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FREE PRODUCT OF TWO ELLIPTIC QUATERNIONIC MÖBIUS TRANSFORMATIONS

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Abstract

Suppose that f and g are two elliptic quaternionic Möbius transformations of orders m and n respectively. If the hyperbolic distance $\delta(f,g)$ between $\mathrm{fix}(f)$ and $\mathrm{fix}(g)$ satisfies

$$\cosh\delta(f,g)\geq\frac{\cos\frac{\pi}{m}\cos\frac{\pi}{n}+1}{\sin\frac{\pi}{m}\sin\frac{\pi}{n}},$$

then the group $\langle f,g\rangle$ is discrete non-elementary and isomorphic to the free product $\langle f\rangle*\langle g\rangle$.

1. Introduction

Elements of the quaternions $\mathbb{H}$ have the form $q=q_0+q_1\mathbf{i}+q_2\mathbf{j}+q_3\mathbf{k}\in\mathbb{H}$, where $q_i\in\mathbb{R}$ and $\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=\mathbf{ijk}=-1$. Let $\overline{q}=q_0-q_1\mathbf{i}-q_2\mathbf{j}-q_3\mathbf{k}$ and $|q|=\sqrt{\overline{q}q}=\sqrt{q_0^2+q_1^2+q_2^2+q_3^2}$ be the conjugate and modulus of q, respectively. We define $\Re(q)=(q+\overline{q})/2$ and $\Im(q)=(q-\overline{q})/2$.

Let $\mathbf{H}^3=\{x+y\mathbf{i}+t\mathbf{j}:x,y,t\in\mathbb{R},t>0\}$ and $\partial\mathbf{H}^3=\overline{\mathbb{C}}$. We associate with each Möbius transformation

$$f(z)=\frac{az+b}{cz+d},\ \ ad-bc=1,$$

the matrix

$$f=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathrm{SL}(2,\mathbb{C}).$$

The Poincaré extension of f is given by

$$f(x+y\mathbf{i}+t\mathbf{j})=\big(a(x+y\mathbf{i}+t\mathbf{j})+b\big)\big(c(x+y\mathbf{i}+t\mathbf{j})+d\big)^{-1}.$$

For $f\in\mathrm{PSL}(2,\mathbb{C})=\mathrm{SL}(2,\mathbb{C})/\{\pm I_2\}$, we define

$$\mathrm{fix}(f)=\{z\in\overline{\mathbf{H}^3}:f(z)=z\}.$$

It is well known that

$$\mathrm{PSL}(2,\mathbb{C})=\mathrm{Isom}(\mathbf{H}^3).$$

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A Kleinian group is a discrete nonelementary subgroup of isometries of Isom($\mathbf{H}^3$). Equivalently such groups are identified with discrete groups of Möbius transformations of the Riemann sphere $\overline{\mathbb{C}} = \partial\mathbf{H}^3$.

We recall the following normalization of two elliptic elements of PSL$(2,\mathbb{C})$ [6].

Proposition 1.1. *Suppose that f and g are elliptic elements of* PSL$(2,\mathbb{C})$ *of order m and n respectively. Let $\delta = \delta(f,g)$ be the hyperbolic distance between* fix(f) *and* fix(g) *and $\phi = \phi(f,g)$ the angle between the spheres or hyperplanes which contain* fix(f) *or* fix(g) *and the common perpendicular of* fix(f) *and* fix(g)*. Then there is an $h \in$* PSL$(2,\mathbb{C})$ *such that*

$$hfh^{-1} = \begin{pmatrix} \cos\frac{\pi}{m} & \mathbf{i}\omega\sin\frac{\pi}{m} \\ \mathbf{i}\sin\frac{\pi}{m}/\omega & \cos\frac{\pi}{m} \end{pmatrix}$$

and

$$hgh^{-1} = \begin{pmatrix} \cos\frac{\pi}{n} & \mathbf{i}\sin\frac{\pi}{n}/\omega \\ \mathbf{i}\omega\sin\frac{\pi}{n} & \cos\frac{\pi}{n} \end{pmatrix},$$

where $\omega^2 = e^{\delta+\mathbf{i}\phi}$.

Based on the above observation, Gehring, Maclachlan and Martin [6] proved the following theorem.

Theorem GMM *Suppose that f and g are elliptic elements of* PSL$(2,\mathbb{C})$ *of order m and n respectively. If*

$$\cosh\delta(f,g) \geq \frac{\cos\frac{\pi}{m}\cos\frac{\pi}{n}+1}{\sin\frac{\pi}{m}\sin\frac{\pi}{n}},$$

then $\langle f,g\rangle$ is discrete non-elementary and isomorphic to the free product $\langle f\rangle * \langle g\rangle$.

In this paper we regard $\Im(\mathbb{H})\times\mathbb{R}^+$ as the model of $\mathbf{H}^4$ and identify Isom($\mathbf{H}^4$) with the group PSp$(1,1)$. The main aim of this paper is to establish an analog of Theorem GMM in PSp$(1,1)$. Our main result is the following theorem.

