrak{g}^{\tau_{K_7}}$ and $\mathfrak{g}^{\tau_{K_3}}\ncong\mathfrak{g}^{\tau_{K_1+K_7}}$, and hence $\tau_{K_3}\approx\tau_{K_2+K_7}\approx\tau_{K_1+K_2+K_7}$.

For the case where $\mathfrak{g}=\mathfrak{e}_7$ and $\sigma=\tau_{(1/2)(K_1+K_6)}$, we can check that τ is conjugate within $\mathrm{Int}(\mathfrak{h})$ to some τ_h where h is one of the following:

$$\begin{aligned}&K_i, \quad K_1+K_6, \quad K_1+K_7, \quad K_6+K_7, \quad K_1+K_j, \quad K_6+K_j, \quad K_7+K_j,\\ &K_1+K_6+K_j, \quad K_1+K_7+K_j, \quad K_6+K_7+K_j, \quad K_1+K_6+K_7+K_j,\\ &i=1,2,3,4,5,6,\ j=2,3,4.\end{aligned}$$

In the Table II, we show some conjugations within $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ between $\tau=\tau_h$ for the above h.

Table II. Conjugations within $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ of $\mathfrak{e}_7$ with $\sigma = \tau_{(1/2)(K_1+K_6)}$.

reflection(s)	conjugation
t_{α_0}	$\tau_{K_1+K_3} \approx \tau_{K_3}$ $\tau_{K_1+K_2+K_7} \approx \tau_{K_2+K_7}$ $\tau_{K_1+K_4+K_7} \approx \tau_{K_4+K_7}$
t_{α_7}	$\tau_{K_6+K_7} \approx \tau_{K_7}$ $\tau_{K_2+K_6+K_7} \approx \tau_{K_2+K_7}$ $\tau_{K_3+K_6+K_7} \approx \tau_{K_3+K_7}$ $\tau_{K_4+K_6+K_7} \approx \tau_{K_4+K_7}$ $\tau_{K_1+K_3+K_6+K_7} \approx \tau_{K_1+K_3+K_7}$
$t_{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7}$	$\tau_{K_2+K_6} \approx \tau_{K_1+K_2}$ $\tau_{K_1+K_3+K_6} \approx \tau_{K_3}$ $\tau_{K_1+K_6+K_7} \approx \tau_{K_7}$ $\tau_{K_1+K_4+K_6+K_7} \approx \tau_{K_4+K_7}$
$t_{\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6} \circ t_{\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6}$	$\tau_{K_6} \approx \tau_{K_1}$
$t_{\alpha_3} \circ t_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5}$	$\tau_{K_1+K_6} \approx \tau_{K_3}$
$t_{\alpha_2} \circ t_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5}$	$\tau_{K_1+K_2+K_6+K_7} \approx \tau_{K_2+K_7}$
$t_{\alpha_4} \circ t_{\alpha_2+\alpha_3+\alpha_4}$	$\tau_{K_1+K_4} \approx \tau_{K_4}$
$t_{\alpha_4} \circ t_{\alpha_3+\alpha_4+\alpha_5}$	$\tau_{K_1+K_4+K_6} \approx \tau_{K_4}$
$t_{\alpha_4} \circ t_{\alpha_2+\alpha_4+\alpha_5}$	$\tau_{K_4+K_6} \approx \tau_{K_4}$

REMARK 6.2. For example, it is easy to see that

$$\begin{aligned} &t_{\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6} \circ t_{\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6}(K_1 + K_2) \\ &= t_{\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6}(K_1 - 2K_2 + K_6) = -K_1 - K_2, \end{aligned}$$

which means that

$$t_{\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6} \circ t_{\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6} \in \mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g}).$$

For the case where $\mathfrak{g} = \mathfrak{e}_6$ and $\sigma = \tau_{(1/2)(K_3+K_5)}$, we can check that τ is conjugate within $\mathrm{Int}(\mathfrak{h})$ to some τ_h where h is one of the following:

$$\begin{aligned} &K_i, \quad K_3 + K_5, \quad K_1 + K_6, \quad K_3 + K_j, \quad K_j + K_5, \quad K_2 + K_k, \\ &K_1 + K_3 + K_6, \quad K_1 + K_5 + K_6, \quad K_1 + K_2 + K_6, \quad K_3 + K_5 + K_j, \\ &K_2 + K_3 + K_k, \quad K_2 + K_5 + K_k, \quad K_2 + K_3 + K_5 + K_k, \\ &K_1 + K_2 + K_3 + K_6, \quad K_1 + K_3 + K_5 + K_6, \quad K_1 + K_2 + K_5 + K_6, \\ &K_1 + K_2 + K_3 + K_5 + K_6, \quad i = 1, 2, 3, 5, 6, \ j = 1, 2, 6, \ k = 1, 6. \end{aligned}$$

