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KNOTTING AND LINKING IN THE PETERSEN FAMILY

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Abstract

This paper extends the work of Nikkuni [4] finding an explicit relationship for the graph $K_{3,3,1}$ between knotting and linking, which relates the sum of the squares of linking numbers of links in the embedding and the second coefficient of the Conway polynomial of certain cycles in the embedding. Then we use this and other similar relationships to better understand the relationship between knotting and linking in the Petersen family. The Petersen family is the set of minor minimal intrinsically linked graphs. We prove that if such a spatial graph is complexly algebraically linked then it is knotted.

1. Introduction

Throughout this paper we will work with finite simple graphs, in the piecewise linear category. A *spatial graph* is an embedding of a graph G in $\mathbb{R}^3$, denoted $f(G)$ or simply f. This paper focuses on the interaction between knotting and linking in spatial graphs. A knot or link is said to be in a spatial graph if the knot or link appears as a subgraph. An embedding f of a graph G is *linked* if there is a nontrivial link in $f(G)$. An embedding f of a graph G is *algebraically linked* if there is a link with nonzero linking number in $f(G)$. We will say an embedding of a graph is *complexly algebraically linked* (*CA linked*) if the embedding contains a 2-component link L with $|\mathrm{lk}(L)| \geq 2$ or (at least) two 2-component links L_1 and L_2 with $\mathrm{lk}(L_i) \neq 0$, where lk denotes the linking number in $\mathbb{R}^3$. An embedding f of a graph G is *knotted* if there is a nontrivial knot in $f(G)$. An embedding that is not knotted is called *knotless*.

A graph G is *intrinsically knotted* if every embedding of G into $\mathbb{R}^3$ contains a non-trivial knot. A graph G is *intrinsically linked* if every embedding of G into $\mathbb{R}^3$ contains a non-split link. The combined work of Conway and Gordon [1], Sachs [7], and Robertson, Seymour, and Thomas [5] fully characterize intrinsically linked graphs. They showed that the Petersen family is the complete set of minor minimal intrinsically linked graphs, i.e. every intrinsically linked graph contains a graph in the Petersen family as a minor. The Petersen family is a set of seven graphs shown in Fig. 1. We will denote this set of graphs by $\mathcal{PF}$. They are related by ∇Y-moves (shown in Fig. 11), as indicated by the arrows in Fig. 1. The set of intrinsically knotted graphs has not been fully characterized. However it is known that every intrinsically knotted graph is intrinsically linked.

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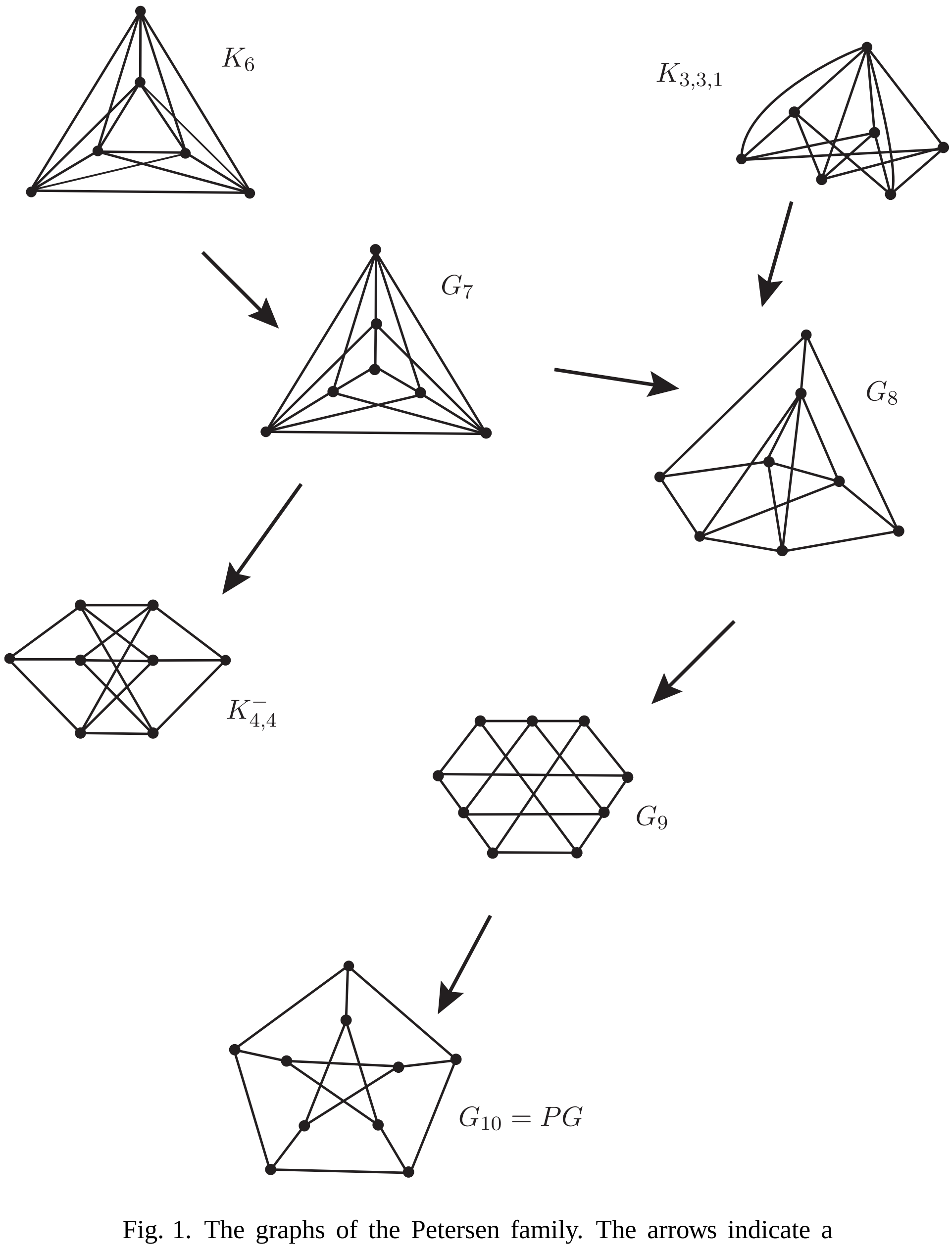


Fig. 1. The graphs of the Petersen family. The arrows indicate a ∇Y-move.

This is a consequence of the work characterizing intrinsically linked graphs [5]. The converse does not hold; there are many graphs that are intrinsically linked graphs that have knotless embeddings. In particular, none of the graphs of $\mathcal{PF}$ are intrinsically knotted.

The study of intrinsically knotted graphs and intrinsically linked graphs began with the work of Sachs, and Conway and Gordon. When Conway and Gordon proved that K_6 is intrinsically linked and that K_7 is intrinsically knotted, they did this by proving for every embedding f of K_6 the following holds, the sum of the linking numbers over all 2-component links in $f(K_6)$ is odd, and for every embedding f of K_7 the following holds, the sum of the second coefficient of the Conway polynomial over all knots in $f(K_7)$ is odd. In the recent work of Ryo Nikkuni, he generalizes these results to get formulae for both K_6 and K_7 explicitly relating knotting and linking in their embeddings, see [4]. A *cycle* γ in a graph G is a subgraph of G homeomorphic to a circle. In particular, γ is called a *k-cycle* if it consists of exactly k edges and a Hamiltonian cycle if it contains all vertices of G. In keeping with the notation of Nikkuni [4], let $\Gamma(G)$ denote the set of all cycles in G, let $\Gamma_H(G)$ be the set of all Hamiltonian cycles in G, let $\Gamma_m(G)$ be the set of all m-cycles in G, let $\Gamma^{(2)}_{s,t}(G)$ be the set of all pairs of disjoint s-cycles and t-cycles, and let $\Gamma^{(2)}(G)$ be the set of all pairs of disjoint cycles. Recently, Nikkuni proved the following theorem relating the linking and knotting in an embedding of K_6:

Theorem 1 ([4]). *For any embedding f of K_6 into $\mathbb{R}^3$ the following holds*:

$$\sum_{\lambda\in\Gamma^{(2)}(K_6)} \mathrm{lk}(f(\lambda))^2 = 2\left(\sum_{\gamma\in\Gamma_H(K_6)} a_2(f(\gamma)) - \sum_{\gamma\in\Gamma_5(K_6)} a_2(f(\gamma))\right) + 1,$$

where a_2 is the second coefficient of the Conway polynomial.