Theorem 1.1. *Suppose that f and g are elliptic elements of* PSp$(1,1)$ *of orders m and n respectively. If the hyperbolic distance $\delta(f,g)$ between* fix(f) *and* fix(g) *satisfies*

$$\cosh\delta(f,g) \geq \frac{\cos\frac{\pi}{m}\cos\frac{\pi}{n}+1}{\sin\frac{\pi}{m}\sin\frac{\pi}{n}},$$

then $\langle f,g\rangle$ is discrete non-elementary and isomorphic to the free product $\langle f\rangle * \langle g\rangle$.

The paper is organized as follows. In Section 2 we shall collect some basic facts in quaternionic hyperbolic geometry. Section 3 is devoted to obtaining some properties of elliptic elements. Section 4 contains the proof of Theorem 1.1.

2. Background

We briefly recall some necessary material on quaternionic hyperbolic geometry here and we refer to [1, 3, 5, 7] for further details.

2.1. Quaternionic Möbius transformations. Let $\mathbb{H}^{1,1}$ be the vector space with the Hermitian form of signature $(1, 1)$ given by

$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* J_1 \mathbf{z} = \overline{w_1} z_2 + \overline{w_2} z_1$$

with the matrix

$$J_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We define

$$\mathrm{Sp}(1, 1) = \{f \in \mathrm{GL}(2, \mathbb{H}) : f^* J_1 f = J_1\}.$$

From this we find $f^{-1} = J_1^{-1} f^* J_1$. That is:

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ f^{-1} = \begin{pmatrix} \overline{d} & \overline{b} \\ \overline{c} & \overline{a} \end{pmatrix}. \tag{1}$$

Use of the identities $ff^{-1} = f^{-1}f = I_2$ proves the following lemma.

Lemma 2.1. $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(1, 1)$ *if and only if*

$$a\overline{d} + b\overline{c} = 1, \ \overline{d}a + \overline{b}c = 1,$$
$$\Re(a\overline{b}) = \Re(c\overline{d}) = \Re(\overline{d}b) = \Re(\overline{c}a) = 0.$$

Following the terminology in [5], we let

$$V_0 = \left\{\mathbf{z} \in \mathbb{H}^{1,1} - \{0\} : \langle \mathbf{z}, \mathbf{z} \rangle = 0\right\}, \ V_- = \left\{\mathbf{z} \in \mathbb{H}^{1,1} : \langle \mathbf{z}, \mathbf{z} \rangle < 0\right\}.$$

It is obvious that V_0 and V_- are invariant under $\mathrm{Sp}(1, 1)$. Let $\mathbb{P} : \mathbb{H}^{1,1} - \{0\} \longrightarrow \mathbb{HP}^1$ be the *right projection* map given by

$$\mathbb{P}(z_1, z_2)^T = z_1 z_2^{-1} \text{ provided } z_2 \neq 0$$

and

$$\mathbb{P}(z_1, 0)^T = \infty, \ \mathbb{P}(0, z_2)^T = o. \tag{2}$$

The Siegel domain model of *quaternionic hyperbolic* 1*-space* is defined to be

$$\mathbf{H}^1_{\mathbb{H}} = \mathbb{P}(V_-), \ \partial\mathbf{H}^1_{\mathbb{H}} = \mathbb{P}(V_0).$$

The Bergman metric on $\mathbf{H}^1_{\mathbb{H}}$ is given by the distance formula

$$\cosh^2 \frac{\rho(z, w)}{2} = \frac{\langle \mathbf{z}, \mathbf{w} \rangle \langle \mathbf{w}, \mathbf{z} \rangle}{\langle \mathbf{z}, \mathbf{z} \rangle \langle \mathbf{w}, \mathbf{w} \rangle}, \tag{3}$$

where $z, w \in \mathbf{H}^1_{\mathbb{H}}, \ \ \mathbf{z} \in \mathbb{P}^{-1}(z), \mathbf{w} \in \mathbb{P}^{-1}(w)$.

The elements f in $\mathrm{Sp}(1, 1)$ act on $\mathbf{H}^n_{\mathbb{H}} \cup \partial\mathbf{H}^n_{\mathbb{H}}$ according to the following formula:

$$f(z) = \mathbb{P} f \mathbb{P}^{-1}(z) = (az + b)(cz + d)^{-1}.$$

We define $\mathrm{PSp}(1, 1) = \mathrm{Sp}(1, 1)/\{\pm I_2\}$, which is the group of holomorphic isometries of $\mathbf{H}^1_{\mathbb{H}}$.

Let $z = v - u = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k} - u$, where $u, X, Y, Z \in \mathbb{R}$. We denote

$$z = (v, u) \in \Im(\mathbb{H}) \times \mathbb{R}.$$

Then $z \in \mathbf{H}^1_{\mathbb{H}}$ if and only if $u > 0$, and $z \in \partial\mathbf{H}^1_{\mathbb{H}}$ if and only if $u = 0$. Thus we have the following identification:

$$\mathbf{H}^1_{\mathbb{H}} = \Im(\mathbb{H}) \times \mathbb{R}^+,$$

where $\mathbb{R}^+ = \{u \in \mathbb{R} : u > 0\}$. Hence we can regard $\mathbf{H}^1_{\mathbb{H}}$ as the upper half-space model of $\mathbf{H}^4$ with the isometric group PSp(1, 1).

