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ON THE GEVREY STRONG HYPERBOLICITY

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Abstract

   In this paper we are concerned with a homogeneous differential operator p of order m of which characteristic set of order m is assumed to be a smooth manifold. We define the Gevrey strong hyperbolicity index as the largest number s such that the Cauchy problem for $p + Q$ is well-posed in the Gevrey class of order s for any differential operator Q of order less than m. We study the case of the largest index and we discuss in which way the Gevrey strong hyperbolicity index relates with the geometry of bicharacteristics of p near the characteristic manifold.

1. Introduction

Let

$$P = D_0^m + \sum_{|\alpha| \leq m, \alpha_0 < m} a_\alpha(x) D^\alpha = p(x, D) + P_{m-1}(x, D) + \cdots$$

be a differential operator of order m defined near the origin of $\mathbb{R}^{n+1}$ where $x = (x_0, \ldots, x_n) = (x_0, x')$ and

$$D_j = -i\partial/\partial x_j, \quad D = (D_0, D'), \quad D' = (D_1, \ldots, D_n).$$

Here $p(x, \xi)$ is the principal symbol of P;

$$p(x, \xi) = \xi_0^m + \sum_{|\alpha| = m, \alpha_0 < m} a_\alpha(x) \xi^\alpha.$$

We assume that the coefficients $a_\alpha(x)$ are in the Gevrey class of order $s > 1$, sufficiently close to 1, which are constant outside $|x'| \leq R$. We say that $f(x) \in \gamma^{(s)}(\mathbb{R}^{n+1})$, the Gevrey class of order s, if for any compact set $K \subset \mathbb{R}^{n+1}$ there exist $C > 0, A > 0$ such that we have

$$|D^\alpha f(x)| \leq C A^{|\alpha|} |\alpha|!^s, \quad x \in K, \quad \forall \alpha \in \mathbb{N}^{n+1}.$$

DEFINITION 1.1. We say that the Cauchy problem for P is $\gamma^{(s)}$ well-posed at the origin if for any $\Phi = (u_0, u_1, \ldots, u_{m-1}) \in (\gamma^{(s)}(\mathbb{R}^n))^m$ there exists a neighborhood U_Φ of the origin such that the Cauchy problem

$$\begin{cases} Pu = 0 & \text{in} \quad U_\Phi, \\ D_0^j u(0, x') = u_j(x'), & j = 0, 1, \ldots, m-1, \quad x' \in U_\Phi \cap \{x_0 = 0\} \end{cases}$$

has a unique solution $u(x) \in C^\infty(U_\Phi)$.

It is a fundamental fact that if $p(x, \xi)$ is strictly hyperbolic near the origin, that is $p(x, \xi_0, \xi') = 0$ has m real distinct roots for any x, near the origin and any $\xi' \neq 0$ then the Cauchy problem for $p + Q$ with any differential operator Q of order less than m is C^∞ well-posed near the origin. In particular, $\gamma^{(s)}$ well-posed for any $s > 1$. On the other hand the Lax-Mizohata theorem in the Gevrey classes asserts:

Proposition 1.1 ([16, Theorem 2.2]). *If the Cauchy problem for P is $\gamma^{(s)}$ ($s > 1$) well-posed at the origin then $p(0, \xi_0, \xi') = 0$ has only real roots ξ_0 for any $\xi' \in \mathbb{R}^n$.*

Taking this result into account we assume, throughout the paper, that $p(x, \xi_0, \xi') = 0$ has only real roots for any x near the origin and any $\xi' \in \mathbb{R}^n$.

DEFINITION 1.2. We define $G(p)$ (the Gevrey strong hyperbolicity index) by

$$G(p) = \sup \left\{ 1 \leq s \,\middle|\, \begin{array}{l} \text{Cauchy problem for } p + Q \text{ is } \gamma^{(s)} \text{ well-posed at the} \\ \text{origin for any differential operator } Q \text{ of order} < m \end{array} \right\}.$$

We first recall a basic result of Bronshtein [4].

