



Title	POLYLOGARITHMIC ANALOGUE OF THE COLEMAN-IHARA FORMULA, I
Author(s)	Sakugawa, Kenji; Wojtkowiak, Zdzisław; Nakamura, Hiroaki
Citation	Osaka Journal of Mathematics. 2017, 54(1), p. 55-74
Version Type	VoR
URL	<a href="https://doi.org/10.18910/61898">https://doi.org/10.18910/61898</a>
rights	
Note	

*The University of Osaka Institutional Knowledge Archive : OUKA*

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

## POLYLOGARITHMIC ANALOGUE OF THE COLEMAN-IHARA FORMULA, I

Dedicated to Professor Yasutaka Ihara  
on the occasion of his 77 th birthday

HIROAKI NAKAMURA, KENJI SAKUGAWA  
and ZDZISŁAW WOJTKOWIAK

(Received October 30, 2014, revised November 17, 2015)

### Abstract

The Coleman-Ihara formula expresses Soule's  $p$ -adic characters restricted to  $p$ -local Galois group as the Coates-Wiles homomorphism multiplied by  $p$ -adic  $L$ -values at positive integers. In this paper, we show an analogous formula that  $\ell$ -adic polylogarithmic characters for  $\ell = p$  restrict to the Coates-Wiles homomorphism multiplied by Coleman's  $p$ -adic polylogarithms at any roots of unity of order prime to  $p$ .

### 1. Introduction

Let  $p$  be an odd prime. In his Annals article [12], Yasutaka Ihara introduced the universal power series for Jacobi sums  $F_\sigma \in \mathbb{Z}_p[[u, v]]^\times$  ( $\sigma \in G_\mathbb{Q} := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ ) and showed a beautiful (local) formula ([12] Theorem C) relating its coefficient characters with  $p$ -adic  $L$ -values  $L_p(m, \omega^{1-m})$  ( $m \geq 3$  : odd) multiplied by the  $m$ -th Coates-Wiles homomorphisms when  $\sigma$  lies in the  $p$ -local subgroup of  $G_{\mathbb{Q}(\mu_{p^\infty})}$ . In the last part of [12] (p.105, (Col2)) documented is that Robert Coleman proved that these coefficient characters are nothing but the restrictions of Soule's cyclotomic elements in  $H^1(G_\mathbb{Q}, \mathbb{Z}_p(m))$ . This motivated later works by Anderson [1], Coleman [7], Ihara-Kaneko-Yukinari [14] toward the explicit (global) formula of  $F_\sigma$  for all  $\sigma \in G_\mathbb{Q}$  (see [13]). Their formula presents particularly a remarkable symmetric form on the main part  $G_{\mathbb{Q}(\mu_{p^\infty})}$  of  $G_\mathbb{Q}$  as:

$$(1.1) \quad F_\sigma(u, v) = \exp \left( \sum_{\substack{m \geq 3 \\ \text{odd}}} \frac{\chi_m(\sigma)}{p^{m-1} - 1} \sum_{\substack{i+j=m \\ i, j \geq 1}} \frac{U^i V^j}{i! j!} \right) \quad (\sigma \in G_{\mathbb{Q}(\mu_{p^\infty})}),$$

where  $1 + u = e^U$ ,  $1 + v = e^V$  and  $\chi_m : G_{\mathbb{Q}(\mu_{p^\infty})} \rightarrow \mathbb{Z}_p(m)$  is the  $m$ -th Soule character ([28]) defined by the properties:

$$(1.2) \quad \left( \prod_{\substack{1 \leq a < p^n \\ p \nmid a}} (1 - \zeta_{p^n}^a)^{a^{m-1}} \right)^{\frac{1}{p^n}(\sigma-1)} = \zeta_{p^n}^{\chi_m(\sigma)} \quad (n \geq 1).$$

This, together with the above mentioned Ihara's local formula [12] Theorem C, implies Coleman's formula presented in [12] p.105 in the form:

$$(1.3) \quad \frac{\chi_m(\text{rec}(\epsilon))}{(p^{m-1} - 1)} = L_p(m, \omega^{1-m}) \phi_m^{CW}(\epsilon) \quad (\epsilon \in \mathcal{U}_\infty)$$

for  $m \geq 3$  : odd. Let us quickly explain the notation used here: For each  $n \geq 1$ , we denote by  $\mathcal{U}_n$  the group of principal units of  $\mathbb{Q}_p(\mu_{p^n})$  and by  $\mathcal{U}_\infty = \varprojlim_n \mathcal{U}_n$  their norm limit. Let  $\Omega_p$  be the maximal abelian pro- $p$  extension of  $\mathbb{Q}(\mu_{p^\infty})$  unramified outside  $p$ . Then, Ihara's power series  $\sigma \mapsto F_\sigma(u, v)$  factors through  $\text{Gal}(\Omega_p/\mathbb{Q})$ . Now, fix an embedding  $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$  and a coherent system of  $p$ -power roots of unity  $\{\zeta_{p^n}\}_{n \geq 1}$  to identify  $\mathbb{Z}_p$  with  $\mathbb{Z}_p(m)$ . This embedding and the local class field theory induce the canonical homomorphism  $\text{rec} : \mathcal{U}_\infty \hookrightarrow \text{Gal}(\Omega_p/\mathbb{Q}(\mu_{p^\infty}))$  called the reciprocity map. On the other side, the system  $\{\zeta_{p^n}\}_n$  determines, for  $m \geq 1$ , the Coates-Wiles homomorphism  $\phi_m^{CW} : \mathcal{U}_\infty \rightarrow \mathbb{Z}_p$ . The coefficient  $L_p(m, \omega^{1-m})$  is the Kubota-Leopoldt  $p$ -adic  $L$ -value at  $m$  with respect to the power of the Teichmüller character  $\omega$ .

Indeed, Coleman's paper [7] proves (1.3) by applying his theory on Hilbert norm residue symbols ([3], [4], [6]) to Jacobi sums which are special values  $F_\sigma(\zeta_{p^n}^a - 1, \zeta_{p^n}^b - 1)$  at Frobenius elements  $\sigma$  over various primes in  $\mathbb{Q}(\mu_{p^n})$  not dividing  $p$ . Especially, it relies on the Tchebotarev density. Consequently, the formula (1.3) results from combination of [7] and (1.1), relying on global arithmetic nature of  $F_\sigma(u, v)$ . The global proof certainly enables us to highlight (1.3) in contexts enriched with many important materials of Iwasawa theory (see, e.g., [11] §3-1). However, (1.3) itself is essentially of local nature, concerning the ratio between the Coates-Wiles homomorphism and Soule's character restricted on the local Galois group; Passing through the Jacobi sum interpolation properties of  $F_\sigma(u, v)$  to derive (1.3) should look rather roundabout.

The purpose of this paper is to give an alternative direct proof of (1.3) and its polylogarithmic variants, where the Soule's characters  $\chi_m$  in LHS are generalized to the ( $l$ -adic) Galois polylogarithms  $\ell i_m(z, \gamma)$  (for the case  $l = p$ ) introduced in [30]-[31]. They are defined as certain coefficients of Galois transforms of a defining path  $\gamma$  from  $\overrightarrow{01}$  to  $z$  on  $\mathbb{P}^1 \setminus \{0, 1, \infty\}$  (cf. § (2.4)). The values  $L_p(m, \omega^{1-m})$  in RHS have obvious generalization to  $\text{Li}_m^{p\text{-adic}}(z)$ , the  $p$ -adic polylogarithms of Coleman (see [4]). As a corollary of our main formula (Theorem 4.4), we obtain:

**Theorem 1.1** (Corollary 4.6, Proposition 5.1, Remark 4.7). *Let  $p$  be an odd prime, and  $F$  a finite unramified extension of  $\mathbb{Q}_p$  with the Frobenius substitution  $\sigma_F \in \text{Gal}(F/\mathbb{Q}_p)$ . Let  $F_\infty := F(\mu_{p^\infty})$  and denote by  $\phi_{m,F}^{CW} : G_{F_\infty} \rightarrow F \otimes \mathbb{Z}_p(m)$  the  $m$ -th Coates-Wiles homomorphism for the local field  $F$  (cf. Definition 3.5). Then, for any root of unity  $z$  contained in  $F$ , there is*

a standard specific path  $\overrightarrow{01} \rightsquigarrow z$  such that

$$(1.4) \quad \ell i_m(z, \gamma)(\sigma) = \frac{-1}{(m-1)!} \text{Tr}_{F/\mathbb{Q}_p} \left( \left\{ \left( 1 - \frac{\sigma_F}{p^m} \right) \text{Li}_m^{p\text{-adic}}(z) \right\} \phi_{m,F}^{CW}(\sigma) \right)$$

holds for  $m \geq 1$  and for  $\sigma \in G_{F_\infty}$ .

Our approach taken in this Part I is to apply the theory of Coleman power series (in a direct but different way from [7]) to the so called Koblitz measure that produces  $\text{Li}_m^{p\text{-adic}}(z)$  ([19]) on one hand, and on another hand, to the explicit formula of  $\ell i_m(z)$  ([23]) generalizing the above Soule's characters (1.2). After the Introduction, in §2, we shall recall basic setup for Galois polylogarithms and  $p$ -adic polylogarithms, and in §3, we introduce and study a special family of Coleman power series that bridges these two kinds of polylogarithms through Coleman's reciprocity law. In §4, we present a general formula for  $p$ -adic polylogarithmic characters on the image of  $\mathcal{U}_\infty(F) \xrightarrow{\text{rec}} G_{F_\infty}^{\text{ab}}$  (Theorem 4.4) and prove Theorem 1.1. A (direct) proof of the original Coleman-Ihara formula (1.3) is also obtained as a special case of  $z = 1$ ,  $F = \mathbb{Q}_p$  (Remark 4.5).

In a subsequent Part II [22], we study a generalization of the above formula to the case of more general  $z \in (\mathbb{P}^1 \setminus \{0, 1, \infty\})(\mathbb{Z}_p^{ur})$ . For the generalization, we switch over to a view on the polylogarithmic torsors of paths in Deligne's Tannakian approach [De], and apply a non-commutative lift of Bloch-Kato's explicit reciprocity law that reverses logarithmic mapping of torsors studied by M.Kim [18], M.Olsson [25]. Here, we partially rely on [26] for some technical details.

We note that Kurihara [21] and Gros [17, II] §IV gave an alternative local approach to the essentially same formula as our above Theorem 1.1 by using syntomic cohomology. See [21] (2.11), (2.12) for the case of  $m < p - 2$  and a comment in (2.15) on extension to general case of  $m > 1$  (in (2.12) of loc. cit., “ $\text{Tr}_{F/\mathbb{Q}_p}$ ” seems lacking in print). Compared to the Gros-Kurihara formulation, our approach in the present paper is of more elementary nature and may be useful to find a source reason behind the formula (1.4) in certain explicit Coleman power series  $f_{z,c}(T)$  given in §3 below. Combined with illustrations in loc. cit. and [20] (in particular, p.425), our above result suggests that the  $l$ -adic Galois polylogarithm  $\ell i_m(z, \gamma)$  stands nearby a shadow of Beilinson's cyclotomic element in  $K_{2m-1}(F) \otimes \mathbb{Q}$  at least when  $z$  is a root of unity of order prime to  $p$ .