Due the nature of the ∇Y-moves, Nikkuni's result for K_6 implies there are similar relations between knotting and linking for all of the graphs that can be obtained from K_6 by ∇Y-moves. This left a single graph $K_{3,3,1}$ of $\mathcal{PF}$ for which it was unknown if there was such a relationship. We prove for every embedding f of $K_{3,3,1}$ that

$$\sum_{\lambda\in\Gamma_{3,4}(K_{3,3,1})} \mathrm{lk}(f(\lambda))^2 = 2\left(\sum_{\gamma\in\Gamma_H} a_2(f(\gamma)) - 2\sum_{\substack{\gamma\in\Gamma_6\\ A\notin\gamma}} a_2(f(\gamma)) - \sum_{\substack{\gamma\in\Gamma_5\\ A\in\gamma}} a_2(f(\gamma))\right) + 1,$$

where A is the single vertex of valance 6 in $K_{3,3,1}$. This gives an explicit connection between linking and knotting in embeddings of $K_{3,3,1}$, completely our understanding of the $\mathcal{PF}$.

In Section 2, we define the Wu invariant and give background on the key ingredients that go into such results. In Section 3, we prove Theorem 7 obtaining the above stated relationship for the graph $K_{3,3,1}$. In Section 4, we further examine the relationship between knotting and linking in the Petersen family.

One might expect that a knotted embedding would be an embedding with more complex linking. However there are knotted embeddings of K_6 that contain only a single Hopf link, see Fig. 12. The question of when complexity in linking of an embedding can guaranty that the embedding is knotted, is much more fruitful. We prove:

Theorem 2. *If f is a CA linked embedding of $G \in \mathcal{PF}$, then f is knotted.*

This result gives an algebraic linking condition on the embedding that will result in a knotted embedding. Another natural question is whether the presences of additional links with linking number 0, or more complex links with linking number ± 1 would guarantee knotting in the embedding. We give examples of embeddings of K_6 suggesting that such geometric linking will not guarantee a knotted embedding.

2. Background on graph homologous embeddings and the Wu invariant

This sections contains a brief description of the Wu invariant, and graph-homologous embeddings, along with useful relationships between the Wu invariant, the *α-invariant*, and the second coefficient of the Conway polynomial.

Let $V(G)$ and $E(G)$ be the set of all vertices and the set of all edges of a graph G, respectively. Let G be a graph with $V(G) = \{v_1, \dots, v_m\}$ (fixed ordering), $E(G) = \{e_1, \dots, e_n\}$ and a fixed orientation on each of the edges. Note, G is a finite one-dimensional simplicial complex. For a simplicial complex X, let

$$P_2(X) = \{s_1 \times s_2 \mid s_1, s_2 \in X,\ s_1 \cap s_2 = \emptyset\}$$

be the *polyhedral residual space* of X. Let σ be the involution on $P_2(X)$, i.e. $\sigma(s_1 \times s_2) = s_2 \times s_1$. Let f be an embedding of G into $\mathbb{R}^3$. The second skew-symmetric cohomology group of the pair $(P_2(G), \sigma)$ is denote $L(G)$. It is known that $L(G)$ is a free abelian group and the Wu invariant of f, denoted $\mathcal{L}(f)$ is in $L(G)$. Next we will focus on computations for graphs. For more background on the Wu invariant and a more general approach see [3, 8, 10, 12].

Following [10], Section 2, there is explicit presentation of $L(G)$. An orientation of a 2-cell $e_i \times e_j \in P_2(G)$ is given by the ordered pair of orientations of e_i and e_j. Let $E_{e_i e_j} = e_i \times e_j + e_j \times e_i \in C_2(P_2(G))$ for $e_i \cap e_j = \emptyset$ $(1 \le i < j \le n)$. The set $\{E_{e_i e_j} \mid 1 \le i < j \le n,\ e_i \cap e_j = \emptyset\}$ is a free basis for $C_2(P_2(G), \sigma)$. Now the set of dual elements $\{E^{e_i e_j} \mid 1 \le i < j \le n,\ e_i \cap e_j = \emptyset\}$ generate $L(G)$. The relations on the generators are given by the coboundary applied to the set $\{V^{e_i v_s} \mid 1 \le i \le n,\ 1 \le s \le m,\ v_s \notin e_i\}$. The coboundary is defined by:

$$\delta^1(V^{e_i v_s}) = \sum_{I(e_j)=v_s} E^{\rho(e_i e_j)} - \sum_{T(e_j)=v_s} E^{\rho(e_i e_j)},$$

where $I(e_i)$ is the initial vertex of e_j, $T(e_j)$ is the terminal vertex of e_j and $\rho(e_i e_j)$ is the standard ordering $e_i e_j$ if $i < j$ and $e_j e_i$ if $j < i$. The Wu invariant $\mathcal{L}(f)$ can be calculated from a projection $\lambda\colon \mathbb{R}^3 \to \mathbb{R}^2$ where $\lambda \circ f$ is a regular projection with finitely many multiple points all of which are transverse double points that occur away from vertices. Let $a_{ij}(f)$ be the sum of the signs of the crossings that occur between $\lambda \circ f(e_i)$ and $\lambda \circ f(e_j)$. Let $W = \sum a_{ij}(f) E^{e_i e_j}$ summed over all pairs of disjoint edges of G. The Wu invariant $\mathcal{L}(f)$ is the coset of the sum W in $L(G)$.

A *spatial graph-homology* (or just *homology*) is an equivalence relation on spatial graphs introduced by Taniyama, see [9] for the precise definition. A result that will be central to obtaining our results is:

Theorem 3 ([10])**.** *Two embeddings f and g of a simple graph G in $\mathbb{R}^3$ are homologous if and only if $\mathcal{L}(f) = \mathcal{L}(g)$.*

Another key insight is that, if two embeddings are spatial graph-homologous then the subgraphs are also spatial graph-homologous. Both linking number and the Wu invariant are spatial graph-homology invariants.

The Wu invariant of $f(K_{3,3})$ can be expressed in this simple combinatorial form [10]:

$$\mathcal{L}(f) = \sum_{(x,y)} \varepsilon(x, y) l(f(x), f(y)),$$

the sum over all unordered disjoint pairs of edges in G, where $l(f(x), f(y))$ is the sum of the signs of the crossing between $f(x)$ and $f(y)$, and $\varepsilon(x, y)$ is a weighting defined,

$$\varepsilon(x, y) = \begin{cases} -1, & \text{for} \quad (c_i, b_l) \quad \text{if} \quad i \quad \text{is odd}, \\ 1, & \text{else}, \end{cases}$$

where the edges of $K_{3,3}$ are labeled as indicated in Fig. 2. This makes sense because the $L(K_{3,3}) \cong \mathbb{Z}$. There is a similar formula for K_5, but it is omitted because it will not be used here. These explicit calculations for the $K_{3,3}$ subgraphs of a graph G are what make it possible to relate the Wu invariant $\mathcal{L}(f)$ and linking in the given embedding f. Then the $\mathcal{L}(f)$ also needs to be related to the second coefficient of the Conway polynomial a_2. This is done via another invariant known as the *α-invariant* of f [8]. For a spatial embedding f of $K_{3,3}$ or K_5:

$$\alpha(f) := \sum_{\gamma \in \Gamma_H} a_2(f(\gamma)) - \sum_{\gamma \in \Gamma_4} a_2(f(\gamma)).$$

There is the following relationship between the *α-invariant* and $\mathcal{L}(f)$:

