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# WILLMORE-LIKE FUNCTIONALS FOR SURFACES IN 3-DIMENSIONAL THURSTON GEOMETRIES

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## Abstract

We find analogues of the Willmore functional for each of the Thurston geometries with 4-dimensional isometry group such that the CMC-spheres in these geometries are critical points of these functionals.

## 1. Introduction

Let  $M$  be a closed orientable surface and  $f: M \rightarrow N$  be an immersion of  $M$  into a 3-dimensional Riemannian manifold  $N$ . Set:

$$\mathcal{W}(f) = \int_M (H^2 + \bar{K}) d\mu,$$

where  $H$  is the mean curvature of the immersed surface, the value of  $\bar{K}$  at a point  $p \in M$  is defined as the sectional curvature of the 2-plane  $f_*(T_p M)$  in  $N$ ,  $d\mu$  is the area element of the induced metric on  $M$ . We will refer to the functional  $\mathcal{W}$  as the Willmore functional. It is known that  $\mathcal{W}(f)$  is a conformal invariant [19].

In the 3-dimensional space forms  $\mathbb{R}^3, \mathbb{H}^3$  and  $\mathbb{S}^3$  the functional  $\mathcal{W}$  enjoys the property that the CMC spheres are critical points of  $\mathcal{W}$ ; recall that in the 3-dimensional space forms the CMC spheres are exactly the round spheres by the Hopf theorem. However, this property for the Willmore functional  $\mathcal{W}$  fails to hold in the other 3-dimensional Thurston geometries.

In this paper we will introduce the Willmore-like functionals for the certain family of Riemannian manifolds  $E(k, \tau)$  that include the model spaces for all Thurston geometries with 4-dimensional group of isometries, i.e., the products  $\mathbb{S}^2 \times \mathbb{R}$  and  $\mathbb{H}^2 \times \mathbb{R}$ , the Heisenberg group  $\text{Nil}$  and the Lie group  $\widetilde{\text{PSL}_2(\mathbb{R})}$ . The functionals to be introduced in these geometries have the form:

$$(1.1) \quad \int_M (H^2 + \alpha \bar{K} + \beta) d\mu,$$

where  $\alpha$  and  $\beta$  are some constants that depend on  $k$  and  $\tau$ . In the case of the Heisenberg group  $\text{Nil}$  (for  $k = 0$  and  $\tau = \frac{1}{2}$ ) the functional:

$$\int_M \left( H^2 + \frac{1}{4} \bar{K} - \frac{1}{16} \right) d\mu$$

was obtained in [5] based on the Weierstrass representation for surfaces in  $\text{Nil}$ . Then it was shown [6] that the CMC spheres in  $\text{Nil}$  are critical points of this functional. In the case of the Lie group  $\widetilde{\text{PSL}_2(\mathbb{R})}$  (for  $k = -1$  and  $\tau = -\frac{1}{2}$ ) it was shown that for the functional:

$$\int_M \left( H^2 + \frac{1}{4} \bar{K} - \frac{5}{16} \right) d\mu,$$

the minimum among the rotationally invariant spheres is attained exactly at the CMC spheres [4].

The main result of the paper is the following theorem.

**Theorem 1.** *The CMC spheres in  $E(k, \tau)$  are critical points of the following Willmore–like functional:*

$$(1.2) \quad E(f) = \int_M \left( H^2 + \frac{1}{4} \bar{K} + \frac{k}{4} - \frac{\tau^2}{4} \right) d\mu.$$

In addition to Theorem 1 we will prove the following theorem.

**Theorem 2.** *For rotationally invariant spheres in  $E(k, \tau)$  the functional  $E(f)$  attains its minimum exactly at the CMC spheres.*

**REMARK 1.** We note that:

$$E(f) = \mathcal{W}(f) + \int_M \left( -\frac{3}{4} \bar{K} + \frac{k}{4} - \frac{\tau^2}{4} \right) d\mu.$$

**REMARK 2.** It can be seen that Theorem 1 agrees with the results obtained earlier for  $\text{Nil}$  [6] and the Lie group  $\widetilde{\text{PSL}_2(\mathbb{R})}$  [4]. In addition, the functional  $E$  for the case  $\mathbb{S}^2 \times \mathbb{R}$  ( $k = 1$  and  $\tau = 0$ ) coincides up to a constant factor with the functional  ${}^1 \int_M (4H^2 + \bar{K} + 1)$  mentioned in [6, § 6.2].