**NOTATION.** In this paper, we let  $p$  be a fixed odd prime. We fix embeddings  $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$  and  $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$ , which determines a coherent system of roots of unity  $\zeta_n \in \mu_n \subset \overline{\mathbb{Q}}$  with  $\zeta_n = \exp(2\pi i/n)$ . Write  $\mathbb{Z}_p^{ur}$  for the ring of integers of the maximal unramified extension  $\mathbb{Q}_p^{ur}$  of  $\mathbb{Q}_p$ . The group of roots of unity  $\mu(\mathbb{Z}_p^{ur})$  is isomorphic to  $\bigcup_{p \nmid N} \mu_N$ .

For any unramified extension  $F$  of  $\mathbb{Q}_p$ , we denote the arithmetic Frobenius substitution on  $F$  by  $\sigma_F : F \rightarrow F$ . We shall use the notation  $F_\infty$  to denote  $F(\mu_{p^\infty})$ , while we prefer in this paper to keep “ $F_n$ ” unused to avoid confusion (with its custom usage ‘ $F_n = F(\mu_{p^{n+1}})$ ’ in Iwasawa theory.) We define  $\mathcal{U}_\infty(F)$  to be the the norm limit of the group of principal units of  $F(\mu_{p^n})$ . The Galois group  $\text{Gal}(F_\infty/F)$  will be written as  $G_\infty$ .

For any local field  $F/\mathbb{Q}_p$ , we write  $\mathcal{O}_F$ ,  $k_F$  for the ring of integers and its residue field respectively. For any field  $F$ , we denote by  $G_F$  the absolute Galois group of  $F$ , and by  $\chi_{\text{cyc}}$  the  $p$ -adic cyclotomic character  $G_F \rightarrow \mathbb{Z}_p^\times$ .

The Bernoulli polynomials and Bernoulli numbers are given by

$$\sum_{n=1}^{\infty} B_n(X) \frac{t^n}{n!} = \frac{te^{Xt}}{(e^t - 1)}$$

and  $B_n := B_n(0)$ .

## 2. Review of Galois and $p$ -adic polylogarithms

**2.1. Galois polylogarithms.** We review  $\ell$ -adic Galois polylogarithms ([30]-[31]) in the case  $\ell = p$ . Let  $\vec{01}$  be the unit tangential base point on  $\mathbb{P}^1 - \{0, 1, \infty\}$ . We denote by  $\pi_1$  the maximal pro- $p$  quotient of the étale fundamental group  $\pi_1^{\text{ét}}(\mathbb{P}_{\mathbb{Q}}^1 \setminus \{0, 1, \infty\}, \vec{01})$  and identify  $\mathbb{Z}_p(1)$  with that of  $\pi_1^{\text{ét}}(\mathbb{P}_{\mathbb{Q}}^1 \setminus \{0, \infty\}, \vec{01})$ . We will always regard these fundamental groups as equipped with the actions of  $G_{\mathbb{Q}}$  determined by  $\vec{01}$ .

Let  $\mathbf{p} : \pi_1 \rightarrow \mathbb{Z}_p(1)$  be the projection homomorphism. We shall focus on the structure of the quotient group  $\pi_1^{\text{pol}} := \pi_1 / [\text{Ker } \mathbf{p}, \text{Ker } \mathbf{p}]$  (the pro- $p$  polylogarithmic quotient) that has the induced projection  $\mathbf{p}' : \pi_1^{\text{pol}} \rightarrow \mathbb{Z}_p(1)$ . By construction,  $\text{Ker } \mathbf{p}'$  is abelian, hence has a conjugate action of  $\text{Im}(\mathbf{p}) = \mathbb{Z}_p(1)$ . In fact, as discussed in [10, Section 16.11-14], it forms a free  $\mathbb{Z}_p \llbracket \mathbb{Z}_p(1) \rrbracket$ -module generated by a generator  $y$  of the inertia subgroup over the puncture 1: There arises a Galois equivariant exact sequence:

$$(2.1) \quad 0 \rightarrow \mathbb{Z}_p \llbracket \mathbb{Z}_p(1) \rrbracket \cdot y \rightarrow \pi_1^{\text{pol}} \xrightarrow{\mathbf{p}'} \mathbb{Z}_p(1) \rightarrow 0.$$

The sequence turns out to split, as the image  $\text{Im}(\mathbf{p}') = \mathbb{Z}_p(1)$  can be lifted to a Galois-invariant inertia subgroup over 0 along  $\vec{01}$ . We shall take standard generators  $x, y$  of those respective inertia subgroups over 0, 1 so as to correspond to our choice of  $\{\zeta_{p^n}\}_n \in \mu_{p^\infty}$ . It follows then that  $\pi_1^{\text{pol}}$  is isomorphic to a semi-direct product:

$$(2.2) \quad \pi_1^{\text{pol}} = x^{\mathbb{Z}_p} \rtimes (\mathbb{Z}_p \llbracket x^{\mathbb{Z}_p} \rrbracket \cdot y) \cong \mathbb{Z}_p(1) \rtimes (\mathbb{Z}_p \llbracket \mathbb{Z}_p(1) \rrbracket(1)).$$

Here, the action in the last semi-direct product is given by the translation of  $\mathbb{Z}_p(1)$ . Let  $\mathcal{P}^{\text{ét}}(\vec{01})$  be the pro-unipotent completion of  $\pi_1^{\text{pol}}$  (which we call the  $p$ -adic étale polylogarithmic group), and let  $\log : \mathcal{P}^{\text{ét}}(\vec{01}) \rightarrow \text{Lie}(\mathcal{P}^{\text{ét}}(\vec{01})) = \mathbb{Q}_p(1) \oplus \prod_{k=1}^{\infty} \mathbb{Q}_p(k)$  its logarithm map. We have then the following Lie expansion map (also denoted  $\log$ ):

$$(2.3) \quad \log : \pi_1^{\text{pol}} \hookrightarrow \mathcal{P}^{\text{ét}}(\vec{01}) \longrightarrow \text{Lie}(\mathcal{P}^{\text{ét}}(\vec{01})) = \mathbb{Q}_p(1) \oplus \prod_{k=1}^{\infty} \mathbb{Q}_p(k).$$

In practice, both of  $\mathcal{P}^{\text{ét}}(\vec{01})$  and  $\text{Lie}(\mathcal{P}^{\text{ét}}(\vec{01}))$  are realized as subsets of the non-commutative power series ring  $\mathbb{Q}_p \langle\langle X, Y \rangle\rangle$  in  $X = \log(x)$ ,  $Y = \log(y)$  modulo the ideal  $I_Y$  generated by the words having  $Y$  twice or more.

Now, take any point  $z \in (\mathbb{P}^1 \setminus \{0, 1, \infty\})(\overline{\mathbb{Q}_p})$  and consider the set  $\pi_1(\vec{01}, z)$  of the pro- $p$  étale paths from  $\vec{01}$  to  $z$ . Set  $F := \mathbb{Q}_p(z)$ . Then  $\pi_1(\vec{01}, z)$  forms a  $\pi_1$ -torsor equipped with a canonical action of  $G_F$ . In the well-known manner, the reduction map  $\pi_1 \twoheadrightarrow \pi_1^{\text{pol}}$  gives rise to a  $\pi_1^{\text{pol}}$ -torsor  $\pi_1^{\text{pol}}(\vec{01}, z)$  with  $G_F$ -action on it.

Choose any path  $\gamma \in \pi_1^{\text{pol}}(\overrightarrow{01}, z)$  and call it the *defining path*. Then, for each  $\sigma \in G_F$ , there is a unique element  $\dagger_\sigma \in \pi_1^{\text{pol}}$  such that  $\gamma$  is written as  $\dagger_\sigma^z * \sigma(\gamma)$ . The  $p$ -adic Galois polylogarithms (associated to  $\gamma$ ) are defined as coefficients of the expansion of the Lie series  $\log(\dagger_\sigma^z)$  in  $\mathbb{Q}_p[[X, Y]]/I_Y$  as:

$$(2.4) \quad \log(\dagger_\sigma^z) \equiv -\kappa_z(\sigma)X - \sum_{k=1}^{\infty} \ell i_k(z)(\sigma) \cdot \text{ad}(X)^{k-1}(Y).$$

The above first coefficient  $\kappa_z : G_F \rightarrow \mathbb{Z}_p$  is the Kummer 1-cocycle

$$(2.5) \quad \zeta_{p^n}^{\kappa_z(\sigma)} = (z^{1/p^n})^{\sigma-1} := \frac{\sigma(z^{1/p^n})}{z^{1/p^n}}$$

over the system  $\{z^{1/p^n}\}_{n=1}^{\infty}$  which is determined by the specialization homomorphism  $\bigcup_{n=1}^{\infty} \overline{F}[t^{1/p^n}] \rightarrow \overline{F}$  along the path  $\gamma$ . The second coefficient  $\ell i_1(z) : G_F \rightarrow \mathbb{Z}_p$  is the Kummer 1-cocycle  $\kappa_{1-z}$  along the composition of the standard path  $0 \rightsquigarrow 1$  with  $\bar{\gamma} : 1 \rightsquigarrow 1 - z$ , where  $\bar{\gamma}$  is the obvious reflection of  $\gamma$ . The other coefficients  $\ell i_k(z) (= \ell i_k(z, \gamma))$  ( $k \geq 2$ ) are in general only 1-cochains  $G_F \rightarrow \mathbb{Q}_p$ .

The following lemma is crucial to understand nature of the LHS of our main statement of Introduction.

**Lemma 2.1** ([30] Theorem 5.3.1). *The fixed field  $H_m$  of the intersection of the kernels of  $\chi_{\text{cyc}}$ ,  $\kappa_z$  and of  $\ell i_1(z, \gamma), \dots, \ell i_{m-1}(z, \gamma)$  is independent of the choice of the defining path  $\gamma : \overrightarrow{01} \rightsquigarrow z$ . Moreover, for  $\sigma \in G_{H_m}$ , the value  $\ell i_m(z, \gamma) \in \mathbb{Q}_p$  is independent of the choice of  $\gamma$ .  $\square$*

Explicit formulas for  $\ell i_k(z)$  have been given in [23] for all  $\sigma \in G_F$ . For simplicity, we present the formula only for  $\sigma \in G_F$  with  $\chi_{\text{cyc}}(\sigma) = 1$  and  $\kappa_z(\sigma) = 0$  in the following

**Proposition 2.2** ([23] §3 Corollary). *For  $m \geq 1$  and  $\sigma \in G_{F(\zeta_{p^\infty}, z^{1/p^\infty})}$ , we have*

$$(2.6) \quad \ell i_m(z)(\sigma) = (-1)^{m-1} \frac{\tilde{\chi}_m^z(\sigma)}{(m-1)!}$$

where  $\tilde{\chi}_m^z(\sigma) \in \mathbb{Z}_p$  is defined by the Kummer properties

$$(2.7) \quad \zeta_{p^n}^{\tilde{\chi}_m^z(\sigma)} = \left( \prod_{a=0}^{p^n-1} (1 - \zeta_{p^n}^a z^{1/p^n})^{\frac{a^{m-1}}{p^n}} \right)^{\sigma-1} \quad (n \geq 1). \quad \square$$

