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THE S^1-EQUIVARIANT COHOMOLOGY RINGS OF $(n-k,k)$ SPRINGER VARIETIES

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Abstract

The main result of this note gives an explicit presentation of the S^1-equivariant cohomology ring of the $(n-k,k)$ Springer variety (in type A) as a quotient of a polynomial ring by an ideal I, in the spirit of the well-known Borel presentation of the cohomology of the flag variety.

Contents

1. Introduction

The Springer variety $\mathcal{S}_N$ associated to a nilpotent operator $N\colon \mathbb{C}^n \to \mathbb{C}^n$ is the subvariety of $\mathit{Flags}(\mathbb{C}^n)$ defined as

$$\mathcal{S}_N = \{V_\bullet \in \mathit{Flags}(\mathbb{C}^n) \mid NV_i \subseteq V_{i-1} \text{ for all } 1 \le i \le n\}$$

where $V_\bullet$ denotes a nested sequence

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{C}^n$$

of subspaces of $\mathbb{C}^n$ and $\dim_{\mathbb{C}} V_i = i$ for all i. When N consists of two Jordan blocks of sizes $n-k$ and k with $n \ge 2k$, we denote $\mathcal{S}_N$ by $\mathcal{S}_{(n-k,k)}$. The cohomology ring of Springer variety $\mathcal{S}_N$ has been much studied due to its relation to representations of the permutation group on n letters ([5], [6]). In fact, the ordinary cohomology ring $H^*(\mathcal{S}_N;\mathbb{Q})$ is known to be the quotient of a polynomial ring by an ideal called Tanisaki's ideal ([7]). In this paper we study the equivariant cohomology ring of $\mathcal{S}_{(n-k,k)}$ with respect to a certain circle action on $\mathcal{S}_N$ which we describe below.

Recall that the n-dimensional compact torus T consisting of diagonal unitary matrices of size n acts on $\mathit{Flags}(\mathbb{C}^n)$ in a natural way. A certain circle subgroup S of T

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leaves $\mathcal{S}_N$ invariant (cf. Section 2). The ring homomorphism

$$H_T^*(Flags(\mathbb{C}^n); \mathbb{Q}) \to H_S^*(\mathcal{S}_N; \mathbb{Q})$$

induced from the inclusions of $\mathcal{S}_N$ into $Flags(\mathbb{C}^n)$ and S into T is known to be surjective (cf. [3]). The main result of this paper is an explicit presentation of $H_S^*(\mathcal{S}_{(n-k,k)};\mathbb{Q})$ as a ring using the epimorphism above (Theorem 3.3). In related work, Dewitt and Harada [1] give a module basis of $H_S^*(\mathcal{S}_{(n-k,k)}; \mathbb{Q})$ over $H^*(BS; \mathbb{Q})$ when $k = 2$ from the viewpoint of Schubert calculus.

Finally, since the restriction map

$$H_S^*(\mathcal{S}_N; \mathbb{Q}) \to H^*(\mathcal{S}_N; \mathbb{Q})$$

is also known to be surjective for any nilpotent operator N, our presentation of $H_S^*(\mathcal{S}_{(n-k,k)}; \mathbb{Q})$ yields a presentation of $H^*(\mathcal{S}_{(n-k,k)}; \mathbb{Q})$ as a ring (Corollary 3.4). However, the resulting presentation is slightly different from the one given in [7].

This paper is organized as follows. We briefly recall the necessary background in Section 2. Our main theorem, Theorem 3.3, is formulated in Section 3 and proved in Section 4.

2. Nilpotent Springer varieties and S^1-fixed points

We begin by recalling the definition of the nilpotent Springer varieties in type A. Since we work exclusively with type A in this paper, we henceforth omit it from our terminology.

The flag variety $Flags(\mathbb{C}^n)$ is the projective variety of nested subspaces in $\mathbb{C}^n$, i.e.

$$Flags(\mathbb{C}^n) = \{V_\bullet = (0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{C}^n) \mid \dim_{\mathbb{C}} V_i = i\}.$$

DEFINITION. Let $N\colon \mathbb{C}^n \to \mathbb{C}^n$ be a nilpotent operator. The (*nilpotent*) *Springer variety* $\mathcal{S}_N$ associated to N is defined as

$$\mathcal{S}_N = \{V_\bullet \in Flags(\mathbb{C}^n) \mid NV_i \subseteq V_{i-1} \text{ for all } 1 \leq i \leq n\}.$$