In [1], Cao, Parker and Wang considered the classification of quaternionic Möbius transformations in the ball model. The Hermitian form of the ball model is given by

$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* J_2 \mathbf{z} = \overline{w_1} z_1 - \overline{w_2} z_2, \quad \text{where } J_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The cayley transformation C between the ball model and the Siegel domain model is

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{4}$$

That is, $C^* J_2 C = C^{-1} J_2 C = J_1$.

For $f \in \mathrm{PSp}(1, 1)$, we define

$$\mathrm{fix}(f) = \{z \in \overline{\mathbf{H}^1_{\mathbb{H}}} : f(z) = z\}.$$

Following Chen and Greenberg [5], we say that a non-trivial element $f \in \mathrm{PSp}(1, 1)$ is:

(i) *elliptic* if it has a fixed point in $\mathbf{H}^1_{\mathbb{H}}$;
(ii) *parabolic* if it has exactly one fixed point which lies in $\partial\mathbf{H}^1_{\mathbb{H}}$;
(iii) *loxodromic* if it has exactly two fixed points which lie in $\partial\mathbf{H}^1_{\mathbb{H}}$.

2.2. Isometric sphere. Let $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSp}(1, 1)$ with $f(\infty) \neq \infty$. We know that $c \neq 0$ and set

$$r_f = \frac{1}{|c|}.$$

As in [3, 7], we define the *isometric sphere* $I(f)$ of $f \in \mathrm{PSp}(1, 1)$ not fixing ∞ as

$$I(f) = \left\{z \in \overline{\mathbf{H}^1_{\mathbb{H}}} : |cz + d| = 1\right\} = \left\{z \in \overline{\mathbf{H}^1_{\mathbb{H}}} : |z + c^{-1}d| = r_f\right\}.$$

By (1) we have

$$I(f^{-1}) = \left\{z \in \overline{\mathbf{H}^1_{\mathbb{H}}} : |\bar{c}z + \bar{a}| = 1\right\} = \left\{z \in \overline{\mathbf{H}^1_{\mathbb{H}}} : |z + \bar{c}^{-1}\bar{a}| = r_f\right\}.$$

We define

$$\mathrm{Ext}\, I(f) = \left\{z \in \overline{\mathbf{H}^1_{\mathbb{H}}} : |cz + d| > 1\right\}, \mathrm{Int}\, I(f) = \left\{z \in \overline{\mathbf{H}^1_{\mathbb{H}}} : |cz + d| < 1\right\}.$$

The following proposition will be used later on.

Proposition 2.1. *Let $f \in \mathrm{PSp}(1, 1)$ of the form (1) with $f(\infty) \neq \infty$. Then*

(i) $f(I(f)) = I(f^{-1})$;
(ii) $f(\mathrm{Ext}\, I(f)) \subset \mathrm{Int}\, I(f^{-1})$;

(iii) $f(\text{Int}\, I(f)) \subset \text{Ext}\, I(f^{-1})$.

Proof. Let $z \in I(f)$. Then $|cz+d| = 1$. By Lemma 2.1, we have

$$|\bar{c}f(z) + \bar{a}| = \frac{|\bar{c}(az+b) + \bar{a}(cz+d)|}{|cz+d|} = 1.$$

This implies that $f(z) \in I(f^{-1})$, which proves (i). The proofs of (ii) and (iii) follow from similar reasoning. □

3. Some properties of elliptic elements

Let f be an elliptic element in $\text{PSp}(1,1)$ with Λ_0 its negative class of eigenvalues and Λ_1 its positive class. According to [5], there are two types of elliptic elements in $\text{PSp}(1,1)$. If $\Lambda_0 = \Lambda_1$, then f is a boundary elliptic element; if $\Lambda_0 \neq \Lambda_1$, then f is a regular elliptic element. As in [2], we have the following conjugacy classification of elliptic elements in $\text{PSp}(1,1)$.

Lemma 3.1. *Let* $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSp}(1,1)$ *and* $\alpha, \beta, \theta \in [0,\pi]$.

(i) f is a boundary elliptic element if f is conjugate to

$$E(\theta) = \begin{pmatrix} e^{\mathbf{i}\theta} & 0 \\ 0 & e^{\mathbf{i}\theta} \end{pmatrix}, \text{ where } \theta \neq 0, \pi; \tag{5}$$

(ii) f is a regular elliptic element if f is conjugate to

$$E(\alpha,\beta) = \begin{pmatrix} \frac{e^{\mathbf{i}\alpha}+e^{\mathbf{i}\beta}}{2} & \frac{e^{\mathbf{i}\alpha}-e^{\mathbf{i}\beta}}{2} \\ \frac{e^{\mathbf{i}\alpha}-e^{\mathbf{i}\beta}}{2} & \frac{e^{\mathbf{i}\alpha}+e^{\mathbf{i}\beta}}{2} \end{pmatrix}, \text{ where } \alpha \neq \beta. \tag{6}$$

Proof. By conjugation, if necessary, we may assume that f fixes o and ∞. That is $f = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$. Then there is a $\mu \in \mathbb{H}$ of unit modulus such that $\mu a \mu^{-1} = e^{\mathbf{i}\theta}$, where $0 < \theta < \pi$. This implies that $hfh^{-1} = E(\theta)$, where $h = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} \in \text{PSp}(1,1)$. This proves (i).