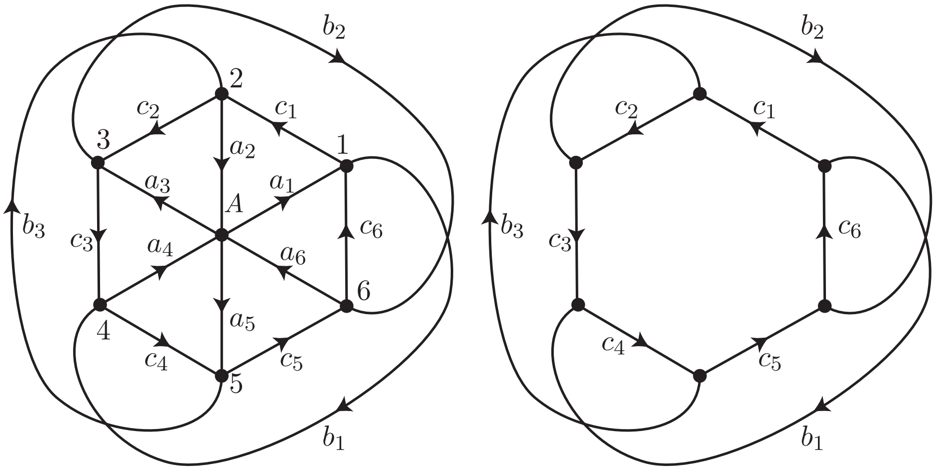


Fig. 2. *On the left*: The graph $K_{3,3,1}$ with edges oriented and the edges and vertices labeled. *On the right*: The graph $K_{3,3}$ with edges oriented and labeled in the standard convention for the Wu invariant.

Proposition 4 ([2]). *Let f be a spatial embedding of $K_{3,3}$ or K_5 then,*

$$\alpha(f) = \frac{\mathcal{L}(f)^2 - 1}{8}.$$

Together this gives the relationship between $\mathcal{L}(f)$ and a_2 of certain cycles in embeddings of $K_{3,3}$ and K_5.

3. Conway–Gordon theorem for $K_{3,3,1}$

In this section we prove the before mentioned relationship between the linking number and a_2 of cycles in embeddings of $K_{3,3,1}$. In the following proposition we determine a standard embedding of $K_{3,3,1}$, which given the correct choice of nine integers is graph-homologous to any other given embedding of $K_{3,3,1}$. We prove this by finding a basis for $L(K_{3,3,1})$. Throughout this paper we indicate the number of half twists between two edges with a box and integer as shown in Fig. 3, with the handedness of the crossings is as shown.

Proposition 5. *Given an embedding f of $K_{3,3,1}$ there exist a choice of the nine integers l_i, m_i, n_i for $i = 1, 2, 3$, such that h is spatial graph-homologous to the embedding f. The embedding h is shown in* Fig. 4.

Proof. We will use the edge and vertex labeling, as well as edge orientation indicated in Fig. 2. The order on the sets is $E(K_{3,3,1}) = \{a_1, \ldots, a_6, b_1, b_2, b_3, c_1, \ldots, c_6\}$

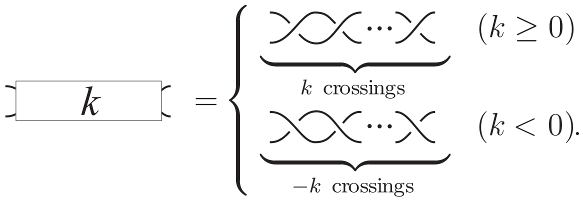


Fig. 3. Crossings between two edges.

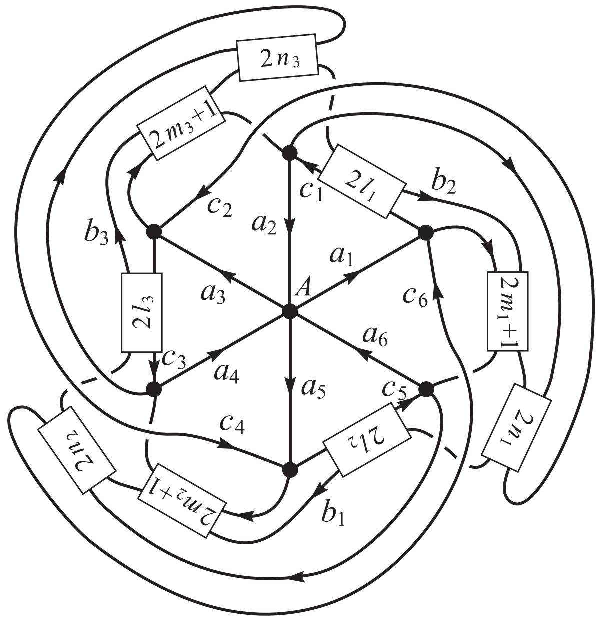


Fig. 4. An embedding h of $K_{3,3,1}$ where the integers in the boxes indicate the number of half twists between the two edges, as shown in Fig. 3.

and $V(K_{3,3,1}) = \{1, 2, 3, 4, 5, 6, A\}$. Let $S = \{E^{b_1b_2}, E^{b_1b_3}, E^{b_2b_3}, E^{b_1c_2}, E^{b_1c_5}, E^{b_2c_1}, E^{b_2c_4}, E^{b_3c_3}, E^{b_3c_6}\}$. In the following we will show that S is a basis for $L(K_{3,3,1})$. The set of dual elements to the basis elements in S come from the pairs of edges with crossings in h. Thus, Theorem 3 implies for any f that $\mathcal{L}(f) = \mathcal{L}(h)$ for some choice of $l_i, m_i, n_i \in \mathbb{Z}$ for $i = 1, 2, 3$.

In the following the coboundary is applied to different sets of $V^{e_i v_s}$ to obtain the relations and express all of the other $E^{e_i e_j}$ in terms of the elements of S. If we consider the coboundary for elements V^{b_1*} we find

$$\delta^1(V^{b_12}) = E^{b_1c_2} + E^{a_2b_1} - E^{b_1b_3} = 0,$$
$$\delta^1(V^{b_13}) = E^{b_1b_2} - E^{a_3b_1} - E^{b_1c_2} = 0,$$
$$\delta^1(V^{b_15}) = E^{b_1c_5} + E^{b_1b_3} - E^{a_5b_1} = 0,$$
$$\delta^1(V^{b_16}) = E^{a_6b_1} - E^{b_1b_2} - E^{b_1c_5} = 0.$$

One can solve for $E^{a_ib_1}$ in each of the above. So we see the elements $E^{a_ib_1}$ (for i such that $b_1 \cap a_i = \emptyset$) can all be expressed as linear combinations elements of S. This is consistent with the additional relation given by $\delta^1(V^{b_1A})$. Similarly, all those elements of the form $E^{a_ib_2}$, and $E^{a_ib_3}$ (for appropriate a_i) can be expressed as linear combinations elements of S. In the same way, if we consider the coboundary for elements V^{a_1*} we find

$$\delta^1(V^{a_12}) = E^{a_1c_2} - E^{a_1b_3},$$
$$\delta^1(V^{a_14}) = E^{a_1c_4} - E^{a_1c_3},$$
$$\delta^1(V^{a_16}) = -E^{a_1c_5} - E^{a_1b_2},$$
$$\delta^1(V^{a_13}) = E^{a_1b_2} + E^{a_1c_3} - E^{a_1c_2},$$
$$\delta^1(V^{a_15}) = E^{a_1c_5} + E^{a_1b_3} - E^{a_1c_4}.$$

Thus, all of the elements of the from $E^{a_1c_i}$ (for i such that $a_1 \cap c_i = \emptyset$) can be expressed as a linear combination of $E^{a_1b_2}$ and $E^{a_1b_3}$, which can in turn be expressed as a linear combination of the elements in S. Similarly, those elements of the form $E^{a_jc_i}$ can be expressed as a linear combination of $E^{a_lb_k}$ for those l and k such that $a_l \cap b_k = \emptyset$. Finally, if we consider the coboundary for elements V^{c_1*} we find

$$\delta^1(V^{c_13}) = E^{c_1c_3} + E^{b_2c_1} - E^{a_3c_1},$$
$$\delta^1(V^{c_14}) = E^{a_4c_1} + E^{c_1c_4} - E^{c_1c_3},$$
$$\delta^1(V^{c_15}) = E^{c_1c_5} - E^{c_1c_4} - E^{a_5c_1},$$
$$\delta^1(V^{c_16}) = E^{a_6c_1} - E^{c_1c_5} - E^{b_2c_1}.$$

So the elements $E^{c_1c_i}$ (for i such that $c_1 \cap c_i = \emptyset$) can be written as a linear combination of $E^{c_1b_2}$ and $E^{a_jc_1}$ (for j such that $a_j \cap c_1 = \emptyset$), which can be written as linear combinations of those elements in S. Similarly, all the remaining elements, $E^{c_i c_j}$, can be written as linear combinations of the elements in S.