Since $\mathcal{S}_{gNg^{-1}}$ is homeomorphic (in fact, isomorphic as algebraic varieties) to $\mathcal{S}_N$ for any $g \in GL_n(\mathbb{C})$, we may assume that N is in Jordan canonical form with Jordan blocks of weakly decreasing sizes. Let λ_N denote the partition of n with entries the sizes of the Jordan blocks of N. The n-dimensional torus T consisting of diagonal unitary matrices of size n acts on $Flags(\mathbb{C}^n)$ in a natural way and the circle subgroup

S of T defined as

$$S = \left\{ \begin{pmatrix} g & & & \\ & g^2 & & \\ & & \ddots & \\ & & & g^n \end{pmatrix} \middle| \, g \in \mathbb{C}, \ |g| = 1 \right\} \tag{2.1}$$

leaves $\mathcal{S}_N \subseteq Flags(\mathbb{C}^n)$ invariant (see [2]). The T-fixed point set $Flags(\mathbb{C}^n)^T$ of $Flags(\mathbb{C}^n)$ is given by

$$\{(\langle e_{w(1)}\rangle \subset \langle e_{w(1)}, e_{w(2)}\rangle \subset \cdots \subset \langle e_{w(1)}, e_{w(2)}, \ldots, e_{w(n)}\rangle = \mathbb{C}^n) \mid w \in S_n\}$$

where $e_1, e_2, \ldots, e_n$ is the standard basis of $\mathbb{C}^n$ and S_n is the permutation group on n letters $\{1, 2, \ldots, n\}$, so we identify $Flags(\mathbb{C}^n)^T$ with S_n as is standard. Also, since the S-fixed point set $Flags(\mathbb{C}^n)^S$ of $Flags(\mathbb{C}^n)$ agrees with $Flags(\mathbb{C}^n)^T$, we have

$$\mathcal{S}_N^S = \mathcal{S}_N \cap Flags(\mathbb{C}^n)^S = \mathcal{S}_N \cap Flags(\mathbb{C}^n)^T \subset S_n.$$

We denote by $\mathcal{S}_{(n-k,k)}$ the Springer variety corresponding to the partition $\lambda_N = (n-k, k)$ with $2k \leq n$. We next describe the S-fixed points in $\mathcal{S}_{(n-k,k)}$. Let $w_{l_1,l_2,\ldots,l_k}$ be an element of S_n defined by

$$w_{l_1,l_2,\ldots,l_k}(i) = \begin{cases} n-k+j & \text{if} \quad i = l_j, \\ i-j & \text{if} \quad l_j < i < l_{j+1}, \end{cases} \tag{2.2}$$

where $l_0 := 0$, $l_{k+1} := n+1$. Note that $w^{-1}_{l_1,l_2,\ldots,l_k}(i) < w^{-1}_{l_1,l_2,\ldots,l_k}(i')$ if $1 \leq i < i' \leq n-k$ or $n-k+1 \leq i < i' \leq n$.

EXAMPLE. Take $n = 4$ and $k = 2$. Using one-line notation, the set of permutations of the form described in (2.2) are as follows:

$$[3, 4, 1, 2], [3, 1, 4, 2], [3, 1, 2, 4], [1, 3, 4, 2], [1, 3, 2, 4], [1, 2, 3, 4].$$

Lemma 2.1. *The S-fixed points $\mathcal{S}^S_{(n-k,k)}$ of the Springer variety $\mathcal{S}_{(n-k,k)}$ is the set*

$$\{w_{l_1,l_2,\ldots,l_k} \in S_n \mid 1 \leq l_1 < l_2 < \cdots < l_k \leq n\}.$$

Proof. Since $\mathcal{S}^S_{(n-k,k)} \subset Flags(\mathbb{C}^n)^T$, any element $V_\bullet$ of $\mathcal{S}^S_{(n-k,k)}$ is of the form

$$V_\bullet = (\langle e_{w(1)}\rangle \subset \langle e_{w(1)}, e_{w(2)}\rangle \subset \cdots \subset \langle e_{w(1)}, e_{w(2)}, \ldots, e_{w(n)}\rangle)$$

for some $w \in S_n$. Since N is the nilpotent operator consisting of two Jordan blocks with weakly decreasing sizes $(n-k, k)$,

$$Ne_i = \begin{cases} 0 & \text{if} \quad i = 1 \quad \text{or} \quad n-k+1, \\ e_{i-1} & \text{otherwise.} \end{cases}$$

Therefore, if $V_\bullet$ belongs to $\mathcal{S}_{(n-k,k)}$, then $w(1) = 1$ or $n-k+1$. If $w(1) = 1$ then $w(2) = 2$ or $n-k+1$. If $w(1) = n-k+1$ then $w(2) = 1$ or $n-k+2$, and so on. This shows that $w = w_{l_1,l_2,\ldots,l_k}$ for some $1 \le l_1 < l_2 < \cdots < l_k \le n$. Conversely, one can easily see that $w_{l_1,l_2,\ldots,l_k} \in \mathcal{S}^S_{(n-k,k)}$. □

3. Main theorem

In this section, we formulate our main theorem which gives an explicit presentation of the S-equivariant cohomology ring of the $(n-k,k)$ Springer variety.