Theorem 1.1 ([4, Theorem 1]). *Let p be a homogeneous differential operator of order m with real characteristic roots. Then for any differential operator Q of order less than m, the Cauchy problem for $p + Q$ is $\gamma^{(m/(m-1))}$ well-posed.*

This implies that for differential operators p of order m with real characteristic roots we have

$$G(p) \geq m/(m - 1).$$

We also recall a result which bounds $G(p)$ from above. The following result is a special case of Ivrii [10, Theorem 1]. Recall that $(x, \xi) \in \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \setminus \{0\})$ is called a characteristic of order r of p if

$$\partial_x^\alpha \partial_\xi^\beta p(x, \xi) = 0, \quad \forall |\alpha + \beta| < r.$$

Theorem 1.2 ([10, Theorem 1]). *Let p be a homogeneous differential operator of order m with real analytic coefficients and let $(0, \bar{\xi})$, $\bar{\xi} = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$ be a characteristic of order m. If the Cauchy problem for $P = p + P_{m-1} + \cdots$ is $\gamma^{(\kappa)}$ well-posed at the origin we have*

$$\partial_\xi^\alpha \partial_x^\beta P_{m-1}(0, \bar{\xi}) = 0$$

for any $|\alpha + \beta| \leq m - 2\kappa/(\kappa - 1)$.

Assume that p has a characteristic $(0, \bar{\xi})$ of order m and that the Cauchy problem for $p + P_{m-1} + \cdots$ is $\gamma^{(\kappa)}$ well-posed for any P_{m-1}. Then from Theorem 1.2 it follows that $m - 2\kappa/(\kappa - 1) < 0$, that is $\kappa < m/(m - 2)$ which yields

$$G(p) \leq m/(m - 2).$$

Let ρ be a characteristic of order m. Then the localization $p_\rho(X)$ of p at ρ is defined by $p(\rho + \mu X) = \mu^m(p_\rho(X) + o(1))$ with $X = (x, \xi)$ as $\mu \to 0$ which is nothing but the first non-vanishing term of the Taylor expansion of p around ρ. Note that p_ρ is a hyperbolic polynomial in X in the direction $(0, \theta) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ where $\theta = (1, \ldots, 0) \in \mathbb{R}^{n+1}$ (for

example [7, Lemma 8.7.2]). The hyperbolic cone Γ_ρ of p_ρ is the connected component of $(0, \theta)$ in the set (for example [7, Lemma 8.7.3])

$$\Gamma_\rho = \{X \in \mathbb{R}^{2(n+1)} \mid p_\rho(X) \neq 0\}$$

and the propagation cone C_ρ of the localization p_ρ is the dual cone with respect to the symplectic two form $\sigma = d\xi \wedge dx = \sum_{j=0}^{n} d\xi_j \wedge dx_j$;

$$C_\rho = \{X \in \mathbb{R}^{2(n+1)} \mid (d\xi \wedge dx)(X, Y) \leq 0, \forall Y \in \Gamma_\rho\}.$$

Let

$$H_p = \sum_{j=0}^{n} (\partial p/\partial \xi_j)\partial/\partial x_j - (\partial p/\partial x_j)\partial/\partial \xi_j$$

be the Hamilton vector field of p then integral curves of H_p, along which $p = 0$, are called bicharacteristics of p. We note that C_ρ is the *minimal* cone including every bicharacteristic which has ρ as a limit point in the following sense:

Lemma 1.1 ([12, Lemma 1.1.1]). *Let $\rho \in \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \setminus \{0\})$ be a multiple characteristic of p. Assume that there are simple characteristics ρ_j and non-zero real numbers γ_j with $\gamma_j p_{\rho_j}(0, \theta) > 0$ such that*

$$\rho_j \to \rho \quad and \quad \gamma_j H_p(\rho_j) \to X, \quad j \to \infty.$$

Then $X \in C_\rho$.

We now introduce assumptions of which motivation will be discussed in the next section. Denote by Σ the set of characteristics of order m of $p(x, \xi)$;

$$\Sigma = \{(x, \xi) \in \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \setminus \{0\}) \mid \partial_x^\alpha \partial_\xi^\beta p(x, \xi) = 0, \forall |\alpha + \beta| < m\}$$

which is assumed to be a $\gamma^{(s)}$ manifold. Note that p_ρ is a function on $\mathbb{R}^{2(n+1)}/T_\rho\Sigma$ because $p_\rho(X + Y) = p_\rho(Y)$ for any $X \in T_\rho\Sigma$ and any $Y \in \mathbb{R}^{2(n+1)}$ where $T_\rho\Sigma$ denotes the tangent space of Σ at $\rho \in \Sigma$. We assume that

(1.1) $\qquad p_\rho$ is a strictly hyperbolic polynomial on $\mathbb{R}^{2(n+1)}/T_\rho\Sigma$, $\rho \in \Sigma$.