In the Table III, we show some conjugations within $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ between $\tau = \tau_h$ for the above h.

Table III. Conjugations within $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ of $\mathfrak{e}_6$ with $\sigma = \tau_{(1/2)(K_3+K_5)}$.

reflection(s)	conjugation
t_{α_1}	$\tau_{K_1+K_3} \approx \tau_{K_1}$
	$\tau_{K_1+K_3+K_5} \approx \tau_{K_1+K_5}$
	$\tau_{K_1+K_2+K_3} \approx \tau_{K_1+K_2}$
	$\tau_{K_1+K_3+K_6} \approx \tau_{K_1+K_6}$
	$\tau_{K_1+K_2+K_3+K_5} \approx \tau_{K_1+K_2+K_5}$
	$\tau_{K_1+K_2+K_3+K_6} \approx \tau_{K_1+K_2+K_6}$
t_{α_6}	$\tau_{K_5+K_6} \approx \tau_{K_6}$
	$\tau_{K_3+K_5+K_6} \approx \tau_{K_3+K_6}$
	$\tau_{K_2+K_5+K_6} \approx \tau_{K_2+K_6}$
	$\tau_{K_1+K_5+K_6} \approx \tau_{K_1+K_6}$
	$\tau_{K_2+K_3+K_5+K_6} \approx \tau_{K_2+K_3+K_6}$
	$\tau_{K_1+K_2+K_5+K_6} \approx \tau_{K_1+K_2+K_6}$
t_{α_0}	$\tau_{K_1+K_2+K_5} \approx \tau_{K_1+K_5}$
$t_{\alpha_1} \circ t_{\alpha_6}$	$\tau_{K_1+K_3+K_5+K_6} \approx \tau_{K_1+K_6}$
	$\tau_{K_1+K_2+K_3+K_5+K_6} \approx \tau_{K_1+K_2+K_6}$
$t_{\alpha_2} \circ t_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6}$	$\tau_{K_2+K_3+K_5} \approx \tau_{K_2}$

Finally we consider an involution $\phi \in \mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ (see (5.3)). Then it is easy to see that

$$\phi(K_1) = K_6, \quad \phi(K_2) = K_2, \quad \phi(K_3) = K_5,$$
$$\phi(K_4) = K_4, \quad \phi(K_5) = K_3, \quad \phi(K_6) = K_1,$$

and therefore ϕ gives the following conjugations:

$$\tau_{K_1} \approx \tau_{K_6}, \quad \tau_{K_3} \approx \tau_{K_5}, \quad \tau_{K_1+K_2} \approx \tau_{K_2+K_6}, \quad \tau_{K_1+K_4} \approx \tau_{K_4+K_6},$$
$$\tau_{K_1+K_5} \approx \tau_{K_3+K_6}, \quad \tau_{K_2+K_5} \approx \tau_{K_2+K_3}, \quad \tau_{K_1+K_2+K_5} \approx \tau_{K_2+K_3+K_6},$$
$$\tau_{K_1+K_4+K_5} \approx \tau_{K_3+K_4+K_6}.$$

For the case where $\mathfrak{g} = \mathfrak{e}_8$ and $\sigma = \tau_{(1/2)(K_1+K_8)}$, we can check that τ is conjugate within $\mathrm{Int}(\mathfrak{h})$ to some τ_h where h is one of the following:

$$K_i, \quad K_1 + K_j, \quad K_j + K_8, \quad K_1 + K_8, \quad K_1 + K_j + K_8,$$
$$i = 1, 2, 3, 4, 5, 8, \ j = 2, 3, 4, 5.$$

In the Table IV, we show some conjugations within $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ between $\tau = \tau_h$ for the above h.