Indeed, in [23], where an embedding  $\overline{F} \hookrightarrow \mathbb{C}$  is fixed and  $\gamma : \overrightarrow{01} \rightsquigarrow z$  is taken to be a continuous curve on  $\mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}$ , we constructed a measure  $\hat{\kappa}_{z,\gamma}(\sigma) \in \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}]]$  called the (adelic) *Kummer-Heisenberg measure*, after specifying standard branches of power roots of  $(1 - \zeta_n^a z^{1/n})$  along  $\gamma$ . (In fact, in loc.cit., we assumed  $z \in F$  a number field, but the argument goes similarly for any subfield  $F$  of  $\mathbb{C}$ .) The  $\mathbb{Z}_p$ -valued measure  $\kappa_{z,\gamma}(\sigma) \in \mathbb{Z}_p[[\mathbb{Z}_p]]$  obtained as the image of  $\hat{\kappa}_{z,\gamma}(\sigma)$  is given by

$$(2.8) \quad \zeta_{p^r}^{\kappa_{z,\gamma}(\sigma)(a+p^n\mathbb{Z}_p)} = \frac{\sigma((1 - \zeta_{p^n}^{\chi_{\text{cyc}}(\sigma)^{-1}a} z^{1/p^n})^{\frac{1}{p^r}})}{(1 - \zeta_{p^n}^a \sigma(z^{1/p^n}))^{\frac{1}{p^r}}} \quad (a \in \mathbb{Z}_p, r \geq 1)$$

for each  $\sigma \in G_F$ . The  $p$ -adic polylogarithmic character  $\tilde{\chi}_m^z : G_F \rightarrow \mathbb{Z}_p$  defined in [23] can be written as the moment integral

$$(2.9) \quad \tilde{\chi}_m^z(\sigma) = \int_{\mathbb{Z}_p} x^{m-1} d\kappa_{z,\gamma}(\sigma)(x) \quad (\sigma \in G_F).$$

In this paper, we shall also consider a restricted version of the above moment integral to  $\mathbb{Z}_p^\times$ , and introduce the *restricted*  $p$ -adic polylogarithmic character  $\chi_m^z : G_F \rightarrow \mathbb{Z}_p$  by

$$(2.10) \quad \chi_m^z(\sigma) = \int_{\mathbb{Z}_p^\times} x^{m-1} d\kappa_{z,\gamma}(\sigma)(x) \quad (\sigma \in G_F).$$

Then, it is easy to see from Proposition 2.2 that the value  $\chi_m^z(\sigma) \in \mathbb{Z}_p$  for  $\sigma \in G_{F(\zeta_{p^\infty}, z^{1/p^\infty})}$  is characterized by the Kummer properties

$$(2.11) \quad \zeta_{p^n}^{\chi_m^z(\sigma)} = \left( \prod_{\substack{1 \leq a \leq p^n \\ p \nmid a}} (1 - \zeta_{p^n}^a z^{1/p^n})^{\frac{m-1}{p^n}} \right)^{\sigma-1} \quad (n \geq 1).$$

When  $z = 1$  and  $F = \mathbb{Q}$ , this is nothing but what is called the  $m$ -th Soule character.

**REMARK 2.3.** Our considering the measure  $\hat{\kappa}_{z,\gamma}(\sigma)$  should be traced back partly to an old idea of O. Gabber (as documented in [23] §3 Remark 1 and [24] Acknowledgments). In [32], it is generalized to a sequence of measures ‘ $K_r(z)$ ’ on  $\mathbb{Z}_p^r$  ( $r \geq 1$ ) that encodes all coefficients of  $\log(\tilde{f}_\sigma^z)$  (which correspond to the multiple polylogarithms) as integrals over  $\mathbb{Z}_p^r$ .

A simple connection between  $\tilde{\chi}_m^z$  and  $\chi_m^z$  can be obtained by comparing (2.7) and (2.11). The following generalizes [23] §2 Remark 2:

**Lemma 2.4** ([32] Proposition 5.1(v)). *For any continuous path  $\gamma : \vec{0}\vec{1} \rightsquigarrow z$  with respect to an embedding  $\overline{F} \hookrightarrow \mathbb{C}$ , any  $\sigma \in G_F$ , and  $m \geq 2$ , we have*

$$\tilde{\chi}_m^z(\sigma) = \chi_m^z(\sigma) + p^{m-1} \chi_m^{z^{1/p}}(\sigma) + \cdots = \sum_{k=0}^{\infty} p^{k(m-1)} \chi_m^{z^{1/p^k}}(\sigma).$$

*Proof.* Denote the path  $\vec{0}\vec{1} \rightsquigarrow z^{1/p^k}$  induced from  $\gamma : \vec{0}\vec{1} \rightsquigarrow z$  by the same symbol  $\gamma$ . Then, it is easy to see from (2.8) that  $\kappa_{z,\gamma}(\sigma)(p^k a + p^{n+k} \mathbb{Z}_p) = \kappa_{z^{1/p^k}, \gamma}(\sigma)(a + p^n \mathbb{Z}_p)$  for all  $k \geq 0$ . Then, decomposing the integral to infinite pieces as

$$\begin{aligned} \int_{\mathbb{Z}_p} x^{m-1} d\kappa_{z,\gamma}(\sigma)(x) &= \sum_{k=0}^{\infty} \int_{p^k \mathbb{Z}_p - p^{k+1} \mathbb{Z}_p} x^{m-1} d\kappa_{z,\gamma}(\sigma)(x) \\ &= \sum_{k=0}^{\infty} \int_{\mathbb{Z}_p^\times} (p^k x)^{m-1} d\kappa_{z^{1/p^k}, \gamma}(\sigma)(x) \end{aligned}$$

immediately yields the desired formula. □

REMARK 2.5. As noted above (after (2.5)), every etale path  $\gamma : \overrightarrow{01} \rightsquigarrow z$  determines the value  $z^{1/p^n} \in \overline{F}$ , hence the above Kummer quantity in (2.8) makes sense for  $\sigma \in G_{F(\zeta_{p^\infty}, z^{1/p^\infty})}$  with no need to mention the choice of  $\overline{F} \hookrightarrow \mathbb{C}$ . So, in what follows, we will consider both versions of  $p$ -adic polylogarithmic characters  $\tilde{\chi}_m^z, \chi_m^z$  for arbitrary  $\gamma \in \pi_1^{\text{pol}}(\overrightarrow{01}, z)$  but only for  $\sigma \in G_{F(\zeta_{p^\infty}, z^{1/p^\infty})}$ . Formulas in Proposition 2.2 and Lemma 2.4 also hold for them.

**2.2. Koblitz measure and  $p$ -adic polylogarithms.** Let  $|\cdot|_p$  denote the standard norm on  $\mathbb{C}_p$  with  $|p|_p = p^{-1}$  and write  $\mathcal{O} := \mathcal{O}_{\mathbb{C}_p} = \{z \in \mathbb{C}_p; |z|_p \leq 1\}$ . For  $z \in \mathbb{C}_p$  with  $|1 - z|_p \geq 1$ , Neal Koblitz introduced in [19, p. 457], an  $\mathcal{O}$ -valued measure  $\mu_z$  on  $\mathbb{Z}_p$  by

$$\mu_z(a + p^n \mathbb{Z}_p) = \frac{z^a}{1 - z^{p^n}} \text{ for all } n \in \mathbb{Z}, 1 \leq a \leq p^n.$$

Note here that  $|1 - z^{p^n}|_p = |z^{p^n}|_p$  if  $|z|_p > 1$  and that  $|1 - z^{p^n}|_p \geq 1$  if  $|z|_p \leq 1$  for all  $n \geq 0$  under our assumption on  $z$ .

**Lemma 2.6.** *Let  $\mathcal{F}_z(T)$  be the element of the Iwasawa algebra  $\mathcal{O}[[T]]$  that corresponds to  $\mu_z$ , and let  $\mu_z^{(p)}$  be the restriction of the measure  $\mu_z$  to  $\mathbb{Z}_p^\times$ . Then,*

$$(1) \quad \mathcal{F}_z(T) = \frac{1}{1 - z(1 + T)} \in \mathcal{O}[[T]],$$

$$(2) \quad \mathcal{F}_z^{(p)}(T) = \frac{1}{1 - z(1 + T)} - \frac{1}{1 - z^p(1 + T)^p}.$$

Proof. The first formula is a consequence of the congruence

$$\mathcal{F}_z(T) = \frac{1}{1 - z^{p^n}} \left( \frac{1 - z^{p^n}}{1 - z(1 + T)} \right) \equiv \sum_{a=0}^{p^n-1} \frac{z^a}{1 - z^{p^n}} (1 + T)^a$$

modulo the ideal  $((1 + T)^{p^n} - 1)$  in  $\mathcal{O}[[T]]$  for every  $n$ . For the second one, recalling a formula [9] §3.4 for the restriction of a measure on  $\mathbb{Z}_p$  to  $\mathbb{Z}_p^\times$ , we obtain  $\mathcal{F}_z^{(p)} = \mathcal{F}_z(T) - \frac{1}{p} \sum_{\xi \in \mu_p} \mathcal{F}_z(\xi(1 + T) - 1)$ . Apply then a general formula  $\sum_{\xi \in \mu_n} \frac{1}{1 - \xi Y} = \frac{n}{1 - Y^n}$ .  $\square$

Note that there is an equality

$$(2.12) \quad \mathcal{F}_z^{(p)}(T) = \sum_{n=0}^{\infty} \left( \int_{\mathbb{Z}_p^\times} x^n d\mu_z(x) \right) \frac{T^n}{n!}$$

in  $\mathbb{C}_p[[X]]$  with  $1 + T = e^X$ .

In [7], Coleman introduced the  $p$ -adic polylogarithm function  $\text{Li}_k^{p\text{-adic}}(z)$  ( $k \geq 1$ ) and its companion function  $\text{Li}_k^{(p)}(z) := \text{Li}_k^{p\text{-adic}}(z) - p^{-k} \text{Li}_k^{p\text{-adic}}(z^p)$  (In his notation,  $\text{Li}_k^{p\text{-adic}}(z) := \ell_k(z)$ ,  $\text{Li}_k^{(p)}(z) := \ell_k^{(p)}(z)$ ). The latter function is given by the Koblitz measure as follows:

$$(2.13) \quad \text{Li}_k^{(p)}(z) = \int_{\mathbb{Z}_p^\times} x^{-k} d\mu_z(x) = \int_{\mathbb{Z}_p^\times} x^{-k} d\mu_z^{(p)}(x)$$

(see [7, Lemma 7.2]). In other words, if  $|1 - z|_p \geq 1$ , we have the equality

$$(2.14) \quad \text{Li}_k^{(p)}(z) = \lim_{n \rightarrow \infty} \left( \sum_{a=1, p \nmid a}^{p^n} \frac{z^a a^{-k}}{1 - z^{p^n}} \right).$$

REMARK 2.7. It is worthwhile to recall the following: Coleman [5] showed that  $\text{Li}_k^{p\text{-adic}}(z)$  has ‘‘analytic continuation along Frobenius’’ to  $\mathbb{P}^1(\mathbb{C}_p) - \{1, \infty\}$  depending on branch parameter of Iwasawa logarithm (cf. e.g., [20, p.425]). This result has been extended to the  $p$ -adic multiple polylogarithms by H. Furusho [16]. See also Remark 2.3 for comparable progress in the theory of Galois polylogarithms.