We also assume that the propagation cone C_ρ is transversal to the characteristic manifold Σ;

(1.2) $\qquad\qquad\qquad\qquad C_\rho \cap T_\rho\Sigma = \{0\}, \quad \rho \in \Sigma.$

Denoting $(T_\rho\Sigma)^\sigma = \{X \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid (d\xi \wedge dx)(X, Y) = 0, \forall Y \in T_\rho\Sigma\}$ we note that (1.2) is equivalent to $\Gamma_\rho \cap (T_\rho\Sigma)^\sigma \neq \emptyset$.

Our aim in this paper is to prove

Theorem 1.3. *Assume (1.1) and (1.2). Then the Cauchy problem for $p + Q$ is $\gamma^{(s)}$ well-posed at the origin for any differential operator Q of order less than m and for any $1 < s < m/(m - 2)$. In particular we have $G(p) = m/(m - 2)$.*

EXAMPLE 1.1. Let

$$q(\zeta) = \zeta_0^m + \sum_{|\alpha|=m, \alpha_0 \leq m-2} c_\alpha \zeta^\alpha, \quad \zeta = (\zeta_0, \zeta_1, \ldots, \zeta_k)$$

be a strictly hyperbolic polynomial in the direction ζ_0 where $k \leq n$. Let $b_j(x, \xi')$, $j = 1, \ldots, k$ be smooth functions in a conic neighborhood of $(0, \hat{\xi}')$ which are homogeneous of degree 1 in ξ' with linearly independent differentials at $(0, \hat{\xi}')$. We define

$$p(x, \xi) = q(b(x, \xi)), \quad b = (b_0, b_1, \ldots, b_k)$$

where we set $b_0(x, \xi) = \xi_0$ for notational convenience. Then it is easy to see that $p(x, \xi)$ verifies the condition (1.1) near $\rho = (0, 0, \hat{\xi}')$ with $\Sigma = \{(x, \xi) \mid b_j(x, \xi) = 0, j = 0, \ldots, k\}$ and $p_\rho(x, \xi) = q(db_\rho(x, \xi))$, that is

$$p_\rho(x, \xi) = q(\hat{b}(x, \xi)), \quad \hat{b} = (\hat{b}_0, \hat{b}_1, \ldots, \hat{b}_k)$$

where $\hat{b}_j(x, \xi)$ is the linear part of $b_j(x, \xi)$ at ρ. Therefor $\Gamma_\rho = \{X \mid \hat{b}(X) \in \Gamma\}$ where Γ is the hyperbolic cone of q. If

(1.3) $(\{b_i, b_j\})_{0 \leq i, j \leq k}$ is non-singular at ρ

then $p(x, \xi)$ verifies the condition (1.2) near ρ where $\{b_i, b_j\}$ denotes the Poisson bracket

$$\sum_{\mu=0}^{n} (\partial b_i / \partial \xi_\mu)(\partial b_j / \partial x_\mu) - (\partial b_i / \partial x_\mu)(\partial b_j / \partial \xi_\mu).$$

Indeed since $(T_\rho \Sigma)^\sigma$ is spanned by $H_{b_0}(\rho), H_{b_1}(\rho), \ldots, H_{b_k}(\rho)$ it suffices to show that there are c_j such that $0 \neq X = \sum_{j=0}^{k} c_j H_{b_j}(\rho) \in \Gamma_\rho$. From

$$\hat{b}_j(X) = \sum_{i=0}^{k} c_i \{b_i, b_j\}(\rho), \quad j = 0, \ldots, k$$

one can choose c_j so that $\hat{b}(X) = (1, 0, \ldots, 0)$ by assumption (1.3) and hence the result.