Table IV. Conjugations within $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ of $\mathfrak{e}_8$ with $\sigma = \tau_{(1/2)(K_1+K_8)}$.

reflection(s)	conjugation
t_{α_0}	$\tau_{K_2+K_8} \approx \tau_{K_2}$, $\tau_{K_5+K_8} \approx \tau_{K_5}$ $\tau_{K_1+K_2+K_8} \approx \tau_{K_1+K_2}$
$t_{\alpha_4} \circ t_{\alpha_2+\alpha_3+2\alpha_4}$	$\tau_{K_1+K_4} \approx \tau_{K_4}$ $\tau_{K_1+K_4+K_8} \approx \tau_{K_4+K_8}$
$t_{\alpha_5} \circ t_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5}$	$\tau_{K_1+K_5} \approx \tau_{K_5}$
$t_{\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7+\alpha_8}$	$\tau_{K_3+K_8} \approx \tau_{K_1+K_3}$ $\tau_{K_1+K_3+K_8} \approx \tau_{K_3}$ $\tau_{K_1+K_5+K_8} \approx \tau_{K_5}$

Table V. Conjugations within $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ of $\mathfrak{f}_4$ with $\sigma = \tau_{(1/2)(K_1+K_4)}$.

reflection(s)	conjugation
t_{α_3}	$\tau_{K_3+K_4} \approx \tau_{K_3}$ $\tau_{K_1+K_3+K_4} \approx \tau_{K_1+K_3}$
t_{α_0}	$\tau_{K_1+K_2} \approx \tau_{K_2}$
$t_{\alpha_2+\alpha_3}$	$\tau_{K_2+K_4} \approx \tau_{K_2}$
$t_{\alpha_2+2\alpha_3+2\alpha_4}$	$\tau_{K_1+K_2+K_4} \approx \tau_{K_2}$

For the case where $\mathfrak{g} = \mathfrak{f}_4$ and $\sigma = \tau_{(1/2)(K_1+K_4)}$, we can check that τ is conjugate within $\mathrm{Int}(\mathfrak{h})$ to some τ_h where h is one of the following:

$$K_i, \quad K_1+K_2, \quad K_1+K_3, \quad K_1+K_4, \quad K_2+K_4,$$
$$K_3+K_4, \quad K_1+K_2+K_4, \quad K_1+K_3+K_4, \quad i = 1, 2, 3, 4.$$

In the Table V, we show some conjugations within $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ between $\tau = \tau_h$ for the above h.

Consequently we have the following proposition.

Proposition 6.3. *Suppose that* $\dim \mathfrak{z} = 1$ *and* $\sigma = \tau_{(1/2)(K_a+K_b)}$ *for some* $\alpha_a, \alpha_b \in \Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ *with* $m_a = m_b = 2$. *Let* τ *be an involution of* $\mathfrak{g}$ *such that* $\tau \circ \sigma = \sigma \circ \tau$. *Then* τ *is conjugate within* $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ *to one of involutions listed in* Table VI.

7. Remarks on conjugations

To complete the classification of involutions τ, we prove the following lemma.

Lemma 7.1. *Suppose that* $\sigma = \tau_{(1/2)(K_a+K_b)}$, $m_a = m_b = 2$. *Then* τ_{K_a} *is not conjugate within* $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ *to* $\tau_{K_a+K_b}$.

Table VI. Involutions of exceptional Lie algebra such that $\dim \mathfrak{z} = 1$, $\sigma = \tau_{(1/2)(K_a+K_b)}$ and $\tau \circ \sigma = \sigma \circ \tau$.