First, we recall an explicit presentation of the T-equivariant cohomology ring of the flag variety. Let E_i be the subbundle of the trivial vector bundle $Flags(\mathbb{C}^n) \times \mathbb{C}^n$ over $Flags(\mathbb{C}^n)$ whose fiber at a flag $V_\bullet$ is just V_i. We denote the T-equivariant first Chern class of the line bundle E_i/E_{i-1} by $\bar{x}_i \in H^2_T(Flags(\mathbb{C}^n); \mathbb{Q})$. The torus T consisting of diagonal unitary matrices of size n has a natural product decomposition $T \cong (S^1)^n$ where S^1 is the unit circle of $\mathbb{C}$. This decomposition identifies BT with $(BS^1)^n$ and induces an identification

$$H^*_T(pt; \mathbb{Q}) = H^*(BT; \mathbb{Q}) \cong \bigotimes H^*(BS^1; \mathbb{Q}) \cong \mathbb{Q}[t_1, \ldots, t_n],$$

where t_i $(1 \le i \le n)$ denotes the element corresponding to a fixed generator t of $H^2(BS^1;\mathbb{Q})$. Then $H^*_T(Flags(\mathbb{C}^n);\mathbb{Q})$ is generated by $\bar{x}_1,\ldots,\bar{x}_n,t_1,\ldots,t_n$ as a ring. We define a ring homomorphism π from the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$ to $H^*_T(Flags(\mathbb{C}^n); \mathbb{Q})$ by $\pi(x_i) = \bar{x}_i$. It is known that π is an epimorphism and $\operatorname{Ker}\pi$ is generated as an ideal by $e_i(x_1, \ldots, x_n) - e_i(t_1, \ldots, t_n)$ for all $1 \le i \le n$, where e_i is the ith elementary symmetric polynomial. Thus, we have an isomorphism:

$$\begin{aligned}&H^*_T(Flags(\mathbb{C}^n); \mathbb{Q})\\ &\cong \mathbb{Q}[x_1, \ldots, x_n, t_1, \ldots, t_n]/(e_i(x_1, \ldots, x_n) - e_i(t_1, \ldots, t_n),\ 1 \le i \le n).\end{aligned}$$

We consider the following commutative diagram:

$$\begin{array}{ccc} H^*_T(Flags(\mathbb{C}^n); \mathbb{Q}) & \xrightarrow{\iota_1} & H^*_T(Flags(\mathbb{C}^n)^T; \mathbb{Q}) = \bigoplus_{w \in S_n} \mathbb{Q}[t_1, \ldots, t_n] \\ {\scriptstyle\pi_1}\downarrow & & \downarrow{\scriptstyle\pi_2} \\ H^*_S(\mathcal{S}_N; \mathbb{Q}) & \xrightarrow{\iota_2} & H^*_S(\mathcal{S}_N^S; \mathbb{Q}) = \bigoplus_{w \in \mathcal{S}_N^S \subset S_n} \mathbb{Q}[t] \end{array} \tag{3.1}$$

where all the maps are induced from inclusion maps, and we have an identification

$$H^*_S(pt; \mathbb{Q}) = H^*(BS; \mathbb{Q}) \cong H^*(BS^1; \mathbb{Q}) \cong \mathbb{Q}[t]$$

where we identify S with S^1 through the map $\operatorname{diag}(g, g^2, \ldots, g^n) \mapsto g$. The maps ι_1 and ι_2 in (3.1) are injective since the odd degree cohomology groups of $Flags(\mathbb{C}^n)$ and

$\mathcal{S}_N$ vanish. The map π_1 in (3.1) is known to be surjective (cf. [3]) and the map π_2 is obviously surjective. Since π_1 is surjective, we have the following lemma. Let τ_i be the image $\pi_1(\bar{x}_i)$ of $\bar{x}_i$ for each i.