In [3], Nikkuni shows for a graph in a class containing $K_{3,3,1}$ that

$$\text{rank}(L(G)) = \frac{1}{2}\left(\beta_1^2 + \beta_1 + 4|\text{E(G)}| - \sum_{v\in \text{V(G)}} (\text{val}(v))^2\right)$$

where β_1 is the first Betti number of G, and val(v) is the valency of v. So we see rank$(L(K_{3,3,1})) = 9$. Thus S is a basis for $L(G)$. □

The following lemma is about the relationship between the sum of the square of the linking number of all of the links in $K_{3,3,1}$ and the sums of the squares of the Wu invariant of subgraphs of $K_{3,3,1}$ that are isotopic to $K_{3,3}$ and $K_{3,3}$ subdivisions. Let the valence 6 vertex of $K_{3,3,1}$ be labeled A.

Let G_i for $i = 1, \ldots, 18$ be the subdivisions of $K_{3,3}$ obtained by deleting three of the edges adjacent to A and then deleting the two edges not adjacent to those already deleted edges, see Figs. 5, 6, and 7. In deleting three edges adjacent to A, the cases where the edge sets $\{1A, 3A, 5A\}$ or $\{2A, 4A, 6A\}$ are deleted must be excluded. Let H_i for $i = 1, \ldots, 6$ be the $K_{3,3}$ subgraphs that are obtained by deleting one vertex $v \neq A$ and deleting the two appropriate additional edges that are adjacent to A, see Fig. 8. Let K be the $K_{3,3}$ subgraph obtained by deleting the vertex A, see Fig. 9.

Lemma 6. *For any embedding f of $K_{3,3,1}$ into $\mathbb{R}^3$ the following holds*

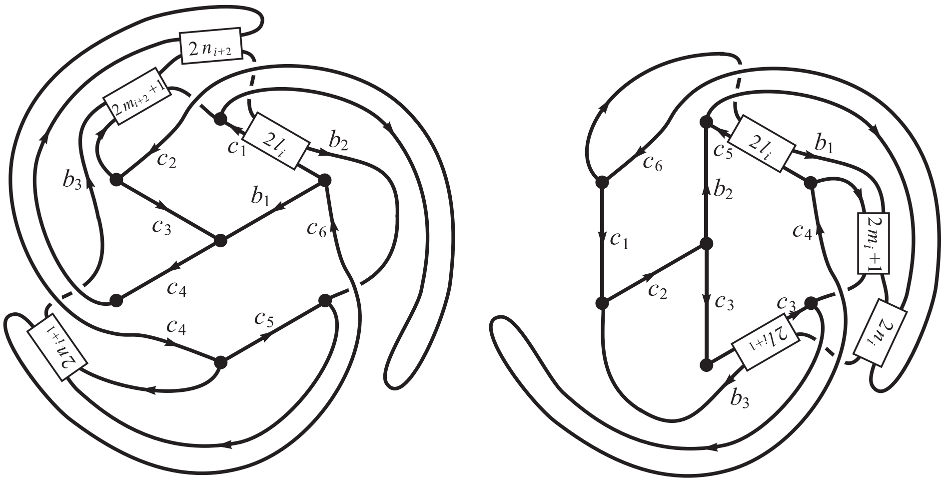
$$\sum_{\gamma\in\Gamma_{3,4}(K_{3,3,1})} \text{lk}(f(\lambda))^2 = \frac{1}{8}\sum_{G_i} \mathcal{L}(f|_{G_i})^2 - \frac{1}{2}\mathcal{L}(f|_K)^2 - \frac{1}{8}\sum_{H_i} \mathcal{L}(f|_{H_i})^2,$$

where G_i, K, H_i are the above described subgraphs.

Proof. From Proposition 5 we know there exists nine integers l_i, m_i, n_i for $i = 1, 2, 3$, such that the embedding h of $K_{3,3,1}$ is spatial graph-homologous to f. If two embeddings are spatial graph-homologous then the subgraphs are also spatial graph-homologous. Both linking number and the Wu invariant are spatial graph-homology invariants. Thus we need only show:

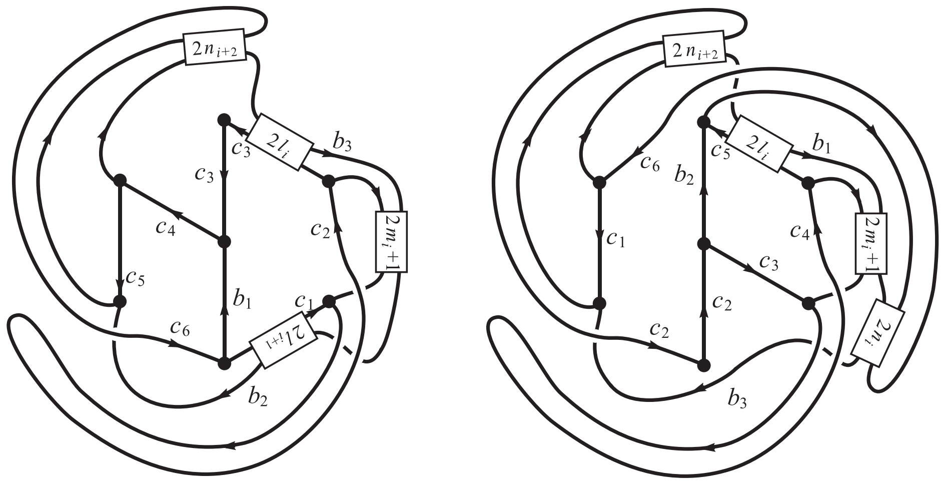
$$\sum_{\Gamma_{3,4}(K_{3,3,1})} \text{lk}(h(\lambda))^2 = \frac{1}{8}\sum_{G_i} \mathcal{L}(h|_{G_i})^2 - \frac{1}{2}\mathcal{L}(h|_K)^2 - \frac{1}{8}\sum_{H_i} \mathcal{L}(h|_{H_i})^2.$$

Let the embedding of $h(G_i)$ be as indicated in Figs. 5, 6, and 7. Let the embedding of $h(H_i)$ be as indicated in Fig. 8. Using the formula give in Section 2 we find the



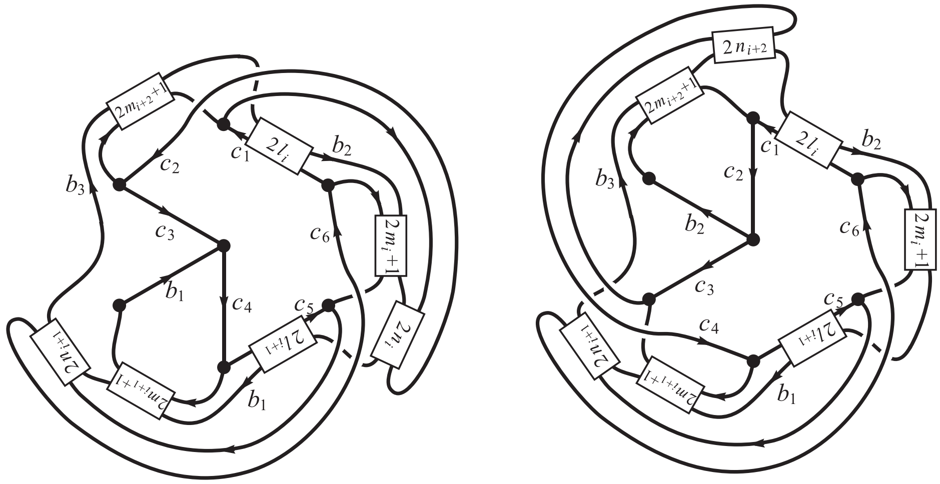
G_i ($i = 1, 2, 3$) G_i ($i = 4, 5, 6$)

Fig. 5. The h embeddings of the G_i subgraphs of $K_{3,3,1}$ for $i = 1, \ldots, 6$. All of the subscripts of l, m, n are given by $i + 3 = i$.