The structure of the remaining part of this paper is as follows. In § 2 we give the description of the Riemannian manifolds  $E(k, \tau)$ . In § 3 we review the characterizations of the CMC spheres in these manifolds. In § 4 we give the details of the proof of Theorem 1. In § 5 we give the details of the proof of Theorem 2.

## 2. The Riemannian manifolds $E(k, \tau)$

The model spaces for the four Thurston geometries:  $\text{Nil}$ ,  $\widetilde{\text{PSL}_2(\mathbb{R})}$ ,  $\mathbb{H}^2 \times \mathbb{R}$  and  $\mathbb{S}^2 \times \mathbb{R}$  belong to the family of Riemannian 3-manifolds  $E(k, \tau)$ ,  $k \in \mathbb{R}$ ,  $\tau \in \mathbb{R}$  that are as follows. If  $k \geq 0$  then  $E(k, \tau)$  is  $\mathbb{R}^3$  with the metric:

$$(2.1) \quad ds^2 = \frac{dx^2 + dy^2}{\left( 1 + \frac{k}{4}(x^2 + y^2) \right)^2} + \left( dz + \frac{\tau(ydx - xdy)}{1 + \frac{k}{4}(x^2 + y^2)} \right)^2.$$

If  $k < 0$  then  $E(k, \tau)$  is the product  $D^2(\frac{2}{\sqrt{-k}}) \times \mathbb{R}$  with the metric (2.1), where  $D^2(\frac{2}{\sqrt{-k}}) = \{(x, y) \mid x^2 + y^2 < \frac{4}{-k}\}$ . The family  $E(k, \tau)$  is also referred to as Bianchi–Cartan–Vranceanu

<sup>1</sup>In [6, § 6.2] the term  $H$  should be considered as  $2H$ .

family [3, 9]. The projection of  $E(k, \tau)$  onto the 2-dimensional domain of constant curvature  $k$  given by the map  $(x, y, z) \mapsto (x, y)$  is a Riemannian fibration. The fibres of such a fiber bundle are geodesics and its unitary tangent vectors  $\frac{\partial}{\partial z}$  form a Killing vector field; this field is also referred to as the vertical vector field. The parameter  $k$  is called the base curvature and  $\tau$  the bundle curvature.

If  $k = -1, \tau = 0$  then  $E(k, \tau)$  is the product  $\mathbb{H}^2 \times \mathbb{R}$ . If  $k = 1, \tau = 0$  then  $E(k, \tau)$  is obtained from the product  $\mathbb{S}^2 \times \mathbb{R}$  by removing one fibre. If  $k = 0, \tau \neq 0$  then  $E(k, \tau)$  is the Heisenberg group Nil with the left-invariant metric determined by the parameter  $\tau$ . If  $k < 0, \tau \neq 0$  then  $E(k, \tau)$  is the Lie group  $\widetilde{\text{PSL}_2(\mathbb{R})}$  with the left-invariant metric determined by the parameters  $k$  and  $\tau$ . For the case  $k > 0, \tau \neq 0$ , the manifolds  $E(k, \tau)$  are obtained from the covering of the Berger spheres by removing one fibre.

For more details we refer the reader to [16, 8]. We will need the following proposition.

**Proposition 1.** *The sectional curvature of a 2-plane in  $E(k, \tau)$  equals:*

$$(2.2) \quad \bar{K} = \tau^2 + (k - 4\tau^2)v^2,$$

where  $v$  is the scalar product of a unit normal vector to the plane and the vertical vector  $\xi = \frac{\partial}{\partial z}$  with respect to the metric (2.1).