EXAMPLE 1.2. Consider

(1.4) $$q(\zeta_0, \zeta_1, \zeta_2) = \prod_{j=1}^{\ell} (\zeta_0 - c_j(\zeta_1^2 + \zeta_2^2))$$

where c_j are real positive constants different from each other and $2\ell = m$. Take $b_1 = (x_0 - x_1)\xi_n$, $b_2 = \xi_1$ and consider

$$p(x, \xi) = \prod_{j=1}^{\ell} (\xi_0^2 - c_j((x_0 - x_1)^2 \xi_n^2 + \xi_1^2))$$

in a conic neighborhood of $\rho = (0, 0, \ldots, 0, 1)$. The 3×3 anti-symmetric matrix $(\{b_i, b_j\})$ is obviously singular. If $\max\{c_j\} = c < 1$ then $C_\rho \cap T_\rho \Sigma = \{0\}$. To see this take any $X = (t, t, x_2, \ldots, x_n, 0, 0, \xi_2, \ldots, \xi_n) \in T_\rho \Sigma$. Assume $X \in C_\rho$ so that $(d\xi \wedge dx)(X, Y) \leq 0$ for any $Y = (y, \eta) \in \Gamma_\rho$, that is for any $(y, \eta) \in \mathbb{R}^{2(n+1)}$ with $\eta_0^2 > c((y_0 - y_1)^2 + \eta_1^2)$ and $\eta_0 > 0$. This implies that $x_2 = \cdots = x_n = 0$, $\xi_2 = \cdots = \xi_n = 0$ and $-t(\eta_0 + \eta_1) \leq 0$ for any $\eta_0 > \sqrt{c}|\eta_1|$. Since $c < 1$ this gives $t = 0$ so that $X = 0$.

On the other hand if $\max\{c_j\} = c \geq 1$ then $C_\rho \cap T_\rho \Sigma \neq \{0\}$. Indeed let $X = (1, 1, 0, \ldots, 0, 0, \ldots, 0) \in T_\rho \Sigma$. Noting that $\eta_0 > \sqrt{c}|\eta_1|$ if $Y = (y, \eta) \in \Gamma_\rho$ we see $(d\xi \wedge dx)(X, Y) = -\eta_0 - \eta_1 < 0$ for any $Y \in \Gamma_\rho$ which proves $X \in C_\rho$.

EXAMPLE 1.3. Take q in (1.4) and choose $b_1 = x_0\xi_n$, $b_2 = \xi_1$ and consider

$$p(x,\xi) = \prod_{j=1}^{\ell} (\xi_0^2 - c_j(x_0^2\xi_n^2 + \xi_1^2))$$

near $\rho = (0,0,\ldots,0,1)$. As remarked in Example 1.2 the matrix $(\{b_i, b_j\})$ is singular. Suppose $X = (0, x_1, \ldots, x_n, 0, 0, \xi_2, \ldots, \xi_n) \in T_\rho\Sigma \cap C_\rho$. As in Example 1.2 we conclude $x_2 = \cdots = x_n = 0$, $\xi_1 = \cdots = \xi_n = 0$ and $-x_1\eta_1 \le 0$ for any $\eta_0^2 > c(y_0^2 + \eta_1^2)$. This gives $x_1 = 0$ so that $X = 0$. Thus we conclude $C_\rho \cap T_\rho\Sigma = \{0\}$.

EXAMPLE 1.4. We specialize Example 1.1 with

$$q(\zeta_0, \zeta_1) = \prod_{j=1}^{m}(\zeta_0 - \alpha_j\zeta_1), \quad q(\zeta_0, \zeta_1) = \prod_{j=1}^{\ell}(\zeta_0^2 - c_j\zeta_1^2)$$

where α_j are real constants different from each other such that $\sum_{j=1}^{m} \alpha_j = 0$ and c_j are positive constant different from each other and $m = 2\ell$. For these q choosing $b_1 = x_0\xi_1$ and $b_1 = x_0|\xi'|$ respectively we get

$$p(x,\xi) = \prod_{j=1}^{m}(\xi_0 - \alpha_j x_0\xi_1), \quad p(x,\xi) = \prod_{j=1}^{\ell}(\xi_0^2 - c_j x_0^2|\xi'|^2).$$

It is clear that $\{b_0, b_1\} = \xi_1 \ne 0$ and $\{b_0, b_1\} = |\xi'| \ne 0$ respectively and hence $C_\rho \cap T_\rho\Sigma = \{0\}$. We find these examples in [5] where they studied Levi type conditions for differential operators of order m with coefficients depending only on the time variable.