$(\mathfrak{g}, \mathfrak{h})$	$h\ (\tau = \tau_h)$	$\mathfrak{k}$	$\mathfrak{h} \cap \mathfrak{k}$
$(\mathfrak{e}_6, \mathfrak{so}(6) \oplus \mathfrak{so}(4) \oplus \mathbb{R})$	K_1	$D_5 \oplus \mathbb{R}$	$D_3 \oplus A_1$
	K_2	$A_5 \oplus A_1$	$D_2 \oplus D_2 \oplus \mathbb{R}^2$
	K_3	$A_5 \oplus A_1$	$D_3 \oplus D_2 \oplus \mathbb{R}$
	$K_1 + K_2$	$D_5 \oplus \mathbb{R}$	$B_2 \oplus B_1 \oplus \mathbb{R}$
	$K_1 + K_5$	$A_5 \oplus A_1$	$A_2 \oplus A_1 \oplus \mathbb{R}$
	$K_1 + K_6$	$D_5 \oplus \mathbb{R}$	$D_3 \oplus \mathbb{R}^3$
	$K_2 + K_3$	$D_5 \oplus \mathbb{R}$	$D_2 \oplus D_2 \oplus \mathbb{R}^2$
	$K_3 + K_5$	$D_5 \oplus \mathbb{R}$	$D_3 \oplus D_2 \oplus \mathbb{R}$
	$K_1 + K_2 + K_5$	$A_5 \oplus A_1$	$B_2 \oplus B_1 \oplus \mathbb{R}$
$(\mathfrak{e}_7, \mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R})$	K_1	$D_6 \oplus A_1$	$A_5 \oplus A_1 \oplus \mathbb{R}$
	K_2	A_7	$A_5 \oplus A_1 \oplus \mathbb{R}$
	K_5	A_7	$A_2 \oplus A_2 \oplus \mathbb{R}^3$
	K_6	$D_6 \oplus A_1$	$A_3 \oplus A_1 \oplus A_1 \oplus \mathbb{R}^2$
	K_7	$E_6 \oplus \mathbb{R}$	$A_4 \oplus \mathbb{R}^4$
	$K_1 + K_2$	$E_6 \oplus \mathbb{R}$	$A_5 \oplus A_1 \oplus \mathbb{R}$
	$K_1 + K_6$	$D_6 \oplus A_1$	$A_3 \oplus A_1 \oplus A_1 \oplus \mathbb{R}^2$
	$K_2 + K_5$	$D_6 \oplus A_1$	$A_2 \oplus A_2 \oplus \mathbb{R}^3$
	$K_2 + K_6$	$E_6 \oplus \mathbb{R}$	$A_3 \oplus A_1 \oplus A_1 \oplus \mathbb{R}^2$
	$K_2 + K_7$	$D_6 \oplus A_1$	$A_4 \oplus \mathbb{R}^4$
	$K_1 + K_2 + K_6$	A_7	$A_3 \oplus A_1 \oplus A_1 \oplus \mathbb{R}^2$
$(\mathfrak{e}_7, \mathfrak{so}(8) \oplus \mathfrak{so}(4) \oplus \mathbb{R})$	K_1	$D_6 \oplus A_1$	$D_4 \oplus D_2 \oplus \mathbb{R}$
	K_2	A_7	$A_3 \oplus D_2 \oplus \mathbb{R}^2$
	K_3	$D_6 \oplus A_1$	$A_3 \oplus A_1 \oplus \mathbb{R}^3$
	K_4	$D_6 \oplus A_1$	$D_2 \oplus D_2 \oplus D_2 \oplus \mathbb{R}$
	K_7	$E_6 \oplus \mathbb{R}$	$D_4 \oplus A_1 \oplus \mathbb{R}$
	$K_1 + K_2$	$E_6 \oplus \mathbb{R}$	$A_3 \oplus D_2 \oplus \mathbb{R}$
	$K_1 + K_6$	$D_6 \oplus A_1$	$D_4 \oplus D_2 \oplus \mathbb{R}$
	$K_2 + K_7$	$D_6 \oplus A_1$	$A_3 \oplus \mathbb{R}^3$
	$K_3 + K_7$	A_7	$A_3 \oplus A_1 \oplus \mathbb{R}$
	$K_4 + K_7$	A_7	$D_2 \oplus D_2 \oplus \mathbb{R}^3$
	$K_1 + K_3 + K_7$	$E_6 \oplus \mathbb{R}$	$A_3 \oplus A_1 \oplus \mathbb{R}$
$(\mathfrak{e}_8, \mathfrak{so}(12) \oplus \mathfrak{su}(2) \oplus \mathbb{R})$	K_1	D_8	$D_6 \oplus A_1 \oplus \mathbb{R}$
	K_2	D_8	$A_5 \oplus \mathbb{R}^3$
	K_3	$E_7 \oplus A_1$	$A_5 \oplus A_1 \oplus \mathbb{R}^2$
	K_4	$E_7 \oplus A_1$	$D_4 \oplus D_2 \oplus A_1 \oplus \mathbb{R}$
	K_5	D_8	$D_3 \oplus D_3 \oplus \mathbb{R}^2$
	K_8	$E_7 \oplus A_1$	$D_6 \oplus A_1 \oplus \mathbb{R}$
	$K_1 + K_2$	$E_7 \oplus A_1$	$A_5 \oplus \mathbb{R}^3$
	$K_1 + K_3$	D_8	$A_5 \oplus A_1 \oplus \mathbb{R}^2$
	$K_1 + K_8$	$E_7 \oplus A_1$	$D_6 \oplus A_1 \oplus \mathbb{R}$
	$K_4 + K_8$	D_8	$D_4 \oplus D_2 \oplus A_1 \oplus \mathbb{R}$
$(\mathfrak{f}_4, \mathfrak{sp}(2) \oplus \mathfrak{sp}(1) \oplus \mathbb{R})$	K_1	$C_3 \oplus A_1$	$C_2 \oplus C_1 \oplus \mathbb{R}$
	K_2	$C_3 \oplus A_1$	$A_1 \oplus \mathbb{R}^3$
	K_3	B_4	$C_1 \oplus C_1 \oplus C_1 \oplus \mathbb{R}$
	K_4	$C_3 \oplus A_1$	$C_2 \oplus C_1 \oplus \mathbb{R}$
	$K_1 + K_3$	$C_3 \oplus A_1$	$C_1 \oplus C_1 \oplus C_1 \oplus \mathbb{R}$
	$K_1 + K_4$	$C_3 \oplus A_1$	$C_2 \oplus C_1 \oplus \mathbb{R}$