Proof. The identity (2.2) can be obtained directly from the general formula for the Riemann curvature tensor of  $E(k, \tau)$  shown in [8, Proposition 2.1].  $\square$

### 3. The CMC spheres in $E(k, \tau)$

The rotationally invariant CMC surfaces in the products  $\mathbb{H}^2 \times \mathbb{R}$ ,  $\mathbb{S}^2 \times \mathbb{R}$  and the Heisenberg group Nil were described in [12, 14] and [7, 11, 17] respectively. The case of the Lie group  $\widetilde{\text{PSL}_2(\mathbb{R})}$  and the Berger spheres were studied in [15] and [18]. In order to describe rotationally invariant CMC surfaces in  $E(k, \tau)$  we will follow the approach used in [11, 17] for  $E(0, \frac{1}{2})$ .

For the cylindrical coordinates  $\rho, \theta, z$  in  $E(k, \tau)$  such that  $x = \rho \cos \theta, y = \rho \sin \theta, z = z$  the metric (2.1) has the form:

$$(3.1) \quad ds^2 = \frac{1}{(1 + \frac{k}{4}\rho^2)^2} d\rho^2 + \frac{\rho^2 + \tau^2\rho^4}{(1 + \frac{k}{4}\rho^2)^2} d\theta^2 - \frac{2\tau\rho^2}{1 + \frac{k}{4}\rho^2} dzd\theta + dz^2.$$

We note that  $\rho \in [0, R]$ , where  $R = \frac{2}{\sqrt{-k}}$  if  $k < 0$  and  $R = \infty$  if  $k \geq 0$ .

The group  $\text{SO}(2)$  acts on  $E(k, \tau)$  by rotations  $\theta \mapsto \theta + \text{const}$  around  $z$ -axis. The rotations are isometries and the factor-space  $E(k, \tau)/\text{SO}(2)$  is the 2-dimensional domain  $B(k, \tau) = \{(u, v) | u \in [0, R], v \in \mathbb{R}\}$  with the metric:

$$(3.2) \quad d\tilde{s}^2 = \frac{1}{(1 + \frac{k}{4}u^2)^2} du^2 + \frac{1}{1 + \tau^2u^2} dv^2,$$

so the projection  $E(k, \tau) \rightarrow B(k, \tau)$  is a Riemannian submersion.

For a given rotationally invariant surface we define by  $\gamma(s) = (u(s), v(s))$  its projection onto  $B(k, \tau)$ , where  $s$  is a natural parameter with respect to the metric (3.2). Let  $\sigma$  be the

angle between  $\dot{\gamma}$  and  $\frac{\partial}{\partial u}$ . It can be verified (cf. [11, eq. 2]) that for the metric (3.2) the geodesic curvature of  $\gamma(s)$  equals:

$$(3.3) \quad \tilde{k} = \dot{\sigma} - \frac{\tau^2 u(1 + \frac{k}{4}u^2)}{(1 + \tau^2 u^2)} \sin \sigma.$$

The mean curvature of a rotationally invariant surface is given by the reduction theorem (cf. [11, p. 178]) as follows:

$$(3.4) \quad H = \frac{1}{2} \left( \tilde{k} - \frac{\partial}{\partial n} \ln \mu \right),$$

where  $n = (-1 + \frac{k}{4}u^2) \sin \sigma, \sqrt{1 + \tau^2 u^2} \cos \sigma$  is a normal vector in  $B(k, \tau)$  to  $\gamma(s)$  and  $\mu = \frac{u \sqrt{1 + \tau^2 u^2}}{1 + \frac{k}{4}u^2}$  is the factor of the volume form for an  $SO(2)$  orbit with respect to the metric (3.1). From (3.3) and (3.4) we obtain:

$$(3.5) \quad H = \frac{1}{2} \left( \dot{\sigma} + \left( \frac{1}{u} - k \frac{u}{4} \right) \sin \sigma \right).$$

Thus, we obtain that for a profile  $\gamma(s) = (u(s), v(s))$  of a rotationally invariant CMC surface the following system of ODE is satisfied:

$$(3.6) \quad \begin{cases} \dot{u} = \left( 1 + \frac{k}{4}u^2 \right) \cos \sigma, \\ \dot{v} = \sqrt{1 + \tau^2 u^2} \sin \sigma, \\ \dot{\sigma} = 2H - \left( \frac{1}{u} - k \frac{u}{4} \right) \sin \sigma. \end{cases}$$

It can be straightforwardly verified that the system (3.6) has the following first integral:

$$(3.7) \quad J = \frac{u}{1 + \frac{k}{4}u^2} (\sin \sigma - Hu).$$

Then we have the following proposition.