2. Motivation, the doubly characteristic case

In this section we provide the motivation to introduce $G(p)$ and assumptions (1.1), (1.2). Let $m = 2$ and we consider differential operators of second order

$$P(x, D) = p(x, D) + P_1(x, D) + P_0(x)$$

of principal symbol $p(x, \xi)$. Let ρ be a double characteristic of p and hence singular (stationary) point of H_p. We linearize the Hamilton equation $\dot{X} = H_p(X)$ at ρ, the linearized equation turns to be $\dot{Y} = F_p(\rho)Y$ where $F_p(\rho)$ is given by

$$F_p(\rho) = \begin{pmatrix} \dfrac{\partial^2 p}{\partial x \partial \xi}(\rho) & \dfrac{\partial^2 p}{\partial \xi \partial \xi}(\rho) \\ -\dfrac{\partial^2 p}{\partial x \partial x}(\rho) & -\dfrac{\partial^2 p}{\partial \xi \partial x}(\rho) \end{pmatrix}$$

and called the Hamilton map (fundamental matrix) of p at ρ.

The following special structure of $F_p(\rho)$ results from the fact that $p(x, \xi_0, \xi') = 0$ has only real roots ξ_0 for any (x, ξ').

Lemma 2.1 ([9, Lemma 9.2, 9.4]). *All eigenvalues of the Hamilton map $F_p(\rho)$ are on the imaginary axis, possibly one exception of a pair of non-zero real eigenvalues.*

We assume that the doubly characteristic set $\Sigma = \{(x, \xi) \mid \partial_\xi^\alpha \partial_x^\beta p(x, \xi) = 0, \forall |\alpha + \beta| < 2\}$

verifies the following conditions:

(2.1)
$$\begin{cases} \Sigma \text{ is a } \gamma^{(s)} \text{ manifold,} \\ p \text{ vanishes on } \Sigma \text{ of order exactly 2,} \\ \text{rank} (d\xi \wedge dx) = \text{const. on } \Sigma. \end{cases}$$

Note that $p_\rho(X)$ is *always a strictly hyperbolic polynomial* on $\mathbb{R}^{2(n+1)}/T_\rho\Sigma$ as far as p vanishes on Σ of order exactly 2. We also assume that the codimension Σ is 3 and no transition of spectral type of F_p occur on Σ, that is we assume

(2.2) either $\text{Ker } F_p^2 \cap \text{Im } F_p^2 = \{0\}$ or $\text{Ker } F_p^2 \cap \text{Im } F_p^2 \neq \{0\}$

throughout Σ. The following table sums up a general picture of the Gevrey strong hyperbolicity for differential operators with double characteristics ([2, 3, 17, 11]) where $W = \text{Ker } F_p^2 \cap \text{Im } F_p^2$.

Spectrum of F_p	W	Geometry of bicharacteristics near Σ	$G(p)$
Exists non-zero real eigenvalue	$W = \{0\}$	At every point on Σ exactly two bicharacteristics intersect Σ transversally	$G(p) = \infty$
No non-zero real eigenvalue	$W \neq \{0\}$	No bicharacteristic intersects Σ	$G(p) = 4$
		Exists a bicharacteristic tangent to Σ	$G(p) = 3$
	$W = \{0\}$	No bicharacteristic intersects Σ	$G(p) = 2$

This table shows that, assuming (2.1), (2.2) and the codimension Σ is 3, the Gevrey strong hyperbolicity index $G(p)$ takes only the values 2, 3, 4 and ∞ and that these values completely determine the structure of the Hamilton map and the geometry of bicharacteristics near Σ and vice versa.

Lemma 2.2 ([6, Corollary 1.4.7], [12, Lemma 1.1.3]). *Let ρ be a double characteristic. Then the following two conditions are equivalent.*
(i) $F_p(\rho)$ has non-zero real eigenvalues,
(ii) $C_\rho \cap T_\rho\Sigma = \{0\}$.

Note that the condition (ii) is well defined for characteristics of *any order* while $F_p(\rho) \equiv 0$ if ρ is a characteristic of order larger than 2.

REMARK 2.1. Based on the table, it is quite natural to ask whether the converse of Theorem 1.3 is true. That is if $G(p) = m/(m-2)$ then (1.1) and (1.2) hold?