$(\mathfrak{g}, \mathfrak{h})$	τ	$\mathfrak{k}$	$\mathfrak{h} \cap \mathfrak{k}$
$(\mathfrak{e}_6, \mathfrak{so}(6) \oplus \mathfrak{so}(4) \oplus \mathbb{R})$	ϕ	F_4	$D_3 \oplus A_1$
	$\phi \circ \tau_{K_2}$	C_4	$B_1 \oplus B_1 \oplus A_1 \oplus \mathbb{R}$
$(\mathfrak{e}_7, \mathfrak{so}(8) \oplus \mathfrak{so}(4) \oplus \mathbb{R})$	ψ	$E_6 \oplus \mathbb{R}$	$B_3 \oplus A_1$
	$\psi \circ \tau_{K_2}$	A_7	$B_2 \oplus B_1 \oplus A_1$

$\phi : E_{\alpha_1} \mapsto E_{\alpha_6},\ E_{\alpha_2} \mapsto E_{\alpha_2},\ E_{\alpha_3} \mapsto E_{\alpha_5},\ E_{\alpha_4} \mapsto E_{\alpha_4}$

$\psi : E_{\alpha_1} \mapsto E_{\alpha_6},\ E_{\alpha_2} \mapsto E_{\alpha_2},\ E_{\alpha_3} \mapsto E_{\alpha_5},\ E_{\alpha_4} \mapsto E_{\alpha_4},\ E_{\alpha_7} \mapsto E_{\alpha_0}$

Proof of Lemma 7.1. Assume that there exists $\mu \in \mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ satisfying

$$\mu \circ \tau_{K_a} \circ \mu^{-1} = \tau_{K_a+K_b}. \tag{7.1}$$

Since $\mu(\mathfrak{h}) = \mathfrak{h}$, we have $\sigma \circ \mu(X) = \mu(X)$ for any $X \in \mathfrak{h}$. In particular $(\mu^{-1} \circ \sigma \circ \mu)|_{\mathfrak{t}} =$ Id which implies that there exists $T \in \mathfrak{t}_{\mathbb{C}}$ such that $\mu^{-1} \circ \sigma \circ \mu = \tau_T$.