**Proposition 2.** *If  $k \leq 0$  then for any  $H$  such that  $H^2 > \frac{-k}{4}$  there exists a rotationally invariant CMC sphere of constant mean curvature  $H$  in  $E(k, \tau)$ ; moreover, if  $H^2 \leq \frac{-k}{4}$  then there exists no CMC sphere of constant mean curvature  $H$  in  $E(k, \tau)$ . If  $k > 0$  then for any  $H \neq 0$  there exists a rotationally invariant CMC sphere of constant mean curvature  $H$  in  $E(k, \tau)$ . For every rotationally invariant CMC sphere in  $E(k, \tau)$  the first integral (3.7) vanishes:  $J = 0$ . The CMC spheres in  $E(k, \tau)$  are unique up to isometries.*

Proof. The proof is based on an analysis of a qualitative behavior of the solutions of (3.6) depending on the values of  $J$  and  $H$ . Such an analysis is straightforward and it was done for  $E(0, \frac{1}{2})$  in [11, 17]; the case of  $E(-1, -\frac{1}{2})$  was shown in [4]. The uniqueness of the CMC spheres was proved in [1, 2]. Also, see [10] for the complete proof of Proposition 2.  $\square$

By Proposition 2 we obtain that on a rotationally invariant CMC sphere in  $E(k, \tau)$  the following equality holds:

$$(3.8) \quad \sin \sigma = Hu.$$

**REMARK 3.** Although there exists no a minimal sphere in  $E(k, \tau)$ , such spheres exist for  $\mathbb{S}^2 \times \mathbb{R}$  and the Berger spheres. We recall that for  $k > 0$  the manifolds  $E(k, \tau)$  are obtained from the corresponding homogeneous manifolds by removing one fibre.

#### 4. The proof of Theorem 1

For an immersion  $f : M \rightarrow E(k, \tau)$  of a closed orientable surface  $M$  into  $E(k, \tau)$  set:

$$(4.1) \quad E_{\alpha, \beta}(f) = \int_M (H^2 + \alpha \bar{K} + \beta) d\mu.$$

Let  $F : M \times [0, 1] \rightarrow E(k, \tau)$  be a normal variation of the immersion  $f$ , i.e.,  $F(p, 0) = f(p)$  for all  $p \in M$  and  $\frac{\partial F(p, t)}{\partial t} = \varphi n$ , where  $n$  is the unit normal vector field to  $M$  and the velocity  $\varphi$  is a smooth function on  $M$ . We will denote by  $\delta$  the operator  $\frac{\partial}{\partial t}|_{t=0}$ . We will need the following proposition.

**Proposition 3.** *Under a normal variation with the velocity  $\varphi$  the following identities hold:*

$$(4.2) \quad \delta d\mu = -2H\varphi d\mu,$$

$$(4.3) \quad \delta n = -\nabla \varphi,$$

$$(4.4) \quad 2\delta H = \Delta \varphi + (4H^2 - 2K_e + \text{Ric}(n, n))\varphi,$$

where  $\nabla$  is the gradient and  $\Delta$  is the Laplace–Beltrami operator on  $M$ ,  $K_e$  is the extrinsic Gauss curvature.