### Bernoulli distribution and Kubota-Leopoldt $L$ -function

Before closing this subsection, we quickly review classically known facts about the Bernoulli distribution for the case  $z = 1$ . Let  $c$  be an integer  $\neq 1$  prime to  $p$ . Then a measure  $E_c$  on  $\mathbb{Z}_p$  is defined as follows (cf. [29] §12.2): For each  $n \geq 1$ , choose any integer  $\bar{c} = \bar{c}_n$  with  $c\bar{c} \equiv 1 \pmod{p^n}$ , and define

$$(2.15) \quad E_c(a + p^n \mathbb{Z}_p) := B_1 \left( \left\{ \frac{a}{p^n} \right\} \right) - c B_1 \left( \left\{ \frac{\bar{c}a}{p^n} \right\} \right),$$

where  $\{*\}$  means the fractional part. This is independent of the choice of  $\bar{c}$ , and forms a  $\mathbb{Z}_p$ -valued measure  $E_c$  on  $\mathbb{Z}_p$ . A well-known calculation ( $e^X = 1 + T$ ):

$$(2.16) \quad \frac{1}{T} - \frac{c}{(1+T)^c - 1} = \sum_{n=1}^{\infty} (1 - c^n) \frac{B_n}{n} \cdot \frac{X^{n-1}}{(n-1)!}$$

as well as the formula  $L_p(1 - n, \omega^n) = -(1 - p^{n-1}) \frac{B_n}{n}$  give

$$(2.17) \quad \int_{\mathbb{Z}_p^\times} x^{m-1} dE_c(x) = (1 - c^m)(1 - p^{m-1}) \frac{B_m}{m} = (c^m - 1) L_p(1 - m, \omega^m).$$

(Cf. proofs of [29] Cor. 5.13, Cor. 12.3). A formula of the  $p$ -adic Mellin transformation reads then as follows:

$$(2.18) \quad \int_{\mathbb{Z}_p^\times} x^{-m} dE_c(x) = (c^{1-m} - 1) L_p(m, \omega^{1-m}).$$

(Cf. [29] Theorem 12.2 with  $d = 1, \chi = \omega^{1-m}, s = -m$ .) Finally, we remark that the power series  $\mathcal{F}_z^{(p)}$  of Lemma 2.6 (2) for  $z = 1$  has a pole at  $T = 0$ . However, its usual ‘ $c$ -correction’ cancel the pole and has an expansion in the form:

$$(2.19) \quad \mathcal{F}_1^{(p)}(T) - c \mathcal{F}_1^{(p)}((1+T)^c - 1) = \sum_{n=0}^{\infty} \left( \int_{\mathbb{Z}_p^\times} x^n dE_c(x) \right) \frac{T^n}{n!}$$

in  $\mathbb{Q}_p \llbracket X \rrbracket$  with  $1 + T = e^X$ . (Note LHS= $Ug(T)$  of [29] §12 p.251-252.)

### 3. Special family of Coleman series

**3.1. Basic setup and  $f_{z,c}(T)$ .** In this subsection, after reviewing some basic notions, we shall introduce a special class of Coleman power series that play important roles in our proof of Theorem 1.1.

Let  $F$  be a finite unramified extension of  $\mathbb{Q}_p$ . In this section, we study Galois and  $p$ -adic polylogarithms  $li_m(z)$  and  $Li_m^{p\text{-adic}}(z)$  for  $z \in F \cap \mu(\mathbb{Z}_p^{ur})$ , a root of unity of order prime to  $p$ . We will introduce and observe behaviors of certain special power series  $f_{z,c}(T) \in \mathcal{O}_F \llbracket T \rrbracket$  closely related to  $\mathcal{F}_z^{(p)}(T)$  of the previous section.

We begin by setting up basic operations on power series. Set  $F_\infty := F(\mu_{p^\infty})$  and denote by  $G_\infty$  the Galois group of  $F_\infty/F$ . Note that, the  $p$ -adic cyclotomic character induces the canonical isomorphism

$$\bar{\chi}_{\text{cyc}} : G_\infty \xrightarrow{\sim} \mathbb{Z}_p^\times$$

as  $F$  is unramified over  $\mathbb{Q}_p$ . For each  $a \in \mathbb{Z}_p^\times$ , we define  $\sigma_a \in G_\infty$  to be  $\bar{\chi}_{\text{cyc}}^{-1}(a)$ .

We set  $\mathcal{O}_F \llbracket T \rrbracket^{\times 1} := \text{Ker}(\mathcal{O}_F \llbracket T \rrbracket^\times \xrightarrow{\text{aug}} \mathcal{O}_F^\times \rightarrow k_F^\times)$ . Here, the first map is the augmentation map and  $k_F$  the residue field of  $F$ . Define the action of  $G_\infty$  on  $\mathcal{O}_F \llbracket T \rrbracket$  by

$$\sigma_a f(T) = f((1+T)^a - 1).$$

that restricts to the action on  $\mathcal{O}_F \llbracket T \rrbracket^{\times 1}$ . Since  $\mathcal{O}_F \llbracket T \rrbracket$  and  $\mathcal{O}_F \llbracket T \rrbracket^{\times 1}$  are compact (additive and multiplicative)  $\mathbb{Z}_p$ -modules, the complete group ring  $\mathbb{Z}_p \llbracket G_\infty \rrbracket$  acts on both topological abelian groups. We regard  $\mathcal{O}_F \llbracket T \rrbracket$  (resp.  $\mathcal{O}_F \llbracket T \rrbracket^{\times 1}$ ) as a left (resp. right)  $\mathbb{Z}_p \llbracket G_\infty \rrbracket$ -module.

REMARK 3.1. We remark the followings:

- (1) The  $\mathbb{Z}_p$ -module structure on  $\mathcal{O}_F \llbracket T \rrbracket^{\times 1}$  is given by  $f(T)^c = \sum_{n=0}^{\infty} \binom{c}{n} (f(T) - 1)^n$  for  $c \in \mathbb{Z}_p$ ,  $f \in \mathcal{O}_F \llbracket T \rrbracket^{\times 1}$ , where  $\binom{x}{n} := \frac{x(x-1)\cdots(x-n+1)}{n!} \in \mathbb{Z}_p$ .
- (2) Since  $\mathcal{O}_F \llbracket T \rrbracket$  has a canonical structure of (additive)  $\mathcal{O}_F$ -module, the complete group ring  $\mathcal{O}_F \llbracket G_\infty \rrbracket$  acts on it. However,  $\mathcal{O}_F \llbracket T \rrbracket^{\times 1}$  does not have a canonical  $\mathcal{O}_F$ -module structure, i.e., there does not exist a canonical action of  $\mathcal{O}_F \llbracket G_\infty \rrbracket$  on it.
- (3) A finite element  $\sum_{a \in \mathbb{Z}_p^\times} c_a \sigma_a$  ( $c_a = 0$  for all but finitely many  $a \in G_\infty$ ) acts on  $f(T)$  in the following forms:  $(\sum_a c_a \sigma_a) \cdot f(T) = \sum_a c_a f((1+T)^a - 1)$ , and  $f(T)^{\sum_a c_a \sigma_a} = \prod_a f((1+T)^a - 1)^{c_a}$ .

**Special Coleman series.** For  $c \in \mathbb{Z}$  ( $c \neq 1$ ,  $p \nmid c$ ) and for  $z \in F \cap \mu(\mathbb{Z}_p^{ur})$ , let us introduce power series  $f_{z,c} \in \mathcal{O}_F \llbracket T \rrbracket$  by

$$(3.1) \quad f_{z,c}(T) := \begin{cases} c \cdot \frac{(1+T)^{-\frac{1}{2}} - (1+T)^{\frac{1}{2}}}{(1+T)^{-\frac{c}{2}} - (1+T)^{\frac{c}{2}}}, & (z = 1); \\ \frac{(1+T)^{-\frac{1}{2}} - z(1+T)^{\frac{1}{2}}}{(1+T)^{-\frac{c}{2}} - z(1+T)^{\frac{c}{2}}}, & (z \in \mu(\mathbb{Z}_p^{ur}) \setminus \{1\}). \end{cases}$$

We denote by  $\sigma_F : F \rightarrow F$  the Frobenius automorphism of  $F$ . It acts on each element of  $\mathcal{O}_F \llbracket T \rrbracket$  by the action on coefficients. Define the *integral logarithm*  $\mathcal{L} : \mathcal{O}_F \llbracket T \rrbracket^\times \rightarrow$

$\mathcal{O}_F \llbracket T \rrbracket$  by the formula

$$\mathcal{L}(f(T)) := \frac{1}{p} \log \left( \frac{f(T)^p}{(\sigma_F f)((1+T)^p - 1)} \right).$$

Further, we define the differential operator  $D$  on  $F \llbracket T \rrbracket$  to be  $(1+T) \frac{d}{dT}$  and the ring homomorphism  $[p] : \mathcal{O}_F \llbracket T \rrbracket \rightarrow \mathcal{O}_F \llbracket T \rrbracket$  by  $([p]f)(T) = f((1+T)^p - 1)$ .

Let  $\mathcal{N} : \mathcal{O}_F \llbracket T \rrbracket^\times \rightarrow \mathcal{O}_F \llbracket T \rrbracket^\times$  (resp.  $\overline{\mathcal{S}} : \mathcal{O}_F \llbracket T \rrbracket \rightarrow \mathcal{O}_F \llbracket T \rrbracket$ ) be the norm operator (resp. the ‘reduced’ trace operator) introduced by Coleman (cf. [3], [4, p.386]). They are characterized by

$$([p]\mathcal{N}f)(T) = \prod_{\xi \in \mu_p} f(\xi(1+T) - 1), \quad ([p]\overline{\mathcal{S}}f)(T) = \frac{1}{p} \sum_{\xi \in \mu_p} f(\xi(1+T) - 1).$$

**Lemma 3.2.** *Let  $F$  be a finite unramified extension of  $\mathbb{Q}_p$  containing  $z \in \mu(\mathbb{Z}_p^{ur})$ . Then,*

- (1)  $f_{z,c}(T) \in 1 + T\mathcal{O}_F \llbracket T \rrbracket$ .
- (2)  $\mathcal{N}(f_{z,c}(T)) = (\sigma_F f_{z,c})(T)$ .
- (3)  $D\mathcal{L}f_{z,c}(T) = -\left(\mathcal{F}_z^{(p)}(T) - c\mathcal{F}_z^{(p)}((1+T)^c - 1)\right)$

*Proof.* (1) This is just claiming  $f_{z,c}(0) = 1$ . Use d’Hôspital rule when  $z = 1$ . (2) For  $a \in \mathbb{Z}_p^\times$ , set  $k_{z,a}(T) := (1+T)^{-\frac{a}{2}} - z(1+T)^{\frac{a}{2}}$ . Then, by the definition of the norm operator, we have:

$$\begin{aligned} ([p]\mathcal{N}k_{z,a})(T) &= \prod_{\xi \in \mu_p} \left( (\xi(1+T))^{-\frac{a}{2}} - z(\xi(1+T))^{\frac{a}{2}} \right) \\ &= \prod_{\xi \in \mu_p} \left( (1+T)^{-\frac{a}{2}} - z\xi^a(1+T)^{\frac{a}{2}} \right) \\ &= (1+T)^{-\frac{pa}{2}} - z^p(1+T)^{\frac{pa}{2}} = ([p]k_{z^p,a})(T). \end{aligned}$$

As  $[p]$  is injective, it follows that  $\mathcal{N}k_{z,a} = k_{z^p,a}$ . Since  $\sigma_F$  acts on  $\mu(\mathbb{Z}_p^{ur})$  by  $p$ -power and the operator  $\mathcal{N}$  is multiplicative, the assertion (2) follows. (3) also follows from a simple calculation

$$D\mathcal{L}(k_{z,a}(T)) = a \left( -\frac{1}{1-z(1+T)^a} + \frac{1}{1-\sigma_F(z)(1+T)^{ap}} \right)$$

with Lemma 2.6 (2), including the case  $z = 1$  where  $\mathcal{F}_1^{(p)}(T) - c\mathcal{F}_1^{(p)}((1+T)^c - 1) \in \mathbb{Z}_p \llbracket T \rrbracket$  is discussed in (2.19). (See also Remark 3.3 below.)  $\square$

**REMARK 3.3.** (1) The integral logarithm  $\mathcal{L}$  on  $\mathcal{O}_F \llbracket T \rrbracket^{\times 1}$  preserves the action of  $\mathbb{Z}_p \llbracket G_\infty \rrbracket$ , namely, it holds that  $\mathcal{L}(f^\lambda) = \lambda \cdot \mathcal{L}(f)$  for any  $\lambda \in \mathbb{Z}_p \llbracket G_\infty \rrbracket$  and  $f \in \mathcal{O}_F \llbracket T \rrbracket^{\times 1}$ . By a simple argument, one can easily see that  $\mathcal{N}(f) = \sigma_F(f)$  implies  $\overline{\mathcal{S}}(\mathcal{L}f) = 0$ . Namely,  $\mathcal{L}$  maps  $(\mathcal{O}_F \llbracket T \rrbracket^\times)^{\mathcal{N}=\sigma_F}$  into  $\mathcal{O}_F \llbracket T \rrbracket^{\overline{\mathcal{S}}=0}$ .