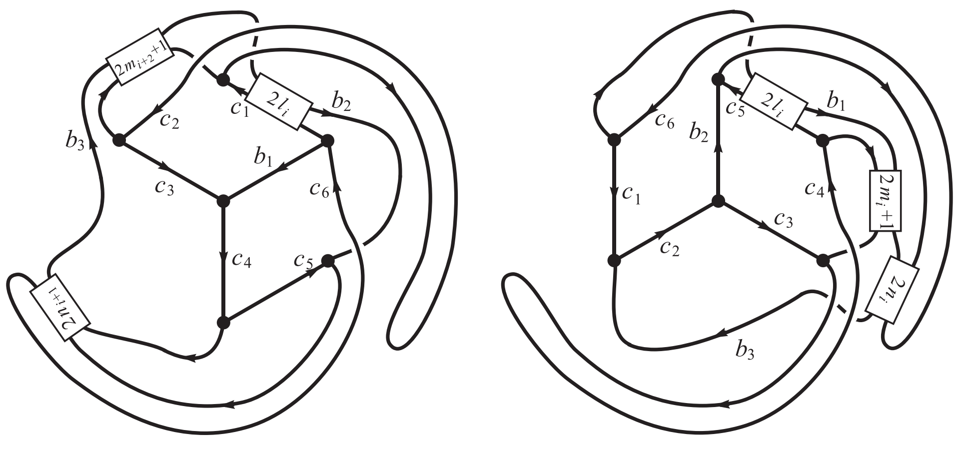
G_i ($i = 7, 8, 9$) G_i ($i = 10, 11, 12$)

Fig. 6. The h embeddings of the G_i subgraphs of $K_{3,3,1}$ for $i = 7, \ldots, 12$. All of the subscripts of l, m, n are given by $i + 3 = i$.



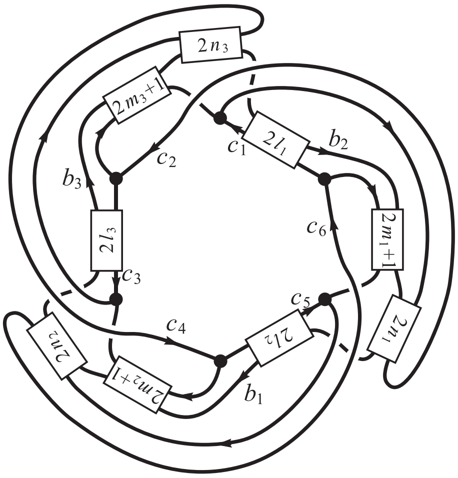
G_i ($i = 13, 14, 15$) G_i ($i = 16, 17, 18$)

Fig. 7. The h embeddings of the G_i subgraphs of $K_{3,3,1}$ for $i = 13, \ldots, 18$. All of the subscripts of l, m, n are given by $i + 3 = i$.



H_i ($i = 1, 2, 3$) H_i ($i = 4, 5, 6$)

Fig. 8. The h embeddings of the H_i subgraphs of $K_{3,3,1}$ for $i = 1, \ldots, 6$. All of the subscripts of l, m, n are given by $i + 3 = i$.



K

Fig. 9. The h embedding of the K subgraphs of $K_{3,3,1}$. All of the subscripts of l, m, n are given by $i + 3 = i$.

Wu invariants are as follows, where all subscripts of l, m, n, are given by $i + 3 = i$:

$$\mathcal{L}(h|_{G_i}) = -2(l_i + m_{i+2} + n_{i+1} + n_{i+2}) - 1 \quad \text{for} \quad i = 1, 2, 3,$$

$$\mathcal{L}(h|_{G_i}) = -2(l_i + l_{i+1} + m_i + n_i) - 1 \quad \text{for} \quad i = 4, 5, 6,$$

$$\mathcal{L}(h|_{G_i}) = -2(l_i + l_{i+1} + m_i + n_{i+2}) - 1 \quad \text{for} \quad i = 7, 8, 9,$$

$$\mathcal{L}(h|_{G_i}) = -2(l_i + m_i + n_i + n_{1+2}) - 1 \quad \text{for} \quad i = 10, 11, 12,$$

$$\mathcal{L}(h|_{G_i}) = -2(l_i + l_{i+1} + m_i + m_{i+1} + m_{i+2} + n_i + n_{i+1}) - 3 \quad \text{for} \quad i = 13, 14, 15,$$

$$\mathcal{L}(h|_{G_i}) = -2(l_i + l_{i+1} + m_i + m_{i+1} + m_{i+2} + n_{i+1} + n_{i+2})) - 3 \quad \text{for} \quad i = 16, 17, 18,$$

$$\mathcal{L}(h|_{H_i}) = -2(l_i + m_{i+2} + n_{i+1}) - 1 \quad \text{for} \quad i = 1, 2, 3,$$

$$\mathcal{L}(h|_{H_i}) = -2(l_i + m_i + n_i) - 1 \quad \text{for} \quad i = 4, 5, 6,$$

$$\mathcal{L}(h|_K) = -2\sum_{i=1}^{3}(l_i + m_i + n_i) - 3.$$

The links in the embedding $h(K_{3,3,1})$ are in three forms, see Fig. 10. For the links we have:

$$\mathrm{lk}(L_i) = l_i \quad \text{for} \quad i = 1, 2, 3,$$

$$\mathrm{lk}(L_i) = n_i \quad \text{for} \quad i = 4, 5, 6,$$

$$\mathrm{lk}(L_i) = l_i + m_i + m_{i+2} + n_{i+2} + 1 \quad \text{for} \quad i = 7, 8, 9.$$