Since $(\tau_T)^4 = (\mu^{-1} \circ \sigma \circ \mu)^4 = \mathrm{Id}$, we have $2T \equiv 0 \pmod{2\Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})}$ and

$$\mu^{-1} \circ \sigma \circ \mu = \tau_{(1/2)T}. \tag{7.2}$$

Considering $\alpha \in \langle 0, 0\rangle$ together with (2.4), we have

$$\alpha_i(K) \equiv 0 \pmod 4 \quad \text{for} \quad i \neq a, b. \tag{7.3}$$

We take a root $\alpha = 2\alpha_a + 2\alpha_b + \sum_{i\neq a,b} m_i \alpha_i \in \langle 2, 2\rangle$ ($m_i \in \mathbb{Z}$). Since $\mu^{-1} \circ \sigma \circ \mu(E_\alpha) = E_\alpha$, it follows from (2.4) and (7.2) that $\alpha(K) \in 4\mathbb{Z}$, which together with (7.3), implies $2\alpha_a(K) + 2\alpha_b(K) \equiv 0 \pmod 4$ and hence $\alpha_a(K) + \alpha_b(K) \equiv 0 \pmod 2$. Thus we can express $K \in \mathfrak{t}$ as follows

$$K = \alpha_a(K)K_a + \alpha_b(K)K_b + \sum_{i\neq a,b} \alpha_i(K)K_i, \quad \alpha_a(K) + \alpha_b(K) \in 2\mathbb{Z}. \tag{7.4}$$

From (7.1) and (7.2) we have

$$\tau_{K_a} = \tau_{\mu^{-1}(K_a+K_b)} = (\tau_{(1/2)\mu^{-1}(K_a+K_b)})^2 = (\mu^{-1} \circ \sigma \circ \mu)^2 = \tau_K. \tag{7.5}$$

Let $\alpha = \alpha_a + \cdots + \alpha_b \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Then we have $\alpha(K_a) = 1$, $\alpha(K) = \alpha_a(K) + \alpha_b(K) \in 2\mathbb{Z}$ and hence

$$\begin{aligned} \tau_{K_a}(E_\alpha) &= e^{\pi\sqrt{-1}} E_\alpha = -E_\alpha, \\ \tau_K(E_\alpha) &= e^{\pi\sqrt{-1}(\alpha_a(K)+\alpha_b(K))} E_\alpha = E_\alpha, \end{aligned}$$

which contradicts (7.5). □

Using Lemma 7.1 we have the following:

For the case where $\mathfrak{g} = \mathfrak{e}_7$ and $\sigma = \tau_{(1/2)(K_1+K_6)}$ we have $\tau_{K_1} \not\approx \tau_{K_1+K_6}$.

For the case where $\mathfrak{g} = \mathfrak{e}_8$ and $\sigma = \tau_{(1/2)(K_1+K_8)}$ we have $\tau_{K_8} \not\approx \tau_{K_1+K_8}$.

For the case where $\mathfrak{g} = \mathfrak{f}_4$ and $\sigma = \tau_{(1/2)(K_1+K_4)}$ we have $\tau_{K_1} \not\approx \tau_{K_1+K_4}$.

Next, for the case where $\mathfrak{g} = \mathfrak{e}_6$ and $\sigma = \tau_{(1/2)(K_3+K_5)}$ we consider $\phi \in \mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ (see (5.3)). Put $\mu_1 := \phi \circ t_{\alpha_0} \circ \psi \in \mathrm{Aut}(\mathfrak{g})$. Then we have

$$\begin{gathered} \mu_1(\alpha_1) = -\alpha_6, \quad \mu_1(\alpha_2) = -\alpha_1 - \alpha_2 - 2\alpha_3 - 3\alpha_4 - 2\alpha_5 - \alpha_6, \\ \mu_1(\alpha_3) = \alpha_5, \quad \mu_1(\alpha_4) = \alpha_4, \quad \mu_1(\alpha_5) = \alpha_3, \quad \mu_1(\alpha_6) = \alpha_1, \end{gathered}$$

which implies that $\mu_1^{-1}(K_3+K_5) = -4K_2+K_3+K_5$ and $\mu_1^{-1}(K_1+K_2) = -2K_2+K_6$. Thus μ_1^{-1} is in $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ and gives a conjugation between $\tau_{K_1+K_2}$ and τ_{K_6}.

8. Classifications

From Proposition 6.3 together with the results in Section 7, we obtain the following theorem which gives the complete classification of involutions τ.