Lemma 3.1. *The S-equivariant cohomology ring $H_S^*(\mathcal{S}_N;\mathbb{Q})$ is generated by $\tau_1,\dots,\tau_n, t$ as a ring where τ_i is the image of $\bar{x}_i$ under the map π_1 in* (3.1).

We next consider relations between $\tau_1, \dots, \tau_n$, and t. We have

$$\iota_2(\tau_i)|_w = w(i)t$$

because $\iota_1(\bar{x}_i)|_w = t_{w(i)}$, $\iota_1(t_i)|_w = t_i$, and $\pi_2(t_i) = it$, where $f|_w$ denotes the w-component of $f \in \bigoplus_{w\in S_n} \mathbb{Q}[t_1, \dots, t_n]$.

Lemma 3.2. *The elements $\tau_1, \dots, \tau_n, t$ satisfy the following relations*:

$$\sum_{1\le i\le n} \tau_i - \frac{n(n+1)}{2}t = 0, \tag{3.2}$$

$$(\tau_i + \tau_{i-1} - (n-k+i)t)(\tau_i - \tau_{i-1} - t) = 0 \quad (1 \le i \le n), \tag{3.3}$$

$$\prod_{0\le j\le k} (\tau_{i_j} - (i_j - j)t)) = 0 \quad (1 \le i_0 < \dots < i_k \le n), \tag{3.4}$$

where $\tau_0 = 0$.

Proof. The relation (3.2) follows from a relation in $H_T^*(Flags(\mathbb{C}^n);\mathbb{Q})$. In fact,

$$\sum_{1\le i\le n} \tau_i - \frac{n(n+1)}{2}t = \pi_1((e_1(\bar{x}_1, \dots, \bar{x}_n) - e_1(t_1, \dots, t_n))) = 0.$$

In the following, we denote $\iota_2(\tau_i)$ by the same notation τ_i for each i. To prove the relation (3.3), it is sufficient to prove either

$$(\tau_i + \tau_{i-1} - (n-k+i)t)|_{w_{l_1,l_2,\dots,l_k}} = 0 \quad \text{or} \quad (\tau_i - \tau_{i-1} - t)|_{w_{l_1,l_2,\dots,l_k}} = 0 \tag{3.5}$$

for any $w_{l_1,l_2,\dots,l_k} \in \mathcal{S}^S_{(n-k,k)}$ since the restriction map ι_2 in (3.1) is injective.

We first treat the case $i = 1$. By the definition of $w_{l_1,l_2,\dots,l_k}$ in (2.2) the following holds:

$$\tau_1|_{w_{l_1,l_2,\dots,l_k}} = w_{l_1,l_2,\dots,l_k}(1)t = \begin{cases} (n-k+1)t & \text{if} \quad l_1 = 1, \\ t & \text{if} \quad l_1 \ne 1. \end{cases}$$

This shows (3.5) for $i = 1$ because $\tau_0 = 0$.

We now treat the case $1 < i \leq n$. Note that

$$(\tau_i - \tau_{i-1})|_{w_{l_1,l_2,\ldots,l_k}} = (w_{l_1,l_2,\ldots,l_k}(i) - w_{l_1,l_2,\ldots,l_k}(i-1))t, \tag{3.6}$$

$$(\tau_i + \tau_{i-1})|_{w_{l_1,l_2,\ldots,l_k}} = (w_{l_1,l_2,\ldots,l_k}(i) + w_{l_1,l_2,\ldots,l_k}(i-1))t. \tag{3.7}$$

We take four cases depending on whether $i-1$ and i appear in $l_1, \ldots, l_k$ or not.

(i) If $l_j = i-1 < i = l_{j+1}$ for some $1 \leq j \leq k-1$, then by (2.2) and (3.6),

$$(\tau_i - \tau_{i-1})|_{w_{l_1,l_2,\ldots,l_k}} = ((n-k+j+1) - (n-k+j))t = t.$$

(ii) If $l_j < i-1 < i < l_{j+1}$ for some $0 \leq j \leq k$, then by (2.2) and (3.6),

$$(\tau_i - \tau_{i-1})|_{w_{l_1,l_2,\ldots,l_k}} = ((i-j) - (i-j-1))t = t.$$

(iii) If $l_j = i-1 < i < l_{j+1}$ for some $1 \leq j \leq k$, then by (2.2) and (3.7),

$$(\tau_i + \tau_{i-1})|_{w_{l_1,l_2,\ldots,l_k}} = ((i-j) + (n-k+j))t = (n-k+i)t.$$

(iv) If $l_{j-1} < i-1 < i = l_j$ for some $1 \leq j \leq k$, then by (2.2) and (3.7),

$$(\tau_i + \tau_{i-1})|_{w_{l_1,l_2,\ldots,l_k}} = ((n-k+j) + (i-j))t = (n-k+i)t.$$

Therefore, (3.5) holds in all cases, proving the relations (3.3).