In the ball model, if f is a regular elliptic element, then f is conjugate to the form

$$\begin{pmatrix} e^{\mathbf{i}\alpha} & 0 \\ 0 & e^{\mathbf{i}\beta} \end{pmatrix}, \text{ where } \alpha \neq \beta.$$

By the connection of the ball model and the Siegel domain model, we have $E(\alpha,\beta) = C\begin{pmatrix} e^{\mathbf{i}\alpha} & 0 \\ 0 & e^{\mathbf{i}\beta} \end{pmatrix}C^{-1}$. This proves (ii). □

By direct computation, we have

$$\text{fix}(E(\theta)) = \{z = X\mathbf{i} - u : u \geq 0\} = \{(v,u) \in \mathbb{R}\mathbf{i} \times \mathbb{R}^{\geq}\}$$

and

$$\text{fix}(E(\alpha,\beta)) = \{1\} = \{(0,1) \in \mathbb{R}\mathbf{i} \times \mathbb{R}^{+}\},$$

where $\mathbb{R}^{\geq} = \{u \in \mathbb{R} : u \geq 0\}$.

Lemma 3.2. *Let g be a regular elliptic element of the form*

$$g = \begin{pmatrix} \frac{e^{\mathbf{i}\alpha}+e^{\mathbf{i}\beta}}{2} & \frac{e^{\mathbf{i}\alpha}-e^{\mathbf{i}\beta}}{2} \\ \frac{e^{\mathbf{i}\alpha}-e^{\mathbf{i}\beta}}{2} & \frac{e^{\mathbf{i}\alpha}+e^{\mathbf{i}\beta}}{2} \end{pmatrix} \in \mathrm{PSp}(1,1), \tag{7}$$

where $\alpha = \frac{\pi}{p}, \beta = \frac{\pi}{q}$, $p \neq q$ and $\mathrm{ord}(g) = n$. Then

$$\frac{|p-q|}{2pq} \geq \frac{1}{n}.$$

Proof. Let C be the cayley transformation given by (4). Then

$$h = C^{-1}gC = \begin{pmatrix} e^{\mathbf{i}\alpha} & 0 \\ 0 & e^{\mathbf{i}\beta} \end{pmatrix}$$

is a regular elliptic element in the ball model. Let the least common multiple of p and q be

$$\mathrm{lcm}(p,q) = l.$$

Without loss of generality, we assume that $p > q$. We only need to consider the following three cases.

Case (1): Both $\frac{l}{p}$ and $\frac{l}{q}$ are odd. In this case, $\mathrm{ord}(g) = l = n$. Let $t = \frac{n}{p}$ and $s = \frac{n}{q}$. Then t and s are odd and $s > t$. This implies that

$$n(p-q) = pq(s-t) \geq 2pq.$$

Case (2): Both $\frac{l}{p}$ and $\frac{l}{q}$ are even. In this case, $\mathrm{ord}(g) = l = n$. Let $t = \frac{n}{p}$ and $s = \frac{n}{q}$. Then t and s are even and $s > t$. This implies that

$$n(p-q) = pq(s-t) \geq 2pq.$$

Case (3): Only one of $\frac{l}{p}$ and $\frac{l}{q}$ is even. In this case, $\mathrm{ord}(g) = 2l = n$. Let $t = \frac{l}{p}$ and $s = \frac{l}{q}$. Then $s > t$. This implies that

$$n(p-q) = 2pq(s-t) \geq 2pq.$$

□

The span of $M \subset \overline{\mathbf{H}^4}$, denoted by $\mathrm{span}(M)$, is the smallest hyperbolic surface containing M. For two elliptic elements of $\mathrm{PSp}(1,1)$, we define $\delta = \delta(f,g)$ to be the hyperbolic distance between $\mathrm{fix}(f)$ and $\mathrm{fix}(g)$.

Definition 3.1. Let f and g be two boundary elliptic elements of $\mathrm{PSp}(1,1)$ with $\mathrm{fix}(f) \cap \mathrm{fix}(g) = \emptyset$. Then there is a unique geodesic γ such that

$$\mathrm{fix}(f) \cap \gamma = \gamma_f,\ \mathrm{fix}(g) \cap \gamma = \gamma_g,\ \rho(\gamma_f, \gamma_g) = \delta(f,g).$$

We define $\phi = \phi(f,g)$ to be the angle between $\mathrm{span}(\mathrm{fix}(f) \cup \gamma)$ and $\mathrm{span}(\mathrm{fix}(g) \cup \gamma)$.