Theorem 8.1. *Let $(G/H, \sigma)$ be a 4-symmetric space such that G is a compact simple Lie group of exceptional type. Suppose that $\dim \mathfrak{z} = 1$ and $\sigma = \tau_{(1/2)(K_a+K_b)}$ for some $\alpha_a, \alpha_b \in \Pi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ with $m_a = m_b = 2$. Then the following Table VII gives the complete list of the conjugation classes within $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ of involutions τ satisfying $\tau \circ \sigma = \sigma \circ \tau$.*

REMARK 8.2. Let $(G/H, \sigma)$ be a compact 4-symmetric space of inner and exceptional type. In the previous paper [6], we classified involutions of G preserving H for the case where the dimension of the center of H is zero, or one and H is a centralizer of a toral subgroup of G. This, together with Theorem 8.1, means that the involution τ of G preserving H satisfied $\tau \circ \sigma = \sigma \circ \tau$ is conjugate within $\mathrm{Aut}_{\mathfrak{h}}(\mathfrak{g})$ to one of involutions listed in Table VII and Tables 7 and 8 in [6].

Table VII. Involutions of exceptional Lie algebra such that $\dim \mathfrak{z} = 1$, $\sigma = \tau_{(1/2)(K_a+K_b)}$ and $\tau \circ \sigma = \sigma \circ \tau$.

$(\mathfrak{g}, \mathfrak{h})$	h $(\tau = \tau_h)$	$\mathfrak{k}$	$\mathfrak{h} \cap \mathfrak{k}$
$(\mathfrak{e}_6, \mathfrak{so}(6) \oplus \mathfrak{so}(4) \oplus \mathbb{R})$	K_1	$\mathfrak{so}(10) \oplus \mathbb{R}$	$\mathfrak{so}(6) \oplus \mathfrak{su}(2)$
	K_2	$\mathfrak{su}(6) \oplus \mathfrak{su}(2)$	$(\mathfrak{so}(4) + \mathfrak{so}(2)) \oplus \mathfrak{so}(4) \oplus \mathbb{R}$
	K_3	$\mathfrak{su}(6) \oplus \mathfrak{su}(2)$	$\mathfrak{so}(6) \oplus \mathfrak{so}(4) \oplus \mathbb{R}$
	$K_1 + K_2$	$\mathfrak{so}(10) \oplus \mathbb{R}$	$\mathfrak{so}(5) \oplus \mathfrak{so}(3) \oplus \mathbb{R}$
	$K_1 + K_6$	$\mathfrak{so}(10) \oplus \mathbb{R}$	$\mathfrak{so}(6) \oplus (\mathfrak{so}(2) + \mathfrak{so}(2)) \oplus \mathbb{R}$
	$K_2 + K_3$	$\mathfrak{so}(10) \oplus \mathbb{R}$	$(\mathfrak{so}(4) + \mathfrak{so}(2)) \oplus \mathfrak{so}(4) \oplus \mathbb{R}$
	$K_3 + K_5$	$\mathfrak{so}(10) \oplus \mathbb{R}$	$\mathfrak{so}(6) \oplus \mathfrak{so}(4) \oplus \mathbb{R}$
	$K_1 + K_2 + K_5$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2)$	$\mathfrak{so}(5) \oplus \mathfrak{so}(3) \oplus \mathbb{R}$
$(\mathfrak{e}_7, \mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R})$	K_1	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	K_2	$\mathfrak{su}(8)$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	K_5	$\mathfrak{su}(8)$	$\mathfrak{s}(\mathfrak{u}(3) + \mathfrak{u}(3)) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
	K_6	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(2)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	K_7	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{s}(\mathfrak{u}(5) + \mathfrak{u}(1)) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
	$K_1 + K_2$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	$K_1 + K_6$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(2)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	$K_2 + K_5$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(3) + \mathfrak{u}(3)) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
	$K_2 + K_6$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(2)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	$K_2 + K_7$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{s}(\mathfrak{u}(5) + \mathfrak{u}(1)) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
	$K_1 + K_2 + K_6$	$\mathfrak{su}(8)$	$\mathfrak{s}(\mathfrak{u}(4) + \mathfrak{u}(2)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_7, \mathfrak{so}(8) \oplus \mathfrak{so}(4) \oplus \mathbb{R})$	K_1	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{so}(8) \oplus \mathfrak{so}(4) \oplus \mathbb{R}$