Finally we prove the relations (3.4). For any $w_{l_1,l_2,\ldots,l_k} \in \mathcal{S}^S_{(n-k,k)}$, there is a positive integer i_j such that $l_j < i_j < l_{j+1}$ for some $0 \leq j \leq k$. Thus, we have

$$w_{l_1,l_2,\ldots,l_k}(i_j) = i_j - j.$$

This means that

$$\prod_{0 \leq j \leq k} (\tau_{i_j} - (i_j - j)t)|_{w_{l_1,l_2,\ldots,l_k}} = 0.$$

Therefore, the relations (3.4) hold, and the proof is complete. □

It follows from Lemma 3.2 that we obtain a well-defined ring homomorphism

$$\varphi\colon \mathbb{Q}[x_1, \ldots, x_n, t]/I \to H^*_S(\mathcal{S}_{(n-k,k)}; \mathbb{Q}) \tag{3.8}$$

where I is the ideal of a polynomial ring $\mathbb{Q}[x_1, \ldots, x_n, t]$ generated by the following three types of elements:

$$\sum_{1 \leq i \leq n} x_i - \frac{n(n+1)}{2}t, \tag{3.9}$$

$$(x_i + x_{i-1} - (n-k+i)t)(x_i - x_{i-1} - t) \quad (1 \leq i \leq n), \tag{3.10}$$

$$\prod_{0 \leq j \leq k} (x_{i_j} - (i_j - j)t) \quad (1 \leq i_0 < \cdots < i_k \leq n), \tag{3.11}$$

where $x_0 = 0$. Moreover, φ is surjective by Lemma 3.1.

The following is our main theorem and will be proved in the next section.

Theorem 3.3. *Let $\mathcal{S}_{(n-k,k)}$ be the $(n-k,k)$ Springer variety with $0 \le k \le n/2$ and let the circle group S act on $\mathcal{S}_{(n-k,k)}$ as described in* Section 2. *Then the S-equivariant cohomology ring of $\mathcal{S}_{(n-k,k)}$ is given by*

$$H_S^*(\mathcal{S}_{(n-k,k)}; \mathbb{Q}) \cong \mathbb{Q}[x_1, \dots, x_n, t]/I$$

where $H_S^*(pt; \mathbb{Q}) = \mathbb{Q}[t]$ and I is the ideal of the polynomial ring $\mathbb{Q}[x_1, \dots, x_n, t]$ generated by the elements listed in (3.9), (3.10), *and* (3.11).

Since the ordinary cohomology ring of $\mathcal{S}_{(n-k,k)}$ can be obtained by taking $t = 0$ in Theorem 3.3, we obtain the following corollary.

Corollary 3.4. *Let $\mathcal{S}_{(n-k,k)}$ be $(n-k,k)$ Springer variety with $0 \le k \le n/2$. Then the ordinary cohomology ring of $\mathcal{S}_{(n-k,k)}$ is given by*

$$H^*(\mathcal{S}_{(n-k,k)}; \mathbb{Q}) \cong \mathbb{Q}[x_1, \dots, x_n]/J$$

where J is the ideal of the polynomial ring $\mathbb{Q}[x_1, \dots, x_n]$ generated by the following three types of elements:

$$\sum_{1 \le i \le n} x_i,$$
$$x_i^2 \quad (1 \le i \le n),$$
$$\prod_{1 \le j \le k+1} x_{i_j} \quad (1 \le i_1 < \cdots < i_{k+1} \le n).$$

REMARK. A ring presentation of the cohomology ring of the Springer variety $\mathcal{S}_N$ is given in [7] for an arbitrary nilpotent operator N. Specifically, it is the quotient of a polynomial ring by an ideal called Tanisaki's ideal. When $\lambda_N = (n-k,k)$, Tanisaki's ideal is generated by the following three types of elements:

$$e_1(x_1, \dots, x_n),$$
$$e_2(x_{i_1}, \dots, x_{i_{n-1}}) \quad (1 \le i_1 < \cdots < i_{n-1} \le n),$$
$$e_{k+1}(x_{i_1}, \dots, x_{i_{k+1}}) \quad (1 \le i_1 < \cdots < i_{k+1} \le n),$$

where e_i is the ith elementary symmetric polynomial. Note that the first and third elements above are the same as those in Corollary 3.4. In fact, one can easily check that Tanisaki's ideal above agrees with the ideal J in Corollary 3.4 although the generators are slightly different.