For each pair of boundary elliptic elements f and g of $\mathrm{PSp}(1,1)$ such that $\mathrm{fix}(f) \cap \mathrm{fix}(g) = \emptyset$, we call the four numbers

$$\mathrm{ord}(f), \mathrm{ord}(g), \phi(f,g), \delta(f,g)$$

the parameters of the 2-generator group $\langle f, g\rangle$ and write

$$\mathrm{par}(\langle f, g\rangle) = (\mathrm{ord}(f), \mathrm{ord}(g), \phi(f, g), \delta(f, g)).$$

Lemma 3.3. *If* $\mathrm{fix}(f) \cap \mathrm{fix}(g) = \emptyset$*, then the group* $\langle f, g\rangle$ *generated by two boundary elliptic elements* f *and* g *is uniquely determined (up to conjugacy) by* $\mathrm{par}(\langle f, g\rangle) = (\mathrm{ord}(f), \mathrm{ord}(g), \phi(f, g), \delta(f, g))$.

Proof. Suppose that f and g are two boundary elliptic elements of $\mathrm{PSp}(1,1)$ of order m and n respectively, with $\mathrm{fix}(f) \cap \mathrm{fix}(g) = \emptyset$. Let $\phi = \phi(f, g)$ and $\delta = \delta(f, g)$. Then there is a unique geodesic γ such that

$$\mathrm{fix}(f) \cap \gamma = \gamma_f,\ \mathrm{fix}(g) \cap \gamma = \gamma_g,\ \rho(\gamma_f, \gamma_g) = \delta.$$

Note that $\mathrm{par}(\langle f, g\rangle) = (m, n, \phi, \delta)$ is invariant under conjugation. We may assume that γ is the u-axis. Then $\mathrm{fix}(f)$ and $\mathrm{fix}(g)$ are two semi-2-spheres. Let

$$F_f = \mathrm{fix}(f) \cap \Im(\mathbb{H}),\ F_g = \mathrm{fix}(g) \cap \Im(\mathbb{H}).$$

Then F_f and F_g are two circles on $\Im(\mathbb{H})$. Let $A_f X + B_f Y + C_f Z = 0$ and $A_g X + B_g Y + C_g Z = 0$ be the planes in $\Im(\mathbb{H})$ containing F_f and F_g, respectively. Note that the angle between the two vectors (A_f, B_f, C_f) and (A_g, B_g, C_g) is just the angle $\phi(f, g)$. Due to the action of $\mathrm{Sp}(1)$ by conjugation in $\mathbb{H}$ coincides with the action of $\mathrm{SO}(3)$, therefore we can find an $h = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} \in \mathrm{PSp}(1,1)$ which maps any two vectors with angle ϕ to two fixed vectors with angle ϕ. This implies that the group $\langle f, g\rangle$ generated by two boundary elliptic elements f and g is uniquely determined (up to conjugacy) by $\mathrm{par}(\langle f, g\rangle) = (m, n, \phi, \delta)$. □

Proposition 3.1. *We have that*

$$\mathrm{par}(\langle f, g\rangle) = (m, n, \phi, \delta)$$

for the following two elliptic elements:

$$f = \begin{pmatrix} \cos\frac{\pi}{m} & \mathbf{k}e^{\delta}\sin\frac{\pi}{m} \\ \mathbf{k}e^{-\delta}\sin\frac{\pi}{m} & \cos\frac{\pi}{m} \end{pmatrix},$$

$$g = \begin{pmatrix} \cos\frac{\pi}{n} & (\mathbf{k}\cos\phi - \mathbf{i}\sin\phi)\sin\frac{\pi}{n} \\ (\mathbf{k}\cos\phi - \mathbf{i}\sin\phi)\sin\frac{\pi}{n} & \cos\frac{\pi}{n} \end{pmatrix}.$$

Proof. Let

$$F = \begin{pmatrix} e^{\mathbf{i}\frac{\pi}{m}} & 0 \\ 0 & e^{\mathbf{i}\frac{\pi}{m}} \end{pmatrix}$$

and

$$G = \begin{pmatrix} \cos\frac{\pi}{n} + (\mathbf{i}\cos\phi + \mathbf{k}\sin\phi)\sin\frac{\pi}{n} & 0 \\ 0 & \cos\frac{\pi}{n} + (\mathbf{i}\cos\phi + \mathbf{k}\sin\phi)\sin\frac{\pi}{n} \end{pmatrix}.$$

Then $F, G \in \mathrm{PSp}(1,1)$ with $\mathrm{ord}(F) = m$ and $\mathrm{ord}(G) = n$. By direct computation, we get

$$\mathrm{fix}(F) = \{(v, u) \in \mathbb{R}\mathbf{i} \times \mathbb{R}^{\geq}\}$$

and

$$\mathrm{fix}(G) = \{z \in \mathbb{R}(\mathbf{i}\cos\phi + \mathbf{k}\sin\phi) : \Re(z) \le 0\} = \{(v,u) \in (\mathbf{i}\cos\phi + \mathbf{k}\sin\phi)\mathbb{R} \times \mathbb{R}^{\ge}\},$$

where $\mathbb{R}(\mathbf{i}\cos\phi + \mathbf{k}\sin\phi)$ is the smallest subfield containing $\mathbb{R}$ and $\mathbf{i}\cos\phi + \mathbf{k}\sin\phi$ [5]. Note that $\mathrm{fix}(F) \cap \Im(\mathbb{H})$ is the X-axis and $\mathrm{fix}(G) \cap \Im(\mathbb{H})$ is the line $Z = \tan\phi X$. The angle between these two lines is ϕ.