REMARK 2.2. Consider the case $C_\rho \subset T_\rho\Sigma$ that would be considered as a opposite case to $C_\rho \cap T_\rho\Sigma = \{0\}$. Here we note

Lemma 2.3 [18, Lemma 2.11] . *We have $C_\rho \subset T_\rho\Sigma$ if and only if $T_\rho\Sigma$ is involutive, that is $(T_\rho\Sigma)^\sigma \subset T_\rho\Sigma$.*

It is also natural to ask whether $G(p) = m/(m-1)$ if $C_\rho \subset T_\rho\Sigma, \rho \in \Sigma$. When Σ is involutive one can choose homogeneous symplectic coordinates x, ξ in a conic neighborhood of $\rho \in \Sigma$ such that Σ is defined by ([8, Theorem 21.2.4], for example)

$$\xi_0 = \xi_1 = \cdots = \xi_k = 0.$$

Thus by conjugation of a Fourier integral operator $p(x, \xi)$ can be written

$$p(x, \xi) = \xi_0^m + \sum_{\alpha_0 \leq m-2, |\alpha|=m} a_\alpha(x, \xi)\tilde{\xi}^\alpha$$

where $\tilde{\xi} = (\xi_0, \xi_1, \ldots, \xi_k)$. Thus $\partial_\xi^\alpha \partial_x^\beta p(0, \bar{\xi}) = 0$ for $|\alpha| < m$ and any β. If the resulting $p(x, D)$ is a *differential operator* so that $a_\alpha(x, \xi) = a_\alpha(x)$ then from [10, Theorem 1] we conclude that if the Cauchy problem for $p + P_{m-1} + \cdots$ is $\gamma^{(\kappa)}$ well-posed then $\partial_\xi^\alpha \partial_x^\beta P_{m-1}(0, \bar{\xi}) = 0$ for any $|\alpha| \leq m - \kappa/(\kappa - 1)$ and any β. This proves $G(p) \leq m/(m - 1)$ and hence $G(p) = m/(m - 1)$.

EXAMPLE 2.1. When $m \geq 3$ the geometry of p with the limit point ρ becomes to be complicated comparing with the case $m = 2$, even (1.1) and (1.2) are satisfied. We give an example. Let us consider

$$p(x, \xi) = \xi_0^3 - 3a\{(x_0^2 + x_1^2)\xi_n^2 + \xi_1^2\}\xi_0 - 2bx_0x_1\xi_1\xi_n^2$$

near $\rho = (0, \ldots, 0, 1)$ which is obtained from Example 1.2 with

$$q(\zeta_0, \zeta_1, \zeta_2, \zeta_3) = \zeta_0^3 - 3a(\zeta_1^2 + \zeta_2^2 + \zeta_3^2)\zeta_0 - 2b\zeta_1\zeta_2\zeta_3$$

and $b_1 = x_0\xi_n$, $b_2 = x_1\xi_n$, $b_3 = \xi_1$ where $a > 0$, b are real constants. Choosing $b = \delta a^{3/2}$ with $|\delta| < 1$ and repeating similar arguments as in Example 1.3 it is easily seen that $p(x, \xi)$ satisfies (1.1) and (1.2).

Consider the Hamilton equations

(2.3) $\dot{x}_j = \partial p/\partial \xi_j, \quad \dot{\xi}_j = -\partial p/\partial x_j, \quad j = 0, \ldots, n.$

Since $\dot{\xi}_n = 0$ we take $\xi_n = 1$ and $x_1 = \xi_0 = 0$ in (2.3) so that the resulting equations reduce to:

(2.4) $\dot{x}_0 = -3a(x_0^2 + \xi_1^2), \quad \dot{\xi}_1 = 2bx_0\xi_1.$

We fix $-1 < \delta < 0$ and take $a > 0$ so that $2b/(3a) < -1$. Then any integral curve of (2.4) passing a point in the cone $|\xi_1| < |1 + (2b/3a)|^{1/2}|x_0|$, $x_0 < 0$ arrives at the origin inside the cone (see, for example [20]). In particular there are infinitely many bicharacteristics with the limit point ρ.