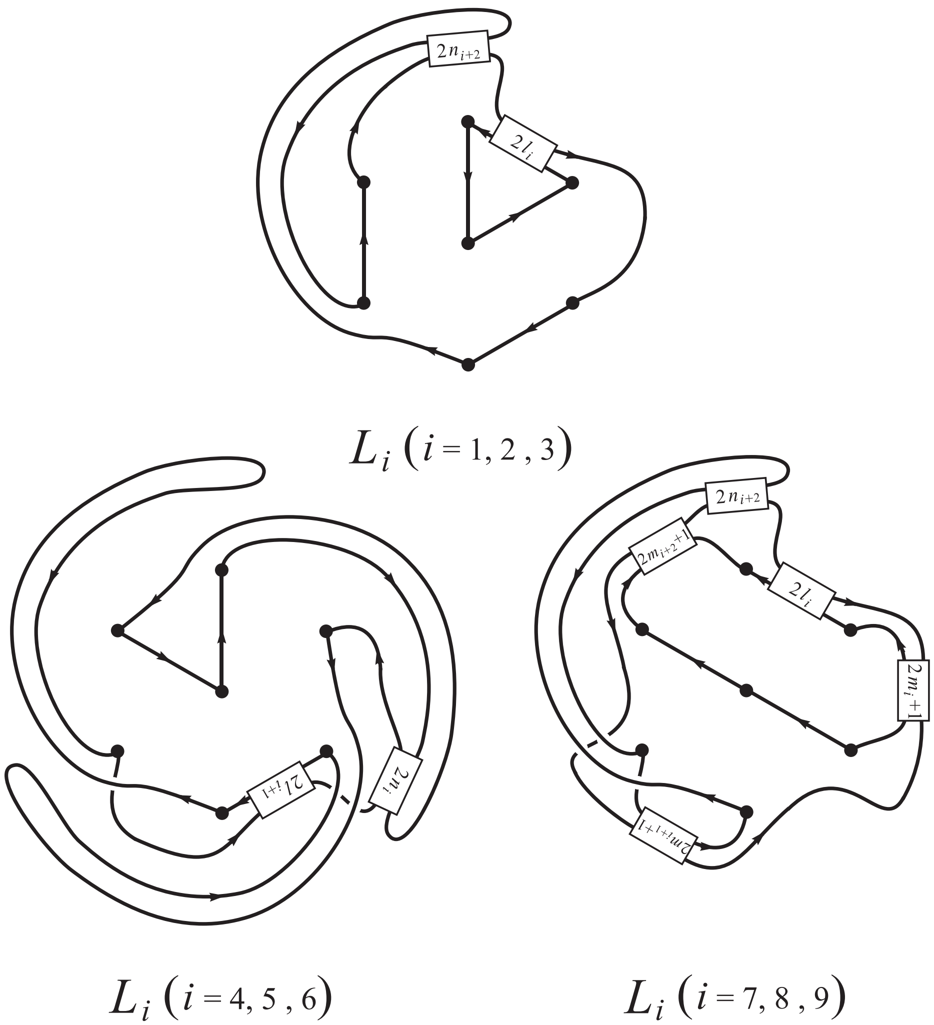


Fig. 10. The three different types of links found in the embedding $h(K_{3,3,1})$. All of the subscripts of l, m, n are given by $i + 3 = i$.

Together these computations give the desired result. $\square$

We have established a relationship between the linking in $K_{3,3,1}$ and the Wu invariant of subgraphs of $K_{3,3,1}$ that are isomorphic to $K_{3,3}$ and its subdivisions. The following theorem will make use of two relations that are known for the Wu invariant of $K_{3,3}$.

Theorem 7. *For every embedding f of $K_{3,3,1}$ into $\mathbb{R}^3$ the following holds*

$$\sum_{\lambda\in\Gamma_{3,4}(K_{3,3,1})} \mathrm{lk}(f(\lambda))^2 = 2\left(\sum_{\gamma\in\Gamma_H} a_2(f(\gamma)) - 2\sum_{\substack{\gamma\in\Gamma_6\\ A\notin\gamma}} a_2(f(\gamma)) - \sum_{\substack{\gamma\in\Gamma_5\\ A\in\gamma}} a_2(f(\gamma))\right) + 1.$$

Proof. Let f be a embedding of $K_{3,3,1}$ into $\mathbb{R}^3$. From Lemma 6 we know,

$$\sum_{\gamma\in\Gamma_{3,4}(K_{3,3,1})} \mathrm{lk}(f(\lambda))^2 = \frac{1}{8}\sum_{G_i} \mathcal{L}(f|_{G_i})^2 - \frac{1}{2}\mathcal{L}(f|_K)^2 - \frac{1}{8}\sum_{H_i} \mathcal{L}(f|_{H_i})^2.$$

Then from Proposition 4 we see that:

$$\mathcal{L}(f)^2 = 8\left(\sum_{\gamma\in\Gamma_H} a_2(f(\gamma)) - \sum_{\gamma\in\Gamma_4} a_2(f(\gamma))\right) + 1.$$

To avoid confusion we note that the application of Proposition 4 to the G_is requires recognizing that this can be applied to such a $K_{3,3}$ subdivision as long as the appropriate cycles are used. We will for the moment think of the G_is as $K_{3,3}$ subgraphs, ignoring the single valance two vertex when describing their cycles. Thus

$$\begin{aligned}
&\frac{1}{8}\sum_{G_i} \mathcal{L}(f|_{G_i})^2 - \frac{1}{2}\mathcal{L}(f|_K)^2 - \frac{1}{8}\sum_{H_i} \mathcal{L}(f|_{H_i})^2 \\
&= \left(\sum_{\substack{\gamma\in\Gamma_H(G_i)\\ G_i\in K_{3,3,1}}} a_2(f(\gamma)) - \sum_{\substack{\gamma\in\Gamma_4(G_i)\\ G_i\in K_{3,3,1}}} a_2(f(\gamma))\right) - 4\left(\sum_{\gamma\in\Gamma_H(K)} a_2(f(\gamma)) - \sum_{\gamma\in\Gamma_4(K)} a_2(f(\gamma))\right) \\
&\quad - \left(\sum_{\substack{\gamma\in\Gamma_H(H_i)\\ H_i\in K_{3,3,1}}} a_2(f(\gamma)) - \sum_{\substack{\gamma\in\Gamma_4(H_i)\\ H_i\in K_{3,3,1}}} a_2(f(\gamma))\right) + \frac{18-4-6}{8}.
\end{aligned}$$

So we need only determine which cycles of $K_{3,3,1}$ are counted in the above sums, and how many times each cycle is counted.

The G_i subgraphs. Recall that the G_is are $K_{3,3}$ subdivisions formed by taking $K_{3,3,1}$ and deleting three of the edges adjacent to A and then deleting the two edges not adjacent to those already deleted edges. This could also be thought of as taking $K_{3,3}$ deleting two adjacent edges and then adding a vertex A and edges from A to each of the vertices that were incident to at least one of the deleted edges. The G_is are subdivisions of $K_{3,3}$ where the valence two vertex was ignored when describing the cycles that were summed. So some of the Hamiltonian cycles counted in the sum of G_is are Hamiltonian cycles of $K_{3,3,1}$ and some are 6-cycles. Similarly the 4-cycles will be 5-cycles and 4-cycles in $K_{3,3,1}$. To count these cycles we will consider different cycles in $K_{3,3,1}$ and determine how many of the G_is contain a given cycle.

Consider an arbitrary Hamiltonian cycle η of $K_{3,3,1}$, to have η be in G_i all of the edges of η must be in G_i. In particular, the two edges incident to A must be in G_i, for this to happen the edge between these two edges, call it e, must be deleted. In addition, another edge which is not incident to A but is adjacent to e must be deleted, there are two such edges which are not in η. Thus two of the eighteen G_i graphs contain η as one of their Hamiltonian cycles. The 6-cycles in $K_{3,3,1}$ can be broken into two sets the ones that contain the vertex A and those that do not. Since two adjacent edges neither of which are incident to A must be deleted to form a G_i, the latter 6-cycle cannot occur. For a 6-cycle in $K_{3,3,1}$ that contains A the two vertices adjacent to A, call them v and w, must be in the same partite set. Thus the two deleted adjacent edges not incident to A must go between v and w. There is one way for this to happen, thus each 6-cycle that contains A appears in one of the G_is as a Hamiltonian cycle.

Every 5-cycle in $K_{3,3,1}$ contains A. To have the edges to the vertex A, the edge between the adjacent vertices must be deleted. As with the Hamiltonian cycles there are two ways to deleted two adjacent edges (not incident to A) and delete the said edge. Thus there are two G_i graphs that contain a given 5-cycle, as a 4-cycle. Next the 4-cycles of $K_{3,3,1}$ can be put into two groups: 4-cycles that contain A and 4-cycles that do not contain A. By similar reasoning one can see that 4-cycles that contain A will appear in two of the G_is and 4-cycles that do not contain A appear in six of the G_is.

The K subgraph. Recall that the subgraph K is the $K_{3,3}$ subgraph obtained by deleting the vertex A. So the Hamiltonian cycles of K are the 6-cycles of $K_{3,3,1}$ that do not contain A. The 4-cycles of K are the 4-cycles of $K_{3,3,1}$ which do not contain A.

The H_i subgraphs. Recall that the H_i subgraphs are the $K_{3,3}$ subgraphs that are obtained from $K_{3,3,1}$ by deleting one vertex $v \neq A$ and the two edges that are adjacent to A as well as those vertices in the same partite set as the vertex v. The Hamiltonian cycles of H_i are all 6-cycles in $K_{3,3,1}$ which contain A, as the H_i are $K_{3,3}$ subgraphs with one vertex $v \neq A$ deleted. Let c be an arbitrary 6-cycle that contains A and does not contain the vertex v. The cycle c will appear in one of the H_is, that is in the H_i which does not contain the vertex v. Next, those 4-cycles that do not contain A will appear in two of the H_i, one for each of the vertices that is not A and is not in the said 4-cycle. In the H_is the vertex A can be thought of as replacing the vertex v that is deleted in the original $K_{3,3}$ subgraph. Now the 4-cycles that contain A, also contain two vertices from one partite set and one from the partite set that A has now joined. Thus there are two H_i graphs that contain each 4-cycle.