Let

$$K_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{j} & 1 \\ 1 & \mathbf{j} \end{pmatrix}, D_\delta = \begin{pmatrix} e^{\delta/2} & 0 \\ 0 & e^{-\delta/2} \end{pmatrix} \in \mathrm{PSp}(1,1).$$

Set

$$f = D_\delta K_1^{-1} F K_1 D_\delta^{-1} = \begin{pmatrix} \cos\frac{\pi}{m} & \mathbf{k}e^{\delta}\sin\frac{\pi}{m} \\ \mathbf{k}e^{-\delta}\sin\frac{\pi}{m} & \cos\frac{\pi}{m} \end{pmatrix},$$

$$g = K_1^{-1} G K_1 = \begin{pmatrix} \cos\frac{\pi}{n} & (\mathbf{k}\cos\phi - \mathbf{i}\sin\phi)\sin\frac{\pi}{n} \\ (\mathbf{k}\cos\phi - \mathbf{i}\sin\phi)\sin\frac{\pi}{n} & \cos\frac{\pi}{n} \end{pmatrix}.$$

We note that $\mathrm{fix}(f)$ is a semi-2-sphere in the XYu space and $\mathrm{fix}(f) \cap \Im(\mathbb{H}) = \{(X,Y) : X^2 + Y^2 = e^{\delta}\}$. We also note that $\mathrm{fix}(g) \cap \Im(\mathbb{H})$ is a circle represented by the parameter equations:

$$X = \frac{-2t\cos\phi}{1+t^2},\ Y = \frac{1-t^2}{1+t^2},\ Z = \frac{-2t\sin\phi}{1+t^2},\ -\infty < t < \infty.$$

Hence $\mathrm{fix}(g) \cap \Im(\mathbb{H})$ lies in the plane given by

$$X\sin\phi - Z\cos\phi = 0.$$

It is obvious that the u-axis is orthogonal to $\mathrm{fix}(f)$ and $\mathrm{fix}(g)$ and intersects with them in the points e^{δ} and 1, respectively. By (3) and Lemma 3.3, we have $\delta(f,g) = \delta$ and $\phi(f,g) = \phi$. □

Lemma 3.4. *Suppose that F is a boundary elliptic element of order m and G is a regular elliptic element of order n with* $\mathrm{fix}(f) \cap \mathrm{fix}(g) = \emptyset$. *Then there is an $h \in \mathrm{PSp}(1,1)$ such that*

$$f = hFh^{-1} = \begin{pmatrix} \cos\frac{\pi}{m} & \mathbf{k}e^{\delta}\sin\frac{\pi}{m} \\ \mathbf{k}e^{-\delta}\sin\frac{\pi}{m} & \cos\frac{\pi}{m} \end{pmatrix}, \tag{8}$$

$$g = hGh^{-1} = \begin{pmatrix} \mu\frac{e^{\mathbf{i}\alpha}+e^{\mathbf{i}\beta}}{2}\mu^{-1} & \mu\frac{e^{\mathbf{i}\alpha}-e^{\mathbf{i}\beta}}{2}\mu^{-1} \\ \mu\frac{e^{\mathbf{i}\alpha}-e^{\mathbf{i}\beta}}{2}\mu^{-1} & \mu\frac{e^{\mathbf{i}\alpha}+e^{\mathbf{i}\beta}}{2}\mu^{-1} \end{pmatrix}, \tag{9}$$

where $\alpha = \frac{\pi}{p}, \beta = \frac{\pi}{q}$, $p \neq q$ and $\mathrm{ord}(g) = n$.

Proof. In the ball model, we assume that g' is of the form

$$\begin{pmatrix} e^{\mathbf{i}\alpha} & 0 \\ 0 & e^{\mathbf{i}\beta} \end{pmatrix}, \text{ where } \alpha \neq \beta$$

and f' is a boundary elliptic element of order n. Without loss of generality, we may assume that $G = E(\alpha,\beta)$ and the u-axis is orthogonal to $\mathrm{fix}(F)$. Similarly as in the proof of Lemma

3.3, we can find an $h = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} \in \mathrm{PSp}(1,1)$ such that hFh^{-1} and hGh^{-1} are of the forms (8) and (9). □

Similarly we can prove the following lemma.