Proof. The proof is standard, one may look it up in [13].  $\square$

It follows from Proposition 1 that the term  $\text{Ric}(n, n)$  equals  $k - 2\tau^2 - (k - 4\tau^2)\nu^2$ . Therefore we have that:

$$(4.5) \quad 2\delta H = \Delta \varphi + (4H^2 - 2K_e + k - 2\tau^2 - (k - 4\tau^2)\nu^2)\varphi.$$

Let  $T$  be the projection of the vertical field  $\xi$  on  $M$ , i.e.,  $T = \xi - \nu n$ . By (4.3) we have that  $\delta \nu = \delta \langle n, \xi \rangle = -\langle \nabla \varphi, \xi \rangle$ . Therefore we obtain:

$$(4.6) \quad \int_M \nu \delta \nu d\mu = \int_M \text{div}(\nu T) \varphi d\mu.$$

By Proposition 1 we have:

$$(4.7) \quad E_{\alpha, \beta}(f) = \int_M (H^2 + \alpha(k - 4\tau^2)\nu^2 + \beta + \alpha\tau^2) d\mu.$$

Then by (4.5), (4.6) and (4.2) we obtain:

$$(4.8) \quad \begin{aligned} \delta E_{\alpha, \beta}(f) = & \int_M (\Delta H + H(2H^2 - 2K_e - (1 + 2\alpha)(k - 4\tau^2)\nu^2 + \\ & + k - 2\tau^2 - 2\beta - 2\alpha\tau^2) + 2\alpha(k - 4\tau^2)\text{div}(\nu T)) \varphi d\mu. \end{aligned}$$

Therefore the Euler–Lagrange equation of the functional (4.1) is as follows:

$$(4.9) \quad \Delta H + H(2H^2 - 2K_e - (1 + 2\alpha)(k - 4\tau^2)\nu^2 + \\ + k - 2\tau^2 - 2\beta - 2\alpha\tau^2) + 2\alpha(k - 4\tau^2)\text{div}(\nu T) = 0.$$

By the Gauss theorem we obtain  $K_e = K - \bar{K} = K - (k - 4\tau^2)\nu^2 - \tau^2$ , where  $K$  is the intrinsic Gauss curvature. Then (4.9) can be rewritten as follows:

$$(4.10) \quad \Delta H + H(2H^2 - 2K + (1 - 2\alpha)(k - 4\tau^2)\nu^2 + \\ + k - 2\beta - \alpha\tau^2) + 2\alpha(k - 4\tau^2)\text{div}(\nu T) = 0.$$

Consider a CMC sphere in  $E(k, \tau)$ . For the coordinates on this sphere we choose  $\theta$  and  $s$ ; recall that  $\theta$  is an angle from the cylindrical coordinate system in  $E(k, \tau)$  and  $s$  is the natural parameter on the projection  $\gamma(s) = (u(s), v(s))$  of this sphere onto  $B(k, \tau)$ . By (3.1), for these coordinates the metric on a CMC sphere is as follows:

$$(4.11) \quad \frac{\rho^2 + \tau^2\rho^4}{(1 + \frac{k}{4}\rho^2)^2} d\theta^2 + ds^2.$$

By (3.2), (3.6) and (4.11) it can be straightforwardly verified that on a CMC sphere in  $E(k, \tau)$ :

$$(4.12) \quad \nu = \frac{\cos \sigma}{\sqrt{1 + \tau^2 u^2}},$$

$$(4.13) \quad K = -\frac{1 + \frac{k}{4}u^2}{u\sqrt{1 + \tau^2 u^2}} \frac{d}{ds} \frac{u\sqrt{1 + \tau^2 u^2}}{1 + \frac{k}{4}u^2},$$

$$(4.14) \quad \text{div}(\nu T) = \frac{1 + \frac{k}{4}u^2}{u\sqrt{1 + \tau^2 u^2}} \frac{d}{ds} \frac{u\cos \sigma \sin \sigma}{(1 + \frac{k}{4}u^2)\sqrt{1 + \tau^2 u^2}}.$$

Put  $\alpha = \frac{1}{4}$  and  $\beta = \frac{k}{4} - \frac{\tau^2}{4}$ . Substituting (4.12), (4.13) and (4.14) into the left hand side of (4.10) we obtain that it equals:

$$(4.15) \quad 2H^3 + H \left( 2 \frac{1 + \frac{k}{4}u^2}{u\sqrt{1 + \tau^2 u^2}} \frac{d}{ds} \frac{u\sqrt{1 + \tau^2 u^2}}{1 + \frac{k}{4}u^2} + \frac{1}{2}(k - 4\tau^2) \frac{\cos^2 \sigma}{1 + \tau^2 u^2} + \frac{k}{2} \right) + \\ + \frac{1}{2}(k - 4\tau^2) \frac{1 + \frac{k}{4}u^2}{u\sqrt{1 + \tau^2 u^2}} \frac{d}{ds} \frac{u\cos \sigma \sin \sigma}{(1 + \frac{k}{4}u^2)\sqrt{1 + \tau^2 u^2}}$$