All together this gives:

$$\frac{1}{8}\sum_{G_i}\mathcal{L}(f|_{G_i})^2-\frac{1}{2}\mathcal{L}(f|_K)^2-\frac{1}{8}\sum_{H_i}\mathcal{L}(f|_{H_i})^2$$

$$\begin{aligned}
&=\Bigg(2\sum_{\gamma\in\Gamma_H}a_2(f(\gamma))+\sum_{\substack{\gamma\in\Gamma_6\\ A\in\gamma}}a_2(f(\gamma))\\
&\qquad-2\sum_{\substack{\gamma\in\Gamma_5\\ A\in\gamma}}a_2(f(\gamma))-2\sum_{\substack{\gamma\in\Gamma_4\\ A\in\gamma}}a_2(f(\gamma))-6\sum_{\substack{\gamma\in\Gamma_4\\ A\notin\gamma}}a_2(f(\gamma))\Bigg)\\
&\qquad-4\Bigg(\sum_{\substack{\gamma\in\Gamma_6\\ A\notin\gamma}}a_2(f(\gamma))-\sum_{\substack{\gamma\in\Gamma_4\\ A\notin\gamma}}a_2(f(\gamma))\Bigg)\\
&\qquad-\Bigg(\sum_{\substack{\gamma\in\Gamma_6\\ A\in\gamma}}a_2(f(\gamma))-2\sum_{\substack{\gamma\in\Gamma_4\\ A\notin\gamma}}a_2(f(\gamma))-2\sum_{\substack{\gamma\in\Gamma_4\\ A\in\gamma}}a_2(f(\gamma))\Bigg)+1\\
&=2\Bigg(\sum_{\gamma\in\Gamma_H}a_2(f(\gamma))-2\sum_{\substack{\gamma\in\Gamma_6\\ A\notin\gamma}}a_2(f(\gamma))-\sum_{\substack{\gamma\in\Gamma_5\\ A\in\gamma}}a_2(f(\gamma))\Bigg)+1. \qquad\square
\end{aligned}$$

The relationship between CA linking and knotting in $K_{3,3,1}$ is an immediate corollary.

Corollary 8. *If an embedding f of $K_{3,3,1}$ is CA linked then f is knotted.*

Proof. If $f(K_{3,3,1})$ is CA linked then $\sum_{\Gamma_{3,4}(K_{3,3,1})}\mathrm{lk}(f(\lambda))^2>1$. Thus at least one of the $a_2(\gamma)\neq 0$ for $\gamma\in\Gamma_H\cup\{\Gamma_6\mid A\notin\gamma\}\cup\{\Gamma_5\mid A\in\gamma\}$. So f is knotted. $\square$

4. Connections between knotting and linking in the Petersen family

In this section we prove, given $G\in\mathcal{PF}$, if $f(G)$ is CA linked, then $f(G)$ is knotted. We also answer a number of other questions in the negative, showing how unique the first result is. We consider the questions about embeddings of the graphs of the

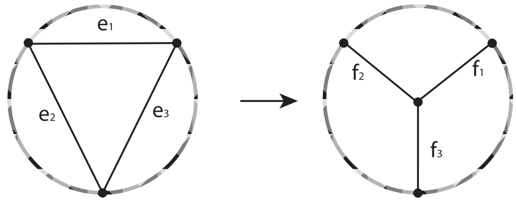


Fig. 11. The ∇Y-moves.

Petersen family: If f is knotted can that imply a level of complexity in the linking? If an embedding is not CA linked but contains more than one link or contains a link that is not the Hopf link would this imply the embedding is knotted? So we show the converse of Theorem 2 does not hold and that more geometric complexity in linking does not guaranty knotting.

To simplify our discussion we will call a graph G *K-linked* when it has the following property: if an embedding f of G is CA linked, then f is knotted. So the main theorem can be restated as: All of the graphs of the Petersen family are K-linked. Recall different abstract graphs can be related by ∇Y-moves (see Fig. 11). In this move three edges that form a cycle (a triangle) are deleted and a vertex is a added along with three edges between the new vertex and the original triangle. Before proving this we need the following lemma.

Lemma 9. *Let G' be obtained from G by a ∇Y-move. If G is K-linked then G' is K-linked.*

Proof. Let $\triangle$ denote the 3-cycle deleted from G in the ∇Y-move, and Y denote the set of three edges and vertex added to G'. Let the subgraphs where the two graphs agree be denoted E and E', respectively.

Now consider an embedding f of G' which is CA linked. Define an embedding $\bar{f}$ of G, where $\bar{f}(E) = f(E')$ and $\triangle$ is mapped onto a tubular neighborhood of $f(\mathrm{Y}) \subset f(G')$. So cycles of $\bar{f}(G)$ are the same simple closed curves as the embedded cycles as $f(G)$, with the addition of $\bar{f}(\triangle)$ which bounds an embedded disk. Since $f(G')$ is CA linked, $\bar{f}(G)$ is also CA linked. By assumption this implies that $\bar{f}(G)$ is knotted. Thus there is some simple closed curve $\gamma \in \bar{f}(G)$ which is nontrivially knotted. The curve γ cannot be $\bar{f}(\triangle)$ since $\bar{f}(\triangle)$ bounds an embedded disk. So $f(G')$ is knotted. Therefore G' is K-linked. $\square$

Having now amassed all the tools needed we will prove:

Theorem 2. *If f is a CA linked embedding of $G \in \mathcal{PF}$, then $f(G)$ is knotted. Which can be restated as*: *All of the graphs of the Petersen family are K-linked.*

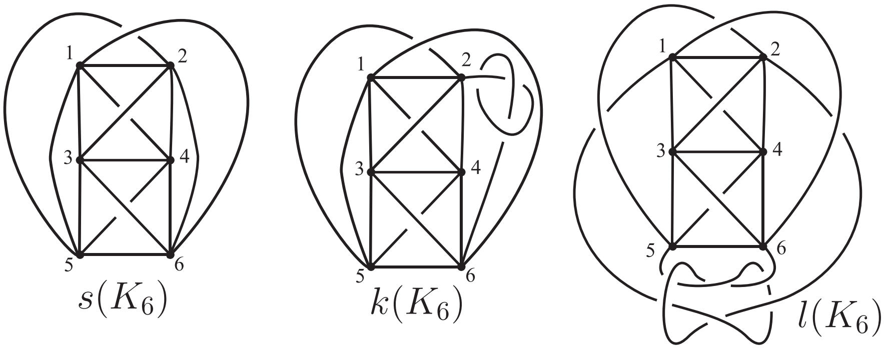


Fig. 12. The standard embedding of the complete graph on six vertices, $s(K_6)$, and two different knotted embeddings $k(K_6)$ and $l(K_6)$.

Proof. Let $G \in \mathcal{PF}$. If $G = K_6$ then for any embedding $f(K_6)$, by Theorem 1 [4] that,

$$\sum_{\lambda \in \Lambda(K_6)} \operatorname{lk}(f(\lambda))^2 = 2\left(\sum_{\gamma \in \Gamma_H} a_2(f(\gamma)) - \sum_{\gamma \in \Gamma_5} a_2(f(\gamma))\right) + 1.$$

If $f(K_6)$ is CA linked then $\sum_{\lambda \in \Lambda(K_6)} \operatorname{lk}(f(\lambda))^2 > 1$. Thus at least one of the $a_2(\gamma) \neq 0$ for $\gamma \in \Gamma_H \cup \Gamma_5$. So f is knotted. Next, if $G = K_{3,3,1}$ then G is K-linked by Corollary 8. If $G \neq K_6$ or $K_{3,3,1}$ the G can be obtained from K_6 or $K_{3,3,1}$ by a series of ∇Y-moves, see Fig. 1. Thus by Lemma 9, G is K-linked. □

Now we will examine other ways that knotting and linking in graph embeddings could be related. We will see through a series of examples that none of these other relationships occur for the graphs of the Petersen family. Let the cycle that is made of the edges v_1v_2, v_2v_3, ..., $v_{i-1}v_i$ and v_1v_i be denoted $v_1v_2v_3 \cdots v_i$. We will look first at knotting implying a greater level of complexity in linking, and next consider other kinds of complexity in linking that could lead to knotting in the embeddings. Our counterexamples come from making changes to a standard embedding of K_6, which we will call $s(K_6)$. See Fig. 12. The embedding $s(K_6)$ contains a single Hopf link in $s(146 \cup 235)$, all other links are trivial, and all cycles are unknots.