(2) If  $f \in \mathcal{O}_F \llbracket T \rrbracket^\times$  is not necessarily in  $\mathcal{O}_F \llbracket T \rrbracket^{\times 1}$ , then  $\log(f)$  has no obvious sense and the convergence (and stay) of  $\mathcal{L}(f)$  in  $\mathcal{O}_F \llbracket T \rrbracket$  involves technical estimate of

coefficients (see [9] Lemma 2.5.1). We have  $\mathcal{L}(fg) = \mathcal{L}(f) + \mathcal{L}(g)$  as long as both  $f, g \in \mathcal{O}_F \llbracket T \rrbracket^\times$ . But, in practical computation, it could happen that we know the existence of  $\mathcal{L}(fg)$  but not of individual  $\mathcal{L}(f)$  or  $\mathcal{L}(g)$ . A possible way to remedy such a computational difficulty (which occurs also in the proof of the above Lemma 3.2 (3) when  $z = 1$ ) is to consider the logarithmic derivative  $D\mathcal{L}(f) = (1 - [p]\sigma_F) \cdot (Df/f)$  (whose existence is often easier to see) and to use  $D\mathcal{L}(fg) = D\mathcal{L}(f) + D\mathcal{L}(g)$ .

- (3) As remarked in [6], Corollary of Theorem 3, the differential operator  $D$  gives an isomorphism  $D : \mathcal{O}_F \llbracket T \rrbracket^{\overline{S}=0} \xrightarrow{\sim} \mathcal{O}_F \llbracket T \rrbracket^{\overline{S}=0}$ . (The proof given in loc. cit. for  $\mathbb{Z}_p \llbracket T \rrbracket$  works also for  $\mathcal{O}_F \llbracket T \rrbracket$  with no obstructions.) This defines the inverse map

$$D^{-1} : \mathcal{O}_F \llbracket T \rrbracket^{\overline{S}=0} \xrightarrow{\sim} \mathcal{O}_F \llbracket T \rrbracket^{\overline{S}=0}.$$

**Lemma 3.4.** *For a power series  $f(T) \in \mathcal{O}_F \llbracket T \rrbracket$ , denote by  $\mu_f$  the corresponding  $\mathcal{O}_F$ -valued measure on  $\mathbb{Z}_p$ . Suppose  $f(T) \in \mathcal{O}_F \llbracket T \rrbracket^{\overline{S}=0}$ . Then, for  $k \in \mathbb{Z}$ , it holds that*

$$D^{-k}(f)(T) = \sum_{n=0}^{\infty} \left( \int_{\mathbb{Z}_p^\times} x^{n-k} d\mu_f(x) \right) \frac{T^n}{n!}$$

in  $F \llbracket X \rrbracket$  with  $e^X = 1 + T$ . In particular, we have

$$D^{-k}(f)(0) = \int_{\mathbb{Z}_p^\times} x^{-k} d\mu_f(x).$$

*Proof.* The case of  $k \leq 0$  is well known (e.g., [9] Lemma 3.3.5). So, assume  $k > 0$ . Consider a linear functional  $L$  on the  $\mathcal{O}_F$ -valued continuous functions on  $\mathbb{Z}_p$  defined by  $h \mapsto \int_{\mathbb{Z}_p} hx^{-k} d\mu_f(x)$ . This is well defined, as  $\overline{S}(f) = 0$  insures the support of the measure  $\mu_f$  is on  $\mathbb{Z}_p^\times$ . Since  $L$  is bounded with support in  $\mathbb{Z}_p^\times$ , it follows that  $L$  is of the form  $h \mapsto \int_{\mathbb{Z}_p} h d\mu_g(x)$  for a unique  $g(T) \in \mathcal{O}_F \llbracket T \rrbracket^{\overline{S}=0}$  that, by the construction, should have the same expansion as RHS of the lemma (cf. [9] p.35 and Lemma 3.3.5). Since  $D = (1 + T) \frac{d}{dT} = \frac{d}{dX}$ , we find  $D^k(g) = f$ . The uniqueness of  $g$  (cf. [6] Corollary of Theorem 3) in  $\mathcal{O}_F \llbracket T \rrbracket^{\overline{S}=0}$  concludes  $D^{-k}(f) = g$ .  $\square$

### 3.2. Coleman's reciprocity law.

**DEFINITION 3.5** (Coates-Wiles homomorphism). Let  $\mathcal{U}_\infty(F)$  be the norm limit of principal units of  $\{F(\mu_{p^n})\}_n$  and denote by

$$[\text{Col}] : \mathcal{U}_\infty(F) \rightarrow \mathcal{O}_F \llbracket T \rrbracket^{\times 1} \cap (\mathcal{O}_F \llbracket T \rrbracket^\times)^{\mathcal{N}=\sigma_F}, \quad \epsilon = (\epsilon_n) \mapsto g_\epsilon(T)$$

the Coleman map which is characterized by

$$(3.2) \quad (\sigma_F^{-n} g_\epsilon)(T)|_{T=\zeta_{p^n}-1} = \epsilon_n.$$

The Coates-Wiles homomorphism  $\phi_{m,F}^{CW} : \mathcal{U}_\infty(F) \rightarrow \mathcal{O}_F$  is defined by the equality

$$(3.3) \quad \log(g_\epsilon(T)) = \sum_{m=0}^{\infty} \frac{\phi_{m,F}^{CW}(\epsilon)}{m!} X^m$$

in  $F \llbracket X \rrbracket$  with  $1 + T = \exp(X)$ .

Let  $\text{rec}_n : F(\mu_{p^n})^\times \rightarrow G_{F(\mu_{p^n})}^{\text{ab}}$  be the reciprocity map of local class field theory ( $n \leq \infty$ ). When  $n = \infty$ , the reciprocity map induces an embedding  $\text{rec}_\infty : \mathcal{U}_\infty(F) \hookrightarrow G_{F_\infty}^{\text{ab}}$  (recall  $F_\infty = F(\mu_{p^\infty})$ , as defined in Notation of Introduction). The above Coates-Wiles homomorphism  $\phi_{m,F}^{CW}$  extends uniquely to a  $G_\infty = \text{Gal}(F_\infty/F)$ -homomorphism from  $G_{F_\infty}^{\text{ab}}$  into  $F(m) := F \otimes_{\mathbb{Z}_p} (m)$ . This extension and its image in  $\text{Hom}_{G_\infty}(G_{F_\infty}^{\text{ab}}, F(m)) \cong H^1(F, F(m))$  will also be denoted by  $\phi_{m,F}^{CW}$  (cf. Bloch-Kato [2] Section 2).

**REMARK 3.6.** In (3.2), we employ Bloch-Kato's normalization ([2], Theorem 2.2) on powers of  $\sigma_F$ , which differs from that in [4]. The constant term of  $g_\epsilon$  is  $\equiv 1 \pmod{p}$ , but may not be 1. This causes our summation in (3.3) to start from  $m = 0$  which modifies [2] (p.344).

The Hilbert norm residue symbol

$$(\cdot, \cdot)_{p^n} : F(\mu_{p^n})^\times \times F(\mu_{p^n})^\times \rightarrow \mu_{p^n}$$

is defined by the formula

$$(a, b)_{p^n} = (a^{1/p^n})^{\text{rec}_n(b)-1}.$$

We shall make use of Coleman's explicit reciprocity law on Hilbert norm residue symbols in the following form: Recall that Coleman [4] introduces a continuous linear functional  $\int_n : F((T))_1 \rightarrow F$  (where  $F((T))_1$  denotes the ring of power series which converge on the unit open ball on  $\mathbb{C}_p$ ) by

$$\int_n f := \frac{1}{p^n} \sum_{\zeta \in \mu_{p^n}} f(\zeta - 1).$$

**Theorem 3.7** (Coleman [4],[7]). *Let  $f(T) \in 1 + T\mathcal{O}_F \llbracket T \rrbracket$  satisfy  $\mathcal{N}(f) = \sigma_F(f)$ , and  $g = g_\epsilon(T) = [\text{Col}](\epsilon)$  be the Coleman power series associated to  $\epsilon = (\epsilon_n) \in \mathcal{U}_\infty$ . Then,*

$$(f(\zeta_{p^n} - 1), \epsilon_n)_{p^n} = \zeta_{p^n}^{\text{Tr}_{F/\mathbb{Q}_p}(\int_n \mathcal{L}(f) \cdot D\mathcal{L}(\sigma_F^{-n} g_\epsilon))}.$$

*Proof.* For reader's convenience, we shall show how to derive this formula from Coleman's work: A direct application of the formula in [4] Theorem 1 tells that the exponent of  $\zeta_{p^n}$  in RHS is

$$\text{Tr}_{F/\mathbb{Q}_p} \left( \int_n \mathcal{L}(f) \cdot D \log(\sigma_F^{-n} g_\epsilon) \right).$$

We remark that there exists no error term in the sense of Coleman (cf. loc.cit.) as  $\mathcal{N}(g_\epsilon) = \sigma_F(g_\epsilon)$  (hence  $k(0) = 0$  in his notation). Since  $\frac{Dg}{g} - D\mathcal{L}(g) = [p]\sigma_F\left(\frac{Dg}{g}\right)$  (Remark 3.3 (2)), it suffices to show  $\int_n \mathcal{L}(f) \cdot [p](\sigma_F \frac{Dg}{g}) = 0$ . This follows from [7] (4.4), as  $\overline{S}(\mathcal{L}(f)) = 0$  when  $\mathcal{N}(f) = \sigma_F(f)$  (cf. Remark 3.3 (1)).  $\square$

#### 4. Proof of Main formula

In this section, we fix a root of unity  $z$  in  $F \cap \mu(\mathbb{Z}_p^{ur})$  (i.e., of order prime to  $p$ ). Note then that  $F(\mu_{p^\infty}, z^{1/p^\infty}) = F_\infty$ , so that the  $p$ -adic polylogarithmic characters  $\tilde{\chi}_m^z(\sigma)$ ,  $\chi_m^z(\sigma)$  are defined for  $\sigma \in G_{F_\infty}$  and for arbitrary etale paths  $\gamma \in \pi_1^{\text{pol}}(\overrightarrow{01}, z)$  as in Remark 2.5. Since the mod  $p$  reduction gives an isomorphism  $\mu(\mathbb{Z}_p^{ur}) \xrightarrow{\sim} \overline{\mathbb{F}}_p^\times$ , we can pick an integer  $d \geq 1$  such that  $z^{p^d} = z$ .