Lemma 3.5. *Suppose that F and G are two regular elliptic elements of order m and n respectively, with* $\mathrm{fix}(f) \cap \mathrm{fix}(g) = \emptyset$. *Then there is an* $h \in \mathrm{PSp}(1,1)$ *such that*

$$f = \begin{pmatrix} \frac{e^{\mathbf{i}\theta}+e^{\mathbf{i}\eta}}{2} & e^{\delta}\frac{e^{\mathbf{i}\theta}-e^{\mathbf{i}\eta}}{2} \\ e^{-\delta}\frac{e^{\mathbf{i}\theta}-e^{\mathbf{i}\eta}}{2} & \frac{e^{\mathbf{i}\theta}+e^{\mathbf{i}\eta}}{2} \end{pmatrix},$$

where $\theta = \frac{\pi}{p'}, \eta = \frac{\pi}{q'}, p' \neq q'$ *and* $\mathrm{ord}(f) = m$;

$$g = hGh^{-1} = \begin{pmatrix} \mu\frac{e^{\mathbf{i}\alpha}+e^{\mathbf{i}\beta}}{2}\mu^{-1} & \mu\frac{e^{\mathbf{i}\alpha}-e^{\mathbf{i}\beta}}{2}\mu^{-1} \\ \mu\frac{e^{\mathbf{i}\alpha}-e^{\mathbf{i}\beta}}{2}\mu^{-1} & \mu\frac{e^{\mathbf{i}\alpha}+e^{\mathbf{i}\beta}}{2}\mu^{-1} \end{pmatrix},$$

where $\alpha = \frac{\pi}{p}, \beta = \frac{\pi}{q}, p \neq q$ *and* $\mathrm{ord}(g) = n$.

4. The proof of the main theorem

In this section we will prove our main theorem. We divide the proof into three cases.

Case (1): Both f and g are boundary elliptic elements. By Lemma 3.3 and Proposition 3.1, we may assume that

$$f = \begin{pmatrix} \cos\frac{\pi}{m} & \mathbf{k}e^{\delta}\sin\frac{\pi}{m} \\ \mathbf{k}e^{-\delta}\sin\frac{\pi}{m} & \cos\frac{\pi}{m} \end{pmatrix},$$

$$g = \begin{pmatrix} \cos\frac{\pi}{n} & (\mathbf{k}\cos\phi - \mathbf{i}\sin\phi)\sin\frac{\pi}{n} \\ (\mathbf{k}\cos\phi - \mathbf{i}\sin\phi)\sin\frac{\pi}{n} & \cos\frac{\pi}{n} \end{pmatrix}.$$

Obviously, the isometric spheres of f and f^{-1} are

$$I(f) = \left\{z \in \overline{\mathbf{H}^1_{\mathbb{H}}} : \left|z - \mathbf{k}e^{\delta}\cot\frac{\pi}{m}\right| = \frac{e^{\delta}}{\sin\frac{\pi}{m}}\right\},$$

$$I(f^{-1}) = \left\{z \in \overline{\mathbf{H}^1_{\mathbb{H}}} : \left|z + \mathbf{k}e^{\delta}\cot\frac{\pi}{m}\right| = \frac{e^{\delta}}{\sin\frac{\pi}{m}}\right\},$$

and the isometric spheres of g and g^{-1} are

$$I(g) = \left\{z \in \overline{\mathbf{H}^1_{\mathbb{H}}} : \left|z - (\mathbf{k}\cos\phi - \mathbf{i}\sin\phi)\cot\frac{\pi}{n}\right| = \frac{1}{\sin\frac{\pi}{n}}\right\},$$

$$I(g^{-1}) = \left\{z \in \overline{\mathbf{H}^1_{\mathbb{H}}} : \left|z + (\mathbf{k}\cos\phi - \mathbf{i}\sin\phi)\cot\frac{\pi}{n}\right| = \frac{1}{\sin\frac{\pi}{n}}\right\}.$$

The fundamental domain for the action of f on $\overline{\mathbf{H}^1_{\mathbb{H}}}$ is the union set of

$$\mathrm{Ext}\, I(f) \cap \mathrm{Ext}\, I(f^{-1})$$

and

$$\operatorname{Int} I(f) \cap \operatorname{Int} I(f^{-1}).$$

It is obvious that $I(g)$ and $I(g^{-1})$ lie in the semi-Euclidean ball $B(0,r) \cap \overline{\mathbf{H}^1_{\mathbb{H}}}$, where

$$r = \frac{1+\cos\frac{\pi}{n}}{\sin\frac{\pi}{n}}$$

and $B(0,r) = \{x \in \mathbb{H} : |x| < r\}$. Additionally, the two isometric sphere $I(f)$ and $I(f^{-1})$ also contain the semi-ball $B(0,s) \cap \overline{\mathbf{H}^1_{\mathbb{H}}}$, where

$$s = \frac{e^{\delta}(1-\cos\frac{\pi}{m})}{\sin\frac{\pi}{m}}.$$

Since

$$\cosh\delta(f,g) = \cosh\delta \geq \frac{\cos\frac{\pi}{m}\cos\frac{\pi}{n}+1}{\sin\frac{\pi}{m}\sin\frac{\pi}{n}},$$

we have

$$e^{\delta} \geq \frac{(1+\cos\frac{\pi}{n})\sin\frac{\pi}{m}}{(1-\cos\frac{\pi}{m})\sin\frac{\pi}{n}},$$

which implies $r \leq s$. We have therefore seen that the exterior of a fundamental domain for $\langle g\rangle$ lies inside a fundamental domain for $\langle f\rangle$. It follows from Proposition 2.1 and the simplest version of the Klein-Maskit combination theorem that the group $\langle f,g\rangle$ is discrete and isomorphic to the free product of cyclic groups:

$$\langle f,g\rangle \cong \langle f\rangle * \langle g\rangle \cong \mathbb{Z}_m * \mathbb{Z}_n.$$