4. Proof of the main theorem

This section is devoted to the proof of Theorem 3.3. More precisely, we will prove that the epimorphism φ in (3.8) is an isomorphism. For this, we first find generators of $\mathbb{Q}[x_1, \ldots, x_n, t]/I$ as a $\mathbb{Q}[t]$-module.

Recall that a *filling* of λ by the alphabet $\{1, \ldots, n\}$ is an injective placing of the integers $\{1, \ldots, n\}$ into the boxes of λ.

DEFINITION. Let λ be a Young diagram with n boxes. A filling of λ is a *permissible filling* if for every horizontal adjacency $\boxed{a \mid b}$ we have $a < b$. Also, a permissible filling is a *standard tableau* if for every vertical adjacency $\begin{array}{|c|}\hline a \\ \hline b \\ \hline \end{array}$ we have $a < b$.

Let T be a permissible filling of $(n-l, l)$ with $0 \leq l \leq k$. Let $j_1, j_2, \ldots, j_l$ be the numbers in the bottom row of T. We define $x_T := x_{j_1} x_{j_2} \cdots x_{j_l}$ and $x_{T_0} := 1$ where T_0 is the standard tableau on (n).

Proposition 4.1. *The set* $\{x_T \mid T$ *standard tableau on* $(n-l, l)$ *with* $0 \leq l \leq k\}$ *generates* $\mathbb{Q}[x_1, \ldots, x_n, t]/I$ *as a* $\mathbb{Q}[t]$*-module.*

Proof. It is sufficient to prove that $x_{b_1} x_{b_2} \cdots x_{b_l}$ $(1 \leq b_1 \leq b_2 \leq \cdots \leq b_l \leq n)$ can be written in $\mathbb{Q}[x_1, \ldots, x_n, t]/I$ as a $\mathbb{Q}[t]$-linear combination of the x_T where T is a standard tableau. We prove this by induction on l. The base case $l = 0$ is clear. Now we assume that $l \geq 1$ and the claim holds for $l - 1$. The relations (3.10) imply that

$$x_i^2 = (n-k+i+1)t x_i + t \sum_{1 \leq p \leq i-1} x_p - \sum_{1 \leq p \leq i} (n-k+p)t^2 \quad (1 \leq i \leq n) \tag{4.1}$$

by an inductive argument on i, so we may assume $b_1 < b_2 < \cdots < b_l$.

To prove the claim for l, we consider two cases: $1 \leq l \leq k$ and $l \geq k+1$.

CASE (i) Suppose $1 \leq l \leq k$. We write $x_{b_1} x_{b_2} \cdots x_{b_l} = x_U$ where

$$U = \begin{array}{|c|c|c|c|c|c|}\hline a_1 & \cdots & a_l & a_{l+1} & \cdots & a_{n-l} \\ \hline b_1 & \cdots & b_l \\ \cline{1-3} \end{array}$$

is a permissible filling of $(n-l, l)$. Let j be the minimal positive integer in the set $\{r \mid a_r > b_r,\ 1 \leq r \leq l\}$, i.e.,

$$a_i < b_i \quad (1 \leq i < j), \tag{4.2}$$

$$a_j > b_j. \tag{4.3}$$

We consider the following equation which follows from the relation (3.9):

$$\begin{aligned}&(-x_{a_1}-x_{a_2}-\cdots-x_{a_{j-1}})^j\cdot x_{b_{j+1}}\cdots x_{b_l}\\ &=\left(x_{b_1}+x_{b_2}+\cdots+x_{b_l}+x_{a_j}+x_{a_{j+1}}+\cdots+x_{a_{n-l}}-\frac{n(n+1)}{2}t\right)^j\cdot x_{b_{j+1}}\cdots x_{b_l}.\end{aligned}\tag{4.4}$$

Claim 1. *The left hand side in* (4.4) *is a* $\mathbb{Q}[t]$*-linear combination of the* x_T *where the* T *are standard tableaux.*