**Lemma 4.1.** *There is an etale path  $\gamma : \overrightarrow{01} \rightsquigarrow z$  on  $\mathbb{P}_{\mathbb{Q}_p}^1 - \{0, 1, \infty\}$  that determines a compatible branch  $z^{1/p^n}$  ( $n = 1, 2, \dots$ ) inside  $\mu(\mathbb{Z}_p^{ur})$  with period  $d$ , i.e.,  $z^{1/p^n} = z^{1/p^{n+d}}$ .*

*Proof.* If one changes the choice of  $\gamma$  by composition with  $x^b$  ( $b \in \mathbb{Z}_p$ ), then the induced branch changes from  $z^{1/p^n}$  to  $z^{1/p^n} \zeta_{p^n}^b$ . As there is only one element in  $(z^{1/p^n} \cdot \mu_{p^n}) \cap \mu(\mathbb{Z}_p^{ur})$  under the assumption, one may find a correct class  $b \pmod{p^n}$  for each  $n \geq 1$ , and hence get a correct  $b$  as their limit.  $\square$

We define the ‘‘weight accelerator’’ homomorphism

$$\mathcal{O}_F \llbracket G_\infty \rrbracket \rightarrow \mathcal{O}_F \llbracket G_\infty \rrbracket; \quad \omega \mapsto \omega(k)$$

for  $k \in \mathbb{Z}$  to be the  $\mathcal{O}_F$ -linear extension of the mapping  $\sigma_a \mapsto a^k \sigma_a$  ( $a \in \mathbb{Z}_p^\times$ ). It is not difficult to see that, for every  $\omega \in \mathcal{O}_F \llbracket G_\infty \rrbracket$ ,

$$(4.1) \quad D \cdot \omega = \omega(1) \cdot D$$

holds as operators on  $\mathcal{O}_F \llbracket T \rrbracket$ . Let us also introduce the basic element

$$\omega_n := \sum_{1 \leq i \leq p^n, p \nmid i} \sigma_i \in \mathbb{Z}_p \llbracket G_\infty \rrbracket.$$

**Proposition 4.2.** *Let  $z \in \mu(\mathbb{Z}_p^{ur})$  and let  $z^{1/p^n}$  ( $n = 1, 2, \dots$ ) be the compatible sequence taken inside  $\mu(\mathbb{Z}_p^{ur})$ . Let  $F$  be a finite unramified extension of  $\mathbb{Q}_p$  containing  $z$  and hence all of  $z^{1/p^n}$ . Then, for  $\epsilon = (\epsilon_n) \in \mathcal{U}_\infty(F)$  and for any positive integer  $m$ , we have:*

$$\begin{aligned} & \left( (f_{z^{1/p^n}, c})^{\omega_n(m-1)} (\zeta_{p^n} - 1), \epsilon_n \right)_{p^n} \\ &= \zeta_{p^n}^{(-1)^{m-1} (c^{1-m} - 1) \text{Tr}_{F/\mathbb{Q}_p} (\text{Li}_m^{(p)}(z) \cdot (1 - p^{m-1} \sigma_F) \phi_{m,F}^{CW}(\epsilon))}. \end{aligned}$$

Here, if  $z = 1$ , we understand  $\text{Li}_m^{(p)}(1)$  represents  $L_p(m, \omega^{1-m})$ .

*Proof.* We denote the left hand side by  $\zeta_{p^n}^\alpha$ . Then, according to Theorem 3.7 and [7, Lemma (4.6)], we have

$$\begin{aligned} \alpha &\equiv \text{Tr}_{F/\mathbb{Q}_p} \left( \int_n \omega_n(m-1) \mathcal{L}(f_{z^{1/p^n}, c}(T)) \cdot D \mathcal{L}(\sigma_F^{-n} g_\epsilon) \right) \\ &\equiv \text{Tr}_{F/\mathbb{Q}_p} \left( (-1)^{m-1} (D^{-(m-1)} \mathcal{L} f_{z^{1/p^n}, c})(0) \cdot (D^m \mathcal{L}(\sigma_F^{-n} g_\epsilon))(0) \right) \pmod{p^n}. \end{aligned}$$

Observe from (3.3) that

$$(D^m \mathcal{L}(\sigma_F^{-n} g_\epsilon))(0) = D^m \mathcal{L}(\sigma_F^{-n} g_\epsilon)|_{T=0} = \sigma_F^{-n} (1 - p^{m-1} \sigma_F) \phi_{m,F}^{CW}(\epsilon).$$

On the other hand, noting that  $(1 - c\sigma_c)(-m) = 1 - c^{1-m}\sigma_c$  and hence  $D^{-m}(1 - c\sigma_c) = (1 - c^{1-m}\sigma_c)D^{-m}$  by (4.1), we obtain for  $z \neq 1$ ,

$$\begin{aligned} D^{-m}D\mathcal{L}(f_{z^{1/p^n},c}(T)) &= D^{-m}(1 - c\sigma_c)(-\mathcal{F}_{z^{1/p^n}}^{(p)}(T)) \quad (\text{Lemma 3.2 (3)}) \\ &= -(1 - c^{1-m}\sigma_c)D^{-m}(\mathcal{F}_{z^{1/p^n}}^{(p)}(T)) \\ &= -(1 - c^{1-m}\sigma_c) \sum_{n=0}^{\infty} \left( \int_{\mathbb{Z}_p^\times} x^{n-m} d\mu_{z^{1/p^n}}(x) \right) \frac{X^n}{n!}, \end{aligned}$$

where Lemma 3.4 is used in the last equality. Hence

$$(D^{-m+1}\mathcal{L}f_{z^{1/p^n},c})(0) = (c^{1-m} - 1)\text{Li}_m^{(p)}(z^{1/p^n}) = (c^{1-m} - 1)(\sigma_F^{-n}\text{Li}_m^{(p)}(z))$$

according to our choice of  $z^{1/p^n}$ . Therefore

$$\begin{aligned} \alpha &\equiv (c^{1-m} - 1)\text{Tr}_{F/\mathbb{Q}_p}(\sigma_F^{-n}\text{Li}_m^{(p)}(z) \cdot \sigma_F^{-n}(1 - p^{m-1}\sigma_F)\phi_{m,F}^{CW}(\epsilon)) \\ &= (c^{1-m} - 1)\text{Tr}_{F/\mathbb{Q}_p}(\text{Li}_m^{(p)}(z) \cdot (1 - p^{m-1}\sigma_F)\phi_{m,F}^{CW}(\epsilon)) \end{aligned}$$

as desired. For  $z = 1$ , we obtain similarly

$$\begin{aligned} D^{-m}D\mathcal{L}(f_{1,c}(T)) &= D^{-m}(1 - c\sigma_c)(-\mathcal{F}_1^{(p)}(T)) \quad (\text{Lemma 3.2 (3)}) \\ &= D^{-m} \sum_{n=0}^{\infty} \left( \int_{\mathbb{Z}_p^\times} x^n dE_c(x) \right) \frac{X^n}{n!} \quad (2.19) \\ &= \sum_{n=0}^{\infty} \left( \int_{\mathbb{Z}_p^\times} x^{n-m} dE_c(x) \right) \frac{X^n}{n!} \quad (\text{Lemma 3.4}), \end{aligned}$$

hence, by (2.18),  $(D^{-m+1}\mathcal{L}f_{1,c})(0) = (c^{1-m} - 1)L_p(m, \omega^{1-m})$ . This completes the proof of the proposition.  $\square$

**Lemma 4.3.** *Let  $m$  be a fixed positive integer. Then, there is an integer  $N_m$  such that for every  $n \geq 1$ ,  $p^n$  divides  $N_m \sum_{\substack{1 \leq a \leq p^n \\ p \nmid a}} a^m$ .*

*Proof.* In fact, from the classical Bernoulli formula of power sums, it follows that

$$\sum_{\substack{1 \leq a \leq p^n \\ p \nmid a}} a^m = \frac{p^n}{m+1} \sum_{k=0}^m \binom{m+1}{k} B_k \cdot (p^{n(m-k)} - p^{(n-1)(m-k)+m}).$$

Thus, any common multiple of  $m+1$  and of the denominators of Bernoulli numbers  $B_0, \dots, B_m$  will do the role of  $N_m$ .  $\square$

Now, we shall show our main formula:

**Theorem 4.4.** For  $z \in \mu(\mathbb{Z}_p^{\text{ur}})$ , let  $\gamma : \overline{01} \rightsquigarrow z \in \mu(\mathbb{Z}_p^{\text{ur}})$  be a specific path of Lemma 4.1, and let  $F$  be a finite unramified extension of  $\mathbb{Q}_p$  containing  $z$ . Suppose that  $\sigma \in G_F$  lies in the image of  $\mathcal{U}_\infty(F) \xrightarrow{\text{rec}_\infty} G_{F_\infty}^{\text{ab}}$ . Then, for  $m \geq 1$ ,

$$(1) \quad \chi_m^z(\sigma) = (-1)^m \text{Tr}_{F/\mathbb{Q}_p}(\text{Li}_m^{(p)}(z) \cdot (1 - p^{m-1} \sigma_F) \phi_{m,F}^{CW}(\sigma)).$$

Moreover, if  $m \geq 2$ , then,

$$(2) \quad \tilde{\chi}_m^z(\sigma) = (-1)^m \text{Tr}_{F/\mathbb{Q}_p}(\text{Li}_m^{(p)}(z) \phi_{m,F}^{CW}(\sigma)).$$

Here, if  $z = 1$ , we understand  $\text{Li}_m^{(p)}(1) = L_p(m, \omega^{1-m})$ .

Proof. (1) Choose  $c \in \mathbb{Z}$  so that  $p \nmid c$  and  $c \neq 1$ . When  $z = 1$ , we moreover impose  $c \in (1 + p\mathbb{Z}_p)^{N_m}$  for some  $N_m$  as in Lemma 4.3. For  $n \geq 1$ , pick an integer  $\bar{c} = \bar{c}_n \in \mathbb{Z}$  such that  $c\bar{c} \equiv 1 \pmod{p^n}$ . Then, by simple computation, it follows that

$$((f_{z^{1/p^n}, c})^{\omega_n(m-1)})(\zeta_{p^n} - 1) \equiv \xi \cdot \prod_{\substack{1 \leq a \leq p^n \\ p \nmid a}} (1 - z^{1/p^n} \zeta_{p^n}^a)^{a^{m-1}(1-\bar{c}^{m-1})}$$

modulo  $F(\zeta_{p^n})^{\times p^n}$  for some constant  $\xi$  that lies in  $\mu_{p^n} \cdot (1 + p\mathbb{Z}_p)^{p^n}$ . Putting this into Proposition 4.2, we obtain

$$(4.2) \quad \begin{aligned} (1 - \bar{c}^{m-1}) \chi_m^z(\sigma) \\ \equiv (-1)^{m-1} (c^{1-m} - 1) \text{Tr}_{F/\mathbb{Q}_p}(\text{Li}_m^{(p)}(z) \cdot (1 - p^{m-1} \sigma_F) \phi_{m,F}^{CW}(\sigma)) \end{aligned}$$

mod  $p^n$ , where, if  $z = 1$ ,  $\text{Li}_m^{(p)}(z)$  stands for  $L_p(m, \omega^{1-m})$ . Replacing  $(1 - \bar{c}^{m-1})$  by  $(1 - c^{1-m})$  in LHS of (4.2), and then passing over  $n \rightarrow \infty$ , we obtain the formula of proposition.