Case (2): f is a boundary elliptic element and g is a regular elliptic element. By Lemma 3.4, we may assume that f has the same representation as in Case (1) and g is of the form given by (9). Then the isometric spheres of g and g^{-1} are

$$I(g) = \left\{ z \in \overline{\mathbf{H}^1_{\mathbb{H}}} : \left| z + \mu\mathbf{i}\mu^{-1}\cot\frac{\beta-\alpha}{2} \right| = \frac{1}{|\sin\frac{\beta-\alpha}{2}|} \right\}$$

and

$$I(g^{-1}) = \left\{ z \in \overline{\mathbf{H}^1_{\mathbb{H}}} : \left| z - \mu\mathbf{i}\mu^{-1}\cot\frac{\beta-\alpha}{2} \right| = \frac{1}{|\sin\frac{\beta-\alpha}{2}|} \right\}.$$

It is obvious that the isometric spheres $I(g)$ and $I(g^{-1})$ lie in the semi-ball $B(0,r) \cap \overline{\mathbf{H}^1_{\mathbb{H}}}$, where

$$r = \frac{1+\cos\frac{|p-q|\pi}{2pq}}{\sin\frac{|p-q|\pi}{2pq}}.$$

It follows from Lemma 3.2 that

$$\frac{|p-q|\pi}{2pq} \geq \frac{\pi}{n}.$$

Hence

$$r \leq \frac{1 + \cos \frac{\pi}{n}}{\sin \frac{\pi}{n}}.$$

As in Case (1), the result follows from the above inequality.

Case (3): Both f and g are regular elliptic elements. By Lemma 3.5, we may assume that g has the same representation as in Case (2) and f is of the form

$$f = \begin{pmatrix} \frac{e^{\mathbf{i}\theta}+e^{\mathbf{i}\eta}}{2} & e^{\delta} \frac{e^{\mathbf{i}\theta}-e^{\mathbf{i}\eta}}{2} \\ e^{-\delta} \frac{e^{\mathbf{i}\theta}-e^{\mathbf{i}\eta}}{2} & \frac{e^{\mathbf{i}\theta}+e^{\mathbf{i}\eta}}{2} \end{pmatrix} \in \mathrm{PSp}(1,1),$$

where $\theta = \frac{\pi}{p'}, \eta = \frac{\pi}{q'}, p' \neq q'$ and $\mathrm{ord}(f) = m$. Note that $\mathrm{fix}(f) = \{e^{\delta}\}$. The isometric spheres of f and f^{-1} are

$$I(f) = \left\{ z \in \overline{\mathbf{H}^1_{\mathbb{H}}} : \left| z + \mathbf{i}e^{\delta} \cot \frac{\eta - \theta}{2} \right| = \frac{e^{\delta}}{|\sin \frac{\eta-\theta}{2}|} \right\}$$

and

$$I(f^{-1}) = \left\{ z \in \overline{\mathbf{H}^1_{\mathbb{H}}} : \left| z - \mathbf{i}e^{\delta} \cot \frac{\eta - \theta}{2} \right| = \frac{e^{\delta}}{|\sin \frac{\eta-\theta}{2}|} \right\}.$$

The two isometric spheres $I(f)$ and $I(f^{-1})$ contain the semi-ball $B(0, s) \cap \overline{\mathbf{H}^1_{\mathbb{H}}}$, where

$$s = \frac{e^{\delta}(1 - \cos \frac{|p'-q'|\pi}{2p'q'})}{\sin \frac{|p'-q'|\pi}{2p'q'}}.$$

It follows from Lemma 3.2 that

$$\frac{|p' - q'|\pi}{2p'q'} \geq \frac{\pi}{m}.$$

Hence

$$s \geq \frac{e^{\delta}(1 - \cos \frac{\pi}{m})}{\sin \frac{\pi}{m}}.$$

As in Cases (1) and (2), the result easily follows from the above inequality. □

REMARK 4.1. As in [4], $\mathrm{SL}(2,\mathbb{C})$ can be embedded as a subgroup of $\mathrm{Sp}(1,1)$ as follows:

$$f \in \mathrm{SL}(2,\mathbb{C}) \hookrightarrow TfT^{-1} \in \mathrm{Sp}(1,1),$$

where

$$T = \begin{pmatrix} \frac{1-\mathbf{j}}{2} & \frac{1-\mathbf{j}}{2} \\ \frac{1+\mathbf{j}}{2} & \frac{-1-\mathbf{j}}{2} \end{pmatrix}.$$

Hence Theorem 1.1 is a natural generalization of the result for real hyperbolic spaces of dimension 2 and 3 [6, 8].

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