Consider the converse of Theorem 2. If $f(G)$ for $G \in \mathcal{PF}$ is knotted, then is $f(G)$ CA linked? This is not the case. The simplest way to produce a counterexample is by having an embedding with a knot that is in one of the edges.

EXAMPLE 1. The embedding $k(K_6)$ is obtained by replacing the edge 26 in $s(K_6)$ with a knotted edge as shown in Fig. 12. The embedding $k(K_6)$ contains a number of knotted cycles, all those cycles that contain the edge 26 are knotted. It is easy to see that

$k(K_6)$ is not CA linked because it is based on the embedding $s(K_6)$. It contains a single link with nonzero linking number, the Hopf link $k(146 \cup 235)$. It does however contain additional nontrivial links, the four links $k(126 \cup 345)$, $k(236 \cup 145)$, $k(246 \cup 135)$ and $k(256 \cup 134)$, which are each the split link of a trivial knot and a trefoil.

While the example of $k(K_6)$ is not CA linked it still has more nontrivial links than that of our standard embedding $s(K_6)$. It should be noted, that the existence of Example 1 and others like it, where the embedding is not CA linked, but at least one of the components of one of the links is knotted, follows immediately from work of Taniyama and Yasuhara. In [11], they showed that there exists an embedding of K_6 that realizes a given set of ten link types L_i as the sublinks if and only if $\sum_i \mathrm{lk}(L_i) \equiv 1 \pmod 2$. So next we consider if $f(G)$ for $G \in \mathcal{PF}$ is knotted, then will $f(G)$ contain more than one nontrivial link with no knotted components? This is also not the case.

EXAMPLE 2. In the slightly more complicated counterexample of $l(K_6)$, the edges 13 and 25 of $s(K_6)$ are replaced as shown in Fig. 12. This is an example of a spatial graph that is knotted but does not contain any more complicated linking than a single Hopf link. This embedding of K_6 contains a single nontrivial link $l(146 \cup 235)$. While all of the 3-cycles are trivial knots, it contains a number of knotted cycles. Many of the knots are the connected sum of two trefoils, an example is $l(1265)$. Thus having a knotted embedding does not imply any increased complexity in the linking.

Next, we consider the possibility that there is some other complexity in the linking in a given embedding that would lead to knotting. The embedding must not be CA linked, so we will look at embeddings where it contains a single link with non zero linking number which is ± 1. We will look at two embeddings of K_6 which are not CA linked but contain links other than the Hopf link and do not contain a nontrivial knot.

EXAMPLE 3. The embedding $f(K_6)$, shown in Fig. 13, contains a Hopf link $f(146 \cup 235)$, and the algebraically split link L in $f(135 \cup 246)$. The 2-component link L has unknotted components, and linking number 0, but is nontrivial. See Fig. 13. It can be verified that L is nontrivial with the Conway polynomial, $\nabla_L(z) = 2z^5 + z^7$. (This was calculated with the assistance of the Mathematica package KnotTheory`.)

Claim 1. *The spatial graph $f(K_6)$ is not knotted.*

Proof. The embedding $f(K_6)$ can be obtained from the embedding $s(K_6)$ in Fig. 12, by replacing the link $s(135 \cup 246)$ with the link L, where L is placed below the other edges. The embedding $s(K_6)$ is not knotted. So for there to be a knot in $f(K_6)$ it must contain some of the edges of L because that is where the embeddings differ. Next the link L was obtained by modifying a 6-component Brunnian link. If

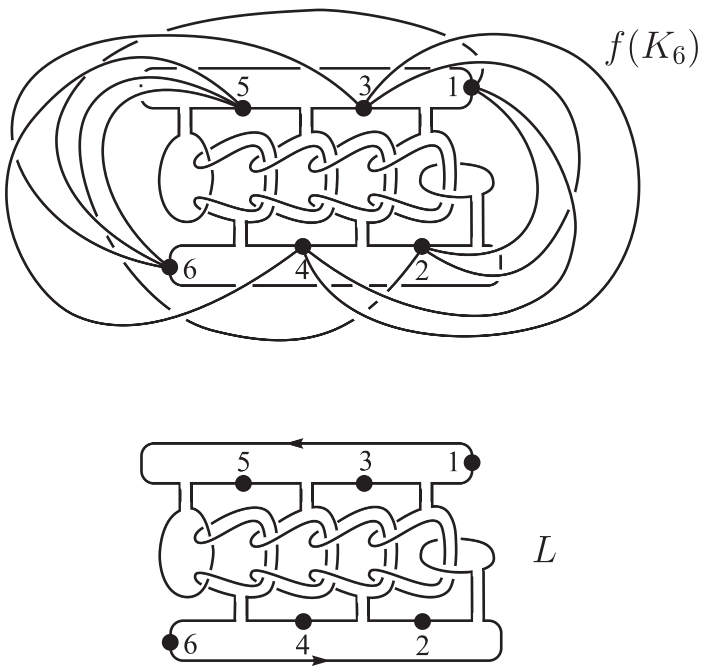
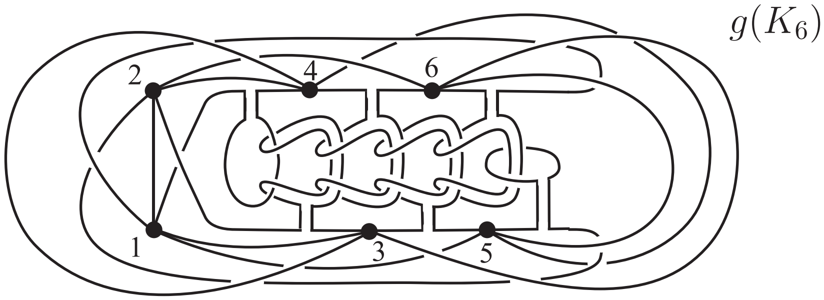


Fig. 13. The embedding $f(K_6)$ which contains both a Hopf link ($f(146 \cup 235)$) and the link L, but is not knotted. The link L.



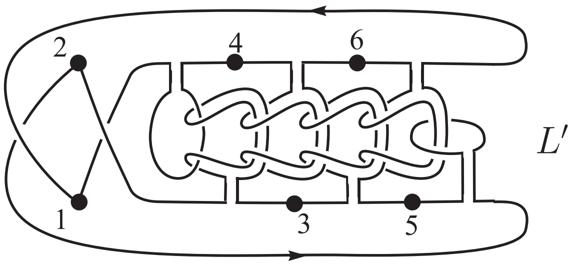


Fig. 14. The embedding $g(K_6)$ which contains link L', but is not knotted. The link L'.

any of the edges of L are deleted the remaining edges can be isotoped with the vertices fixed and without moving the edges over or around the vertices, to a projection there are no crossings in the remaining edges. So the only way to have additional crossings from those edges in L is to have all of them, but together all of the edges make the link L. □

EXAMPLE 4. The second embedding $g(K_6)$, shown in Fig. 14, contains a single nontrivial link L' in $g(135 \cup 246)$. The 2-component link L' has unknotted components, and linking number -1, but is not the Hopf link. See Fig. 14. It can be verified that L' is not the Hopf link with the Conway polynomial, $\nabla_{L'}(z) = -z + 2z^5 + z^7 - z^9$. (This was calculated with the assistance of the Mathematica package KnotTheory`.) In a similar way, it can be seen that $g(K_6)$ is not knotted.

These two examples show embeddings where there is more complex linking but there is not higher linking number, however neither are knotted. Thus the addition of complexity in these embeddings is not enough to result in a knotted embedding.

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