(2) Suppose  $z^{p^d} = z$  for  $d \in \mathbb{Z}_{>0}$ . Then, our specific choice of  $\gamma$  makes the sequence  $\{z^{1/p^k}\}_{k \in \mathbb{Z}}$  cyclic of period  $d$ . By using Lemma 2.4, we can relate  $\tilde{\chi}_m^z(\sigma)$  to  $\chi_m^z(\sigma)$  in such a way that  $\tilde{\chi}_m^z(\sigma) = \sum_{k=0}^{d-1} \frac{p^{(m-1)k}}{1-p^{(m-1)d}} \chi_m^{z^{1/p^k}}(\sigma)$  (cf. also [32] Proposition 5.3 (i)). Putting the formula (1) into this, and replacing  $\text{Li}_m^{(p)}(z^{1/p^k}) \cdot \sigma_F(\phi_{m,F}^{CW}(\sigma))$  by  $\sigma_F(\text{Li}_m^{(p)}(z^{1/p^{k+1}}) \cdot \phi_{m,F}^{CW}(\sigma))$ , we may rewrite  $\tilde{\chi}_m^z(\sigma)$  with  $\alpha_k := p^{(m-1)k} (\text{Li}_m^{(p)}(z^{1/p^k}) \cdot \phi_{m,F}^{CW}(\sigma))$  as follows:

$$\begin{aligned} \tilde{\chi}_m^z(\sigma) &= \frac{(-1)^m}{1 - p^{(m-1)d}} \sum_{k=0}^{d-1} \text{Tr}_{F/\mathbb{Q}_p}(\alpha_k - \sigma_F(\alpha_{k+1})) \\ &= \frac{(-1)^m}{1 - p^{(m-1)d}} \sum_{k=0}^{d-1} (\text{Tr}_{F/\mathbb{Q}_p}(\alpha_k) - \text{Tr}_{F/\mathbb{Q}_p}(\alpha_{k+1})) \\ &= \frac{(-1)^m}{1 - p^{(m-1)d}} \text{Tr}_{F/\mathbb{Q}_p}(\alpha_0 - \alpha_d). \end{aligned}$$

But since  $z^{1/p^d} = z$ , we have  $\alpha_d = p^{(m-1)d} \alpha_0$ . This enables us to cancel the denominator  $(1 - p^{(m-1)d})^{-1}$  in the above expression, and hence to conclude  $\tilde{\chi}_m^z(\sigma) = (-1)^m \text{Tr}_{F/\mathbb{Q}_p}(\alpha_0)$  as asserted in (2).  $\square$

REMARK 4.5. The original Coleman-Ihara formula (1.3) in Introduction results from the above Theorem 4.4 (1) with  $z = 1$  and  $F = \mathbb{Q}_p$ . Note  $(-1)^m = -1$  for odd  $m$ .

We are arriving at the main Theorem 1.1 of Introduction:

**Corollary 4.6.** *Let  $\gamma : \vec{01} \rightsquigarrow z \in \mu(\mathbb{Z}_p^{ur})$  be a specific path of Lemma 4.1, and let  $F$  be a finite unramified extension of  $\mathbb{Q}_p$  containing  $z$ . Suppose that  $\sigma \in G_F$  lies in the image of  $\mathcal{U}_\infty(F) \xrightarrow{\text{rec}_\infty} G_{F_\infty}^{\text{ab}}$ . Then, for  $m \geq 2$ ,*

$$\ell i_m(z, \gamma)(\sigma) = \frac{-1}{(m-1)!} \text{Tr}_{F/\mathbb{Q}_p} \left( \left\{ \left( 1 - \frac{\sigma_F}{p^m} \right) \text{Li}_m^{p\text{-adic}}(z) \right\} \phi_{m,F}^{CW}(\sigma) \right).$$

Here, if  $z = 1$ , we understand  $(1 - \frac{\sigma_F}{p^m}) \text{Li}_m^{p\text{-adic}}(1) = L_p(m, \omega^{1-m})$ .

Proof. Theorem 4.4 (2) and Proposition 2.2 enable us to derive:

$$\begin{aligned} \ell i_m(z)(\sigma) &= \frac{(-1)^{m-1}}{(m-1)!} \tilde{\chi}_m^z(\sigma) \\ &= \frac{-1}{(m-1)!} \text{Tr}_{F/\mathbb{Q}_p} (\text{Li}_m^{(p)}(z) \cdot \phi_{m,F}^{CW}(\sigma)). \end{aligned}$$

Recalling  $\text{Li}_m^{(p)}(z) = \text{Li}_m^{p\text{-adic}}(z) - \frac{1}{p^m} \text{Li}_m^{p\text{-adic}}(z^p)$  by definition, where now  $z^p = \sigma_F(z)$  for  $z \in \mu(\mathbb{Z}_p^{ur})$ , we conclude the corollary.  $\square$

REMARK 4.7. Let  $P := G_F^{\text{ab}}$  and  $U$  the image of  $\mathcal{U}_\infty(F)$  in  $P$ . Then, as in [2, p.342],  $P/U$  is isomorphic to  $\hat{\mathbb{Z}}$  acted on trivially by  $G := \text{Gal}(F(\zeta_{p^\infty})/F)$  so as to fit in the canonical identification

$$H^1(F, F(m)) \cong \text{Hom}_G(P, F(m)) \cong \text{Hom}_G(U, F(m))$$

for  $F(m) := F \otimes_{\mathbb{Z}_p} (m)$  ( $m \geq 1$ ). From this, we obtain a canonical extension of the RHS of the above corollary to a  $G$ -homomorphism from  $P$  to  $\mathbb{Q}_p(m)$ . On the other hand, if we choose a path  $\gamma : \vec{01} \rightsquigarrow z$  so that  $\ell i_m(z, \gamma) \in \text{Hom}_G(P, \mathbb{Q}_p(m))$  (this is the case for  $\gamma$  in Lemma 4.1) then, it annihilates an inverse image  $u_0$  of  $1 \in \hat{\mathbb{Z}}$  in  $P$ . Then, these two extended homomorphisms on  $P$  turn out to coincide with each other, as they both coincide on  $U$  and kill  $u_0$ .

If moreover  $\ell i_m(z, \gamma)$  is a 1-cocycle on  $G_F$ , then they should give the same cohomology class in  $H^1(F, \mathbb{Q}_p(m))$ , as the restriction map from  $F$  to  $F(\zeta_{p^\infty})$  is injective.

## 5. Appendix: The Kummer level case

For completeness, we shall examine the case  $m = 0, 1$  in Theorem 1.1. We consider the  $\ell$ -adic Galois polylogarithms  $\ell i_0, \ell i_1$  as the Kummer 1-cocycles  $\kappa_z, \kappa_{1-z}$  respectively as in (2.4), (2.5) (cf. also [24] §5.2). On the other  $p$ -adic side, we may regard  $\text{Li}_0^{p\text{-adic}}(z) = -\log_p(z)$ ,  $\text{Li}_1^{p\text{-adic}}(z) = -\log_p(1-z)$  (cf. [16]). In fact, we have the following:

**Proposition 5.1.** *Let  $p$  be an odd prime, and  $F$  a finite unramified extension of  $\mathbb{Q}_p$  with the Frobenius substitution  $\sigma_F \in \text{Gal}(F/\mathbb{Q}_p)$ . Let  $F_\infty := F(\mu_{p^\infty})$  and, for  $a \in \mathcal{O}_F^\times$ , let  $\kappa_a : G_{F_\infty} \rightarrow \mathbb{Z}_p(1)$  be the Kummer character for  $p$ -power roots of  $a$ . Then,*

$$(5.1) \quad \kappa_a(\sigma) = \text{Tr}_{F/\mathbb{Q}_p} \left( \left\{ \left( 1 - \frac{\sigma_F}{p} \right) \log_p(a) \right\} \phi_{1,F}^{CW}(\sigma) \right)$$

holds for  $\sigma \in G_{F_\infty}$ , where  $\phi_{1,F}^{CW} : G_{F_\infty} \rightarrow F \otimes \mathbb{Z}_p(1)$  the 1-st Coates-Wiles homomorphism for the local field  $F$  (cf. Definition 3.5).

Following the notation system in §3.2, we first consider the case when  $\sigma = \text{rec}_\infty(\epsilon)$  for some norm compatible system  $\epsilon = (\epsilon_n)_n \in \mathcal{U}_\infty(F)$ . Let  $g_\epsilon(T) \in \mathcal{O}_F \llbracket T \rrbracket^\times$  be the associated Coleman power series such that  $\epsilon_n = \sigma_F^{-n} g_\epsilon(\zeta_{p^n} - 1)$  for  $n = 1, 2, \dots$ .

The explicit reciprocity law of S. Sen computes the Hilbert norm residue symbol by the formula \*

$$(5.2) \quad (\beta, \alpha)_{p^n} = \zeta_{p^n}^{-\frac{1}{p^n} \text{Tr}_{F/\mathbb{Q}_p}(\log_p(\alpha) \cdot \text{Tr}_{F'/F}(\delta_m(\beta')))}$$

for  $\alpha \in 1 + (\zeta_p - 1)^2 \mathcal{O}_F$  in [27, Theorem 3 (b)] and more generally for  $\alpha \in 1 + (\zeta_p - 1) \mathcal{O}_F$  in [27, II; Theorem 1], where  $F' = F(\zeta_{p^m})$  ( $m$  : big enough) and  $\beta' \in F'$  are taken so that  $\alpha^{p^{m-n}} \in 1 + (\zeta_p - 1)^2 \mathcal{O}_F$  and  $N_{F'/F}(\beta') = \beta$ . Finally  $\delta_m(\beta')$  means  $\frac{\zeta_{p^m}}{g'(\pi)} \cdot \frac{f'(\pi)}{f(\pi)}$  where  $f(T), g(T) \in \mathcal{O}_F[T]$  are polynomials with  $f(\pi) = \beta, g(\pi) = \zeta_{p^m}$  for any prime  $\pi$  of  $F'$ . Note that, in our case, we may and do set  $\pi = \zeta_{p^m} - 1$  so that  $g'(\pi) = 1$  (cf. [4, Cor. 15]).

Now, let us apply (5.2) for  $\alpha := a \in 1 + p\mathcal{O}_F$  and  $\beta := \epsilon_n$ . As  $(\alpha, \beta)_{p^n} = (\beta, \alpha)_{p^n}^{-1}$  (cf. [8, p.352]), writing  $(a, \epsilon_n)_{p^n} = \zeta_{p^n}^{[a, \epsilon_n]}$  and letting  $F' = F(\zeta_{p^m})$  as above, we obtain the following congruence modulo  $p^n$ :

$$(5.3) \quad [a, \epsilon_n] \equiv \text{Tr}_{F/\mathbb{Q}_p} \left( \log_p(a) \text{Tr}_{F'/F} \left( \frac{\zeta_{p^m}}{p^m} \cdot \frac{\sigma_F^{-m} g'_\epsilon(\zeta_{p^m} - 1)}{\epsilon_m} \right) \right).$$

**Lemma 5.2.** *Notations being as above, we have*

$$\text{Tr}_{F'/F} \left( \frac{\zeta_{p^m}}{p^m} \cdot \frac{\sigma_F^{-m} g'_\epsilon(\zeta_{p^m} - 1)}{\epsilon_m} \right) = \left( 1 - \frac{\sigma_F^{-1}}{p} \right) \cdot \phi_{1,F}^{CW}(\epsilon).$$

*Proof.* Since  $\mathcal{N}(g_\epsilon) = \sigma_F g_\epsilon$ , we have  $(\sigma_F g_\epsilon)((1+T)^p - 1) = \prod_{i=0}^{p-1} g_\epsilon(\zeta_p^i(1+T) - 1)$ . Substituting  $\zeta_{p^k}^j(1+T) - 1$  or  $(1+T)^{p^k} - 1$  ( $0 \leq j \leq p-1, k = 1, 2, \dots$ ) for  $T$  in it, and combining resulted equations in certain multiple ways, one obtains

$$(5.4) \quad (\sigma_F^k g_\epsilon)((1+T)^{p^k} - 1) = \prod_{i=0}^{p^k-1} g_\epsilon(\zeta_{p^k}^i(1+T) - 1) \quad (k \geq 1).$$

\*According to [27, II, p.69, line 12-13], the Hilbert symbol in loc. cit. coincides with that in [15]. On the other hand, we employ the Hilbert symbol discussed in Coleman's papers. According to [7, p.59, line -6], it is the inverse of the symbol used in [15]. Therefore, a minus sign is put in the exponent of RHS in (5.2).