Using the equation (3.8) and the system (3.6), it can be verified that the expression (4.15) vanishes on a CMC sphere. Theorem 1 is proved.

## 5. The proof of Theorem 2

Let us substitute (2.2) into the formula (1.2) for the functional  $E(f)$ . Then we have:

$$(5.1) \quad E(f) = \int_M \left( H^2 + \left( \frac{k}{4} - \tau^2 \right) v^2 + \frac{k}{4} \right) d\mu.$$

Let us consider a rotationally invariant sphere in  $E(k, \tau)$  defined by a curve  $\gamma(s) = (u(s), v(s)) \subset B(k, \tau)$ . By (3.5) we can represent  $H^2$  as follows:

$$(5.2) \quad H^2 = \frac{1}{4} \left( \dot{\sigma} - \left( \frac{1}{u} + k \frac{u}{4} \right) \sin \sigma \right)^2 + \left( \frac{\dot{\sigma} \sin \sigma}{u} - k \frac{\sin^2 \sigma}{4} \right).$$

For a rotationally invariant surface we have:

$$(5.3) \quad v^2 = \frac{\cos^2 \sigma}{1 + \tau^2 u^2},$$

$$(5.4) \quad d\mu = \frac{u \sqrt{1 + \tau^2 u^2}}{1 + \frac{k}{4} u^2} ds.$$

Substituting (5.2), (5.3) and (5.4) into (5.1) we obtain:

$$(5.5) \quad \begin{aligned} E(f) &= 2\pi \int_{\gamma} \frac{1}{4} \left( \dot{\sigma} - \left( \frac{1}{u} + k \frac{u}{4} \right) \sin \sigma \right)^2 \frac{u \sqrt{1 + \tau^2 u^2}}{1 + \frac{k}{4} u^2} ds + \\ &2\pi \int_{\gamma} \left( \frac{\dot{\sigma} \sin \sigma}{u} - k \frac{\sin^2 \sigma}{4} + \left( \frac{k}{4} - \tau^2 \right) \frac{\cos^2 \sigma}{1 + \tau^2 u^2} + \frac{k}{4} \right) \frac{u \sqrt{1 + \tau^2 u^2}}{1 + \frac{k}{4} u^2} ds. \end{aligned}$$

By (3.6) we have:  $\cos \sigma = \frac{\dot{u}}{1 + \frac{k}{4} u^2}$ . Substituting this into the integrand of the second summand in (5.5) we obtain:

$$(5.6) \quad -\frac{\ddot{u} \sqrt{1 + \tau^2 u^2}}{(1 + \frac{k}{4} u^2)^2} + \frac{u \dot{u}^2}{(1 + \frac{k}{4} u^2)^3} \left( \frac{3}{4} k \sqrt{1 + \tau^2 u^2} + \left( \frac{k}{4} - \tau^2 \right) \frac{1}{\sqrt{1 + \tau^2 u^2}} \right).$$

It can be verified that the expression (5.6) is equal to  $\frac{d}{ds} \left[ -\frac{\dot{u} \sqrt{1 + \tau^2 u^2}}{(1 + \frac{k}{4} u^2)^2} \right]$ . Therefore, for a rotationally invariant sphere the second summand in (5.5) is equal to  $4\pi$ .

The first summand in (5.5) is nonnegative. It vanishes iff the following holds:

$$(5.7) \quad \dot{\sigma} - \left( \frac{1}{u} + k \frac{u}{4} \right) \sin \sigma = 0.$$

It follows from (3.6) that (5.7) holds iff a rotationally invariant sphere is CMC. Theorem 2 is proved.

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