3. Preliminaries

Choosing a new system of local coordinates leaving $x_0 = $ const. to be invariant one can assume that

$$p(x, \xi) = \xi_0^m + a_2(x, \xi')\xi_0^{m-2} + \cdots + a_m(x, \xi')$$

and hence $\Sigma \subset \{\xi_0 = 0\}$. Thus near ρ we may assume that Σ is defined by $b_0(x, \xi) = \cdots =$

$b_k(x, \xi) = 0$ where $b_0 = \xi_0$, $b_j = b_j(x, \xi')$, $1 \leq j \leq k$ and db_j are linearly independent at ρ' where ρ' stands for $(\bar{x}, \bar{\xi}')$ when $\rho = (\bar{x}, \bar{\xi})$. Recall that the localization $p_\rho(x, \xi)$ is a homogeneous hyperbolic polynomial of degree m in (x, ξ) in the direction $(0, \theta) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$.

Lemma 3.1 ([12, Lemma 1.1.3]). *The next two conditions are equivalent.*
(i) $C_\rho \cap T_\rho\Sigma = \{0\}$,
(ii) $\Gamma_\rho \cap (T_\rho\Sigma)^\sigma \cap \langle(0, \theta)\rangle^\sigma \neq \emptyset$ where $\langle(0, \theta)\rangle = \{t(0, \theta) \mid t \in \mathbb{R}\}$.

Assume $C_\rho \cap T_\rho\Sigma = \{0\}$ then thanks to Lemma 3.1 there exists $0 \neq X \in \Gamma_\rho \cap (T_\rho\Sigma)^\sigma \cap \langle(0, \theta)\rangle^\sigma$. Since $(T_\rho\Sigma)^\sigma$ is spanned by $H_{b_j}(\rho)$, $j = 0, \ldots, k$ one can write

$$(3.1) \qquad\qquad X = \sum_{j=0}^{k} \alpha_j H_{b_j}(\rho)$$

where $\alpha_0 = 0$ because $X \in \langle(0, \theta)\rangle^\sigma$. This proves $\partial_{x_0} b_j(\rho') \neq 0$ with some $1 \leq j \leq k$. Indeed if not we would have $X = (x, 0, \xi')$ while denoting $p_\rho(x, \xi) = \prod_{j=1}^{m}(\xi_0 - \Lambda_j(x, \xi'))$ we see $\Gamma_\rho = \{(x, \xi) \mid \xi_0 > \max_j \Lambda_j(x, \xi')\}$ (for example [7, Lemma 8.7.3]) and we would have $\Lambda_j(x, \xi') < 0$ which contradicts $\sum_{j=1}^{m} \Lambda_j(x, \xi') = 0$. Renumbering, if necessary, one can assume $\partial_{x_0} b_1(\rho') \neq 0$ so that

$$b_1(x, \xi') = (x_0 - f_1(x', \xi'))e_1(x, \xi'), \quad e_1(x, \xi') \neq 0.$$

Writing $b_j(x, \xi') = b_j(f_1(x', \xi'), x_1, \ldots, x_n, \xi') + c_j(x, \xi')b_1(x, \xi')$ we may assume $b_j(x, \xi')$, $2 \leq j \leq k$ are independent of x_0. Since $p(x, \xi)$ vanishes on Σ of order m one can write with $b = (b_0, b_1, \ldots, b_k) = (b_0, b')$

$$(3.2) \qquad\qquad p(x, \xi) = b_0^m + \sum_{|\alpha|=m, \alpha_0 \leq m-2} \tilde{a}_\alpha(x, \xi')b(x, \xi)^\alpha.$$

Let $\hat{b}_j$ be defined by $b_j(\rho + \mu X) = \mu\hat{b}_j(X) + O(\mu^2)$ and with $\hat{b} = (\xi_0, \hat{b}_1, \ldots, \hat{b}_k)$ we have $p_\rho(X) = q(\hat{b}(X))$ where

$$q(\zeta) = \zeta_0^m + \sum_{|\alpha|=m, \alpha_0 \leq m-2} \tilde{a}_\alpha(\rho')\zeta^\alpha, \quad \zeta = (\zeta_0, \zeta_1, \ldots, \zeta_k) = (\zeta_0, \zeta')$$

is a strictly hyperbolic polynomial in the direction $(1, 0, \ldots, 0) \in \mathbb{R}^{k+1}$ by (1.1). Denote $\tilde{q}(\zeta; x, \xi') = q(\zeta) + \sum a_\alpha(x, \xi')\zeta^\alpha$ with $a_\alpha(x, \xi') = \tilde{a}_\alpha(x, \xi') - \tilde{a}(\rho')$ and hence we have $p(x, \xi) = \tilde{q}(b(x, \xi); x, \xi')$.