Proof. We expand the left hand side in (4.4). Then any monomial which appears in the expansion is of the form

$$x_{a_1}^{\alpha_1}\cdots x_{a_{j-1}}^{\alpha_{j-1}}x_{b_{j+1}}\cdots x_{b_l}$$

where $\sum_{i=1}^{j-1}\alpha_i=j$ and $\alpha_i\geq 0$. Note that $\alpha_i>1$ for some i since $\sum_{i=1}^{j-1}\alpha_i=j$ and $\alpha_i\geq 0$. Therefore, using the relations (4.1), the monomial above turns into a sum of elements of the form

$$f(t)\cdot x_{c_1}\cdots x_{c_h}$$

where $h<l$, $1\leq c_1<\cdots<c_h\leq n$, and $f(t)\in\mathbb{Q}[t]$, and by the induction assumption the term above can be written as a $\mathbb{Q}[t]$-linear combination of the x_T where T is a standard tableau. This proves Claim 1. □

Claim 2. *The right hand side in* (4.4) *can be written as a* $\mathbb{Q}[t]$*-linear combination of* x_U *and monomials* x_T *and* $x_{U'}$ *where the coefficient of* x_U *is equal to* 1, T *is a standard tableau on shape* $(n-l,l)$ *and* U' *is a permissible filling of* $(n-l,l)$ *such that each of the leftmost* j *columns are strictly increasing* (*i.e.* $a_r<b_r$, $1\leq r\leq j$).

Proof. We expand the right hand side in (4.4). A monomial which appears in this expansion is of the form

$$x_{b_{p_1}}^{\beta_1}\cdots x_{b_{p_m}}^{\beta_m}x_{a_{q_1}}^{\alpha_1}\cdots x_{a_{q_h}}^{\alpha_h}x_{b_{j+1}}\cdots x_{b_l}$$

where $\sum_{i=1}^m\beta_i+\sum_{i=1}^h\alpha_i\leq j$, $\beta_i\geq 1$, $\alpha_i\geq 1$ and $1\leq p_1<\cdots<p_m\leq l$, $j\leq q_1<\cdots<q_h\leq n-l$. It is enough to consider the case $\sum_{i=1}^m\beta_i+\sum_{i=1}^h\alpha_i=j$ since if $\sum_{i=1}^m\beta_i+\sum_{i=1}^h\alpha_i<j$ then it follows from the induction assumption that the above form can be written as a $\mathbb{Q}[t]$-linear combination of the x_T where T is a standard tableau. If $p_m\geq j+1$ or some β_i or α_i is more than 1, then it follows from the relations (4.1) and the induction assumption that the monomial above can be written as a linear combination of x_T's over $\mathbb{Q}[t]$ where T is a standard tableau. If $p_m\leq j$ and all β_i and α_i are equal to 1, then $h=j-m$ and the monomial above is of the form

$$x_{b_{p_1}}\cdots x_{b_{p_m}}x_{a_{q_1}}\cdots x_{a_{q_{j-m}}}x_{b_{j+1}}\cdots x_{b_l}$$

where $1 \leq p_1 < \cdots < p_m \leq j \leq q_1 < \cdots < q_{j-m} \leq n-l$. This monomial is associated to a permissible filling U' given by

$$U' = \begin{array}{|c|c|c|c|c|c|} \hline c_1 & \cdots & c_l & c_{l+1} & \cdots & c_{n-l} \\ \hline d_1 & \cdots & d_l \\ \cline{1-3} \end{array}$$

where

$$d_i = \begin{cases} b_{p_i} & \text{if} \quad 1 \leq i \leq m, \\ \min\{\{a_{q_1}, \ldots, a_{q_{j-m}}, b_{j+1}, \ldots, b_l\} - \{d_{m+1}, \ldots, d_{i-1}\}\} & \text{if} \quad m < i \leq l, \end{cases}$$

and

$$c_i = \min\{\{a_1, \ldots, a_{n-l}, b_1, \ldots, b_j\} - \{a_{q_1}, \ldots, a_{q_{j-m}}, b_{p_1}, \ldots, b_{p_m}, c_1, \ldots, c_{i-1}\}\}$$

for $1 \leq i \leq n-l$. Note that $x_{U'} = x_U$ if and only if $m = j$, since $m = j \Leftrightarrow d_i = b_i$ for $1 \leq i \leq l$. We consider the case $m < j$. Since $j \leq q_1$ and $a_j > b_j$ by (4.3), we have

$$c_i = \min\{\{a_1, \ldots, a_{j-1}, b_1, \ldots, b_j\} - \{b_{p_1}, \ldots, b_{p_m}, c_1, \ldots, c_{i-1}\}\}$$