Taking derivatives of both sides and putting  $T = 0$ , we obtain then

$$(5.5) \quad p^k \phi_{1,F}^{CW}(\epsilon) = \sum_{i=0}^{p^k-1} \zeta_{p^k}^i \frac{\sigma_F^{-k} g'_\epsilon(\zeta_{p^k}^i - 1)}{\sigma_F^{-k} g_\epsilon(\zeta_{p^k}^i - 1)} = \sum_{i=0}^{p^k-1} \zeta_{p^k}^i \frac{\sigma_F^{-k} g'_\epsilon(\zeta_{p^k}^i - 1)}{\epsilon_k}.$$

Using (5.5) for  $k = m, m-1$ , we see that the LHS of Lemma equals to

$$\begin{aligned} & \frac{1}{p^m} \sum_{\substack{0 \leq i < p^m \\ p \nmid i}} \frac{\sigma_F^{-m} g'_\epsilon(\zeta_{p^m}^i - 1)}{\sigma_F^{-m} g_\epsilon(\zeta_{p^m}^i - 1)} \\ &= \frac{1}{p^m} \left( \sum_{i=0}^{p^m} \frac{\sigma_F^{-m} g'_\epsilon(\zeta_{p^m}^i - 1)}{\sigma_F^{-m} g_\epsilon(\zeta_{p^m}^i - 1)} - \sum_{j=0}^{p^{m-1}} \sigma_F^{-1} \left( \frac{\sigma_F^{-(m-1)} g'_\epsilon(\zeta_{p^m}^{p^j} - 1)}{\sigma_F^{-(m-1)} g_\epsilon(\zeta_{p^m}^{p^j} - 1)} \right) \right) \\ &= \phi_{1,F}^{CW}(\epsilon) - \frac{1}{p} \sigma_F^{-1}(\phi_{1,F}^{CW}(\epsilon)). \end{aligned}$$

This completes the proof of the lemma.  $\square$

Plugging this lemma into (5.3), we find

$$\begin{aligned} [a, \epsilon_n] &\equiv \text{Tr}_{F/\mathbb{Q}_p} \left( \log_p(a) \phi_{1,F}^{CW}(\epsilon) - \frac{\log_p(a)}{p} \sigma_F^{-1}(\phi_{1,F}^{CW}(\epsilon)) \right) \\ &\equiv \text{Tr}_{F/\mathbb{Q}_p} \left( \log_p(a) \phi_{1,F}^{CW}(\epsilon) - \sigma_F^{-1} \left\{ \sigma_F \left( \frac{\log_p(a)}{p} \phi_{1,F}^{CW}(\epsilon) \right) \right\} \right) \\ &\equiv \text{Tr}_{F/\mathbb{Q}_p} \left( \log_p(a) \left( 1 - \frac{\sigma_F}{p} \right) \phi_{1,F}^{CW}(\epsilon) \right) \pmod{p^n}. \end{aligned}$$

This settles Proposition 5.1 in the case  $a \in 1 + p\mathcal{O}_F$  and  $\sigma = \text{rec}_\infty(\epsilon)$ , since by definition  $\kappa_a(\sigma) \equiv [a, \epsilon_n] \pmod{p^n}$ . The general case for  $a \in \mathcal{O}^\times$  follows immediately from observing that  $\log_p(a) = \log_p(a')$  when  $a = \zeta a'$  for any  $\zeta \in \mu(\mathbb{Z}_p^{\text{ur}}) \cap F$ . The extension of the statement to all  $\sigma \in G_{F(\zeta_{p^\infty})}$  follows from Remark 4.7. This settles the proof of Proposition 5.1.  $\square$

## References

- [1] G. Anderson: *The hyperadelic gamma function*, Invent. Math. **95** (1989), 63–131.
- [2] S. Bloch and K. Kato: *L-functions and Tamagawa numbers of motives*, The Grothendieck Festschrift Vol. I, (P.Cartier et. al. eds.), Progr. Math. **86** (1990), 333–400.
- [3] R. Coleman: *Division values in local fields*, Invent. Math. **53** (1979), 91–116.
- [4] R. Coleman: *The dilogarithm and the norm residue symbol*, Bull. Soc. Math. France **109** (1981), 373–402.
- [5] R. Coleman: *Dilogarithms, regulators, and p-adic L-functions*, Invent. Math. **69** (1982), 171–208.
- [6] R. Coleman: *Local units modulo circular units*, Proc. Amer. Math. Soc. **89** (1983), 1–7.
- [7] R. Coleman: *Anderson-Ihara theory: Gauss sums and circular units*, in “Algebraic number theory – in honor of K.Iwasawa” (J.Coates, R.Greenberg, B.Mazur, I.Satake eds.), Adv. Studies in Pure Math. **17** (1989), 55–72.
- [8] J.W.S. Cassels and A. Fröhlich: *Algebraic Number Theory*, 2nd edition, London Mathematical Society, 2010.
- [9] J. Coates and R. Sujatha: *Cyclotomic fields and zeta values*, Springer-Verlag, Berlin, 2006.

- [10] P. Deligne: *Le groupe fondamental de la droite projective moins trois points*, in “Galois groups over  $\mathbb{Q}$ ” (Y.Ihara, K.Ribet, J.-P.Serre eds.), Math. Sci. Res. Inst. Publ. **16** (1989), 79–297.
- [11] F. Ichimura and K. Sakaguchi: *The nonvanishing of a certain Kummer character  $\chi_m$  (after C. Soulé), and some related topics*, in “Galois representations and arithmetic algebraic geometry” (Y. Ihara ed.), Adv. Studies in Pure Math. **12** (1987), 53–64.
- [12] Y. Ihara: *Profinite braid groups, Galois representations and complex multiplications*, Ann. of Math. **123** (1986), 43–106.
- [13] Y. Ihara: *Braids, Galois groups, and some arithmetic functions*, Proc. Intern. Congress of Math. Kyoto, 99–120, 1990.
- [14] Y. Ihara, M. Kaneko and A. Yukinari: *On some properties of the universal power series for Jacobi sums*, in “Galois representations and arithmetic algebraic geometry” (Y. Ihara ed.), Adv. Studies in Pure Math. **12** (1987), 65–86.
- [15] K. Iwasawa: *On explicit formulas for the norm residue symbol*, J. Math. Soc. Japan **20** (1968), 151–165.
- [16] H. Furusho:  *$p$ -adic multiple zeta values I:  $p$ -adic multiple polylogarithms and the  $p$ -adic KZ equation*, Invent. Math. **155** (2004), 223–286; *II: Tannakian interpretations*, Amer. Journal of Math. **129** (2007), 1105–1144.
- [17] M. Gros: *Régulateurs syntomiques et valeurs de fonctions  $L$   $p$ -adiques, I*, (with an appendix by Masato Kurihara), Invent. Math. **99** (1990), 293–320; *II*, Invent. Math. **115** (1994), 61–79.
- [18] M. Kim: *The unipotent Albanese map and Selmer varieties for curves*, Publ. Res. Inst. Math. Sci. **45** (2009), 89–133.
- [19] N. Koblitz: *A new proof of certain formulas for  $p$ -adic  $L$ -functions*, Duke Math. J. **46** (1979), 455–468.
- [20] M. Kolster and T. Nguyen Quang Do: *Syntomic regulators and special values of  $p$ -adic  $L$ -functions*, Invent. math. **133** (1998), 417–447.
- [21] M. Kurihara: *Computation of the syntomic regulator in the cyclotomic case*, Appendix to [Gr, I].
- [22] H. Nakamura, K. Sakugawa and Z. Wojtkowiak: *Polylogarithmic analogue of the Coleman-Ihara formula, II*, RIMS Kôkyûroku Bessatsu (to appear).
- [23] H. Nakamura and Z. Wojtkowiak: *On explicit formulae for  $l$ -adic polylogarithms*, in “Arithmetic fundamental groups and noncommutative algebra”, (M.Fried, Y.Ihara eds.) Proc. Sympos. Pure Math. **70** (2002), 285–294.
- [24] H. Nakamura and Z. Wojtkowiak: *Tensor and homotopy criteria for functional equations of  $l$ -adic and classical iterated integrals*, in “Non-abelian fundamental groups and Iwasawa theory” (J.Coates, M.Kim, F.Pop, M.Saïdi, P.Schneider eds.), London Math. Soc. Lecture Note Ser. **393** (2012), 258–310.
- [25] M.C. Olsson: *Towards Non-Abelian  $p$ -adic Hodge Theory in the Good Reduction Case*, Memoirs of A.M.S. **210**, 2011.
- [26] K. Sakugawa: *On a non-abelian generalization of the Bloch-Kato exponential map*, Math. J. Okayama Univ. **59** (2017), 41–70.
- [27] S. Sen: *On explicit reciprocity laws*, J. reine angew. math. **313** (1980), 1–26; *Part II*, J. reine angew. math. **323** (1981), 68–87.
- [28] C. Soulé: *On higher  $p$ -adic regulators*, Lecture Notes in Mathematics **854** (1981), 372–401.
- [29] L.C. Washington: *Introduction to Cyclotomic Fields*, 2nd Edition, Springer 1997.
- [30] Z. Wojtkowiak: *On  $l$ -adic iterated integrals I: Analog of Zagier conjecture*, Nagoya Math. J. **176** (2004), 113–158.
- [31] Z. Wojtkowiak: *On  $l$ -adic iterated integrals II: Functional equations and  $l$ -adic polylogarithms*, Nagoya Math. J. **177** (2005), 117–153.
- [32] Z. Wojtkowiak: *On  $l$ -adic Galois  $L$ -functions*, (preprint 2014) arXiv:1403.2209.

Hiroaki Nakamura  
Department of Mathematics  
Osaka University  
Osaka 560-0043  
Japan  
e-mail: nakamura@math.sci.osaka-u.ac.jp

Kenji Sakugawa  
Department of Mathematics  
Osaka University  
Osaka 560-0043  
Japan  
e-mail: sakugawa.kenji@gmail.com

Zdzisław Wojtkowiak  
Université de Nice-Sophia Antipolis  
Dépt. of Math.  
Laboratoire Jean Alexandre Dieudonné  
U.R.A. au C.N.R.S., No 168, Parc Valrose  
-B.P.N° 71, 06108 Nice Cedex 2  
France  
e-mail: wojtkow@math.unice.fr