	K_2	$\mathfrak{su}(8)$	$\mathfrak{su}(4) \oplus \mathfrak{so}(4) \oplus \mathbb{R}^2$
	K_3	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^3$
	K_4	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$(\mathfrak{so}(4) + \mathfrak{so}(4)) \oplus \mathfrak{so}(4) \oplus \mathbb{R}$
	K_7	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{so}(8) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^2$
	$K_1 + K_2$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{su}(4) \oplus \mathfrak{so}(4) \oplus \mathbb{R}^2$
	$K_1 + K_6$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{so}(8) \oplus \mathfrak{so}(4) \oplus \mathbb{R}$
	$K_2 + K_7$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(4) \oplus (\mathfrak{so}(2) + \mathfrak{so}(2)) \oplus \mathbb{R}^2$
	$K_3 + K_7$	$\mathfrak{su}(8)$	$\mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^3$
	$K_4 + K_7$	$\mathfrak{su}(8)$	$(\mathfrak{so}(4) + \mathfrak{so}(4)) \oplus (\mathfrak{so}(2) + \mathfrak{so}(2)) \oplus \mathbb{R}$
	$K_1 + K_3 + K_7$	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^3$
$(\mathfrak{e}_8, \mathfrak{so}(12) \oplus \mathfrak{su}(2) \oplus \mathbb{R})$	K_1	$\mathfrak{so}(16)$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	K_2	$\mathfrak{so}(16)$	$\mathfrak{su}(6) \oplus \mathfrak{so}(2) \oplus \mathbb{R}^2$
	K_3	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^2$
	K_4	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$(\mathfrak{so}(8) + \mathfrak{so}(4)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	K_5	$\mathfrak{so}(16)$	$(\mathfrak{so}(6) + \mathfrak{so}(6)) \oplus \mathfrak{so}(2) \oplus \mathbb{R}$
	K_8	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	$K_1 + K_2$	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{su}(6) \oplus \mathfrak{so}(2) \oplus \mathbb{R}^2$
	$K_1 + K_3$	$\mathfrak{so}(16)$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^2$
	$K_1 + K_8$	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
	$K_4 + K_8$	$\mathfrak{so}(16)$	$(\mathfrak{so}(8) + \mathfrak{so}(4)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{f}_4, \mathfrak{sp}(2) \oplus \mathfrak{sp}(1) \oplus \mathbb{R})$	K_1	$\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$	$\mathfrak{sp}(2) \oplus \mathfrak{sp}(1) \oplus \mathbb{R}$
	K_2	$\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(2) \oplus \mathbb{R}^3$
	K_3	$\mathfrak{so}(9)$	$(\mathfrak{sp}(1) + \mathfrak{sp}(1)) \oplus \mathfrak{sp}(1) \oplus \mathbb{R}$
	K_4	$\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$	$\mathfrak{sp}(2) \oplus \mathfrak{sp}(1) \oplus \mathbb{R}$
	$K_1 + K_3$	$\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$	$(\mathfrak{sp}(1) + \mathfrak{sp}(1)) \oplus \mathfrak{sp}(1) \oplus \mathbb{R}$
	$K_1 + K_4$	$\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$	$\mathfrak{sp}(2) \oplus \mathfrak{sp}(1) \oplus \mathbb{R}$

$(\mathfrak{g}, \mathfrak{h})$	τ	$\mathfrak{k}$	$\mathfrak{h} \cap \mathfrak{k}$
$(\mathfrak{e}_6, \mathfrak{so}(6) \oplus \mathfrak{so}(4) \oplus \mathbb{R})$	ϕ	$\mathfrak{f}_4$	$\mathfrak{so}(6) \oplus \mathfrak{su}(2)$
	$\phi \circ \tau_{K_2}$	$\mathfrak{sp}(4)$	$(\mathfrak{so}(3) + \mathfrak{so}(3)) \oplus \mathfrak{su}(2) \oplus \mathbb{R}$
$(\mathfrak{e}_7, \mathfrak{so}(8) \oplus \mathfrak{so}(4) \oplus \mathbb{R})$	ψ	$\mathfrak{e}_6 \oplus \mathbb{R}$	$\mathfrak{so}(7) \oplus \mathfrak{su}(2)$
	$\psi \circ \tau_{K_2}$	$\mathfrak{su}(8)$	$(\mathfrak{so}(5) + \mathfrak{so}(3)) \oplus \mathfrak{su}(2)$

ϕ and ψ are the same involution as in Table VI.

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