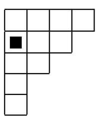
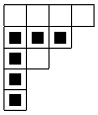
for $1 \leq i \leq j$. If $1 \leq i \leq m$, we have $c_i \leq a_i < b_i \leq b_{p_i} = d_i$. If $m < i \leq j$, we have $c_i \leq \max\{a_{j-1}, b_j\} < \min\{a_j, b_{j+1}\} \leq d_i$ by (4.2), (4.3), and $j \leq q_1$. Thus, U' is a permissible filling of $(n-l, l)$ such that each of the leftmost j columns are strictly increasing (i.e. $a_r < b_r$, $1 \leq r \leq j$). This proves Claim 2. □

Claims 1 and 2 show that x_U can be written as a $\mathbb{Q}[t]$-linear combination of $x_{U'}$ and x_T, where U' and T are as above. Applying the above discussion for $x_{U'}$ in place of x_U, we see that $x_{U'}$ can be written as a $\mathbb{Q}[t]$-linear combination of $x_{U''}$ and x_T where U'' is a permissible filling of $(n-l, l)$ such that each of the leftmost $j+1$ columns are strictly increasing (i.e. $a_r < b_r$, $1 \leq r \leq j+1$) and T is a standard tableau. Repeating this procedure, we can finally express x_U as a $\mathbb{Q}[t]$-linear combination of the x_T where T is a standard tableau.

CASE (ii) If $l \geq k+1$, it follows from the relations (3.11) and the induction assumption that $x_{b_1} x_{b_2} \cdots x_{b_l}$ can be expressed as a $\mathbb{Q}[t]$-linear combination of the x_T where T is a standard tableau.

This completes the induction step and proves the proposition. □

Recall that for a box b in the ith row and jth column of a Young diagram λ, $h(i, j)$ denote the number of boxes in the hook formed by the boxes below b in the jth column, the boxes to the right of b in the ith row, and b itself.

EXAMPLE. For the Young diagram  and the box in the (2, 1) location, the hook is  and $h(2, 1) = 6$.

Lemma 4.2. *Let λ be a Young diagram. Let f^{λ} denote the number of standard tableaux on λ. Then*

$$\binom{n}{k} = \sum_{0 \le l \le k} f^{(n-l,l)}.$$

Proof. We prove the lemma by induction on k. As the case $k = 0$ is clear, we assume that $k \ge 1$ and that the lemma holds for $k - 1$. We use the following hook length formula:

$$f^{\lambda} = \frac{n!}{\Pi_{(i,j)\in\lambda} h(i, j)}.$$

Using the induction assumption and the hook length formula, we have

$$\begin{aligned}\sum_{0 \le l \le k} f^{(n-l,l)} &= \sum_{0 \le l \le k-1} f^{(n-l,l)} + f^{(n-k,k)} \\ &= \binom{n}{k-1} + \frac{n!\,(n-2k+1)}{(n-k+1)!\,k!} \\ &= \binom{n}{k}.\end{aligned}$$

This completes the induction step and proves the lemma. □

It follows from Proposition 4.1 and Lemma 4.2 that

$$\operatorname{rank}_{\mathbb{Q}[t]} \mathbb{Q}[x_1, \ldots, x_n, t]/I \le \sum_{0 \le l \le k} f^{(n-l,l)} = \binom{n}{k}.$$

On the other hand, since the odd degree cohomology groups of $\mathcal{S}_N$ vanish, we have an isomorphism $H_S^*(\mathcal{S}_N; \mathbb{Q}) \cong \mathbb{Q}[t] \otimes H^*(\mathcal{S}_N; \mathbb{Q})$ as $\mathbb{Q}[t]$-modules, and the cellular decomposition of $\mathcal{S}_N$ given by Spaltenstein [4] (cf. also Hotta–Springer [3]) implies that

$$\dim H^*(\mathcal{S}_N; \mathbb{Q}) = \binom{n}{\lambda_N} := \binom{n}{\lambda_1!\,\lambda_2! \cdots \lambda_r!}$$

where $\lambda_N = (\lambda_1, \lambda_2, \ldots, \lambda_r)$. These show

$$\operatorname{rank}_{\mathbb{Q}[t]} H_S^*(\mathcal{S}_{(n-k,k)}; \mathbb{Q}) = \dim_{\mathbb{Q}} H^*(\mathcal{S}_{(n-k,k)}; \mathbb{Q}) = \binom{n}{k}.$$

Therefore, we have

$$\mathrm{rank}_{\mathbb{Q}[t]}\,\mathbb{Q}[x_1,\ldots,x_n,t]/I \le \mathrm{rank}_{\mathbb{Q}[t]}\,H_S^*(\mathcal{S}_{(n-k,k)};\mathbb{Q}).$$

This means that the epimorphism φ in (3.8) is actually an isomorphism, proving Theorem 3.3.

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