Lemma 3.2. *There are m real valued functions $\lambda_1(x, \xi') \leq \lambda_2(x, \xi') \leq \cdots \leq \lambda_m(x, \xi')$ defined in a conic neighborhood of ρ' such that*

$$p(x, \xi) = \prod_{j=1}^{m}(\xi_0 - \lambda_j(x, \xi')), \quad |\lambda_j(x, \xi')| \leq C|b'(x, \xi')|,$$

$$|\lambda_i(x, \xi') - \lambda_j(x, \xi')| \geq c\,|b'(x, \xi')|, \quad (i \neq j)$$

with some $c > 0$, $C > 0$.

Proof. The first assertion is clear because $p(x, \xi)$ is a hyperbolic polynomial in the direction ξ_0. Note that $\tilde{q}(\zeta; \rho') = 0$ has m real distinct roots for $\zeta' \neq 0$ then by Rouché's

theorem $\tilde{q}(\zeta_0, \zeta'; x, \xi') = 0$ has m real distinct roots $\zeta_0 = \lambda_j(\zeta'; x, \xi')$ if $|\xi' - \rho'|$ is sufficiently small which are of homogeneous of degree 1 in ζ' and 0 in ξ'. It is easy to check that $|\lambda_j(\zeta'; x, \xi')| \leq C|\zeta'|$ and $|\lambda_i(\zeta'; x, \xi') - \lambda_j(\zeta'; x, \xi')| \geq c|\zeta'|$ $(i \neq j)$ with some $c > 0, C > 0$. Since $\{|\zeta'| = 1\}$ is compact we end the proof. $\qquad\qquad\square$

4. Basic weights (energy estimates)

We first introduce symbol classes of pseudodifferential operators which will be used in this paper. Denote $\langle \xi \rangle_\gamma^2 = \gamma^2 + |\xi|^2$ where $\gamma \geq 1$ is a positive parameter.

DEFINITION 4.1. Let $W = W(x, \xi; \gamma) > 0$ be a positive function and let $s > 1$, $0 \leq \delta \leq \rho \leq 1$. We define $S_{\rho,\delta}^{(s)}(W)$ to be the set of all $a(x, \xi; \gamma) \in C^\infty(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ such that one can find $A, C > 0$ so that

$$(4.1) \qquad |\partial_x^\beta \partial_\xi^\alpha a(x, \xi; \gamma)| \leq CA^{|\alpha+\beta|}|\alpha + \beta|!^s W \langle \xi \rangle_\gamma^{-\rho|\alpha|+\delta|\beta|}, \quad \forall \alpha, \beta \in \mathbb{N}^{n+1}$$

holds with some $A, C > 0$ independent of $\gamma \geq 1$ and $S_{\rho,\delta}(W)$ to be the set of all $a(x, \xi; \gamma)$ satisfying (4.1) with $C_{\alpha\beta}$ in place of $CA^{|\alpha+\beta|}|\alpha + \beta|!^s$ which may depend on α, β but not on $\gamma \geq 1$. We denote $S_{1,0}^{(s)}(W)$, $S_{1,0}(W)$ simply by $S^{(s)}(W)$, $S(W)$ respectively. We define $S_{\rho,\delta}^{\langle 1,s \rangle}(W)$ to be the set of all $a(x, \xi; \gamma)$ such that we have

$$(4.2) \qquad |\partial_x^\beta \partial_\xi^\alpha a| \leq CA^{|\alpha+\beta|} W(|\alpha + \beta| + |\alpha + \beta|^s \langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha+\beta|} \langle \xi \rangle_\gamma^{-\rho|\alpha|+\delta|\beta|}$$

for any $\alpha, \beta \in \mathbb{N}^{n+1}$ with positive constants $C, A > 0$ independent of $\gamma \geq 1$. If $a(x, \xi; \gamma)$ satisfies (4.1) (resp.(4.2)) in a conic open set $U \subset \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \setminus \{0\})$ we say $a(x, \xi; \gamma) \in S_{\rho,\delta}^{(s)}(W)$ (resp. $S_{\rho,\delta}^{\langle 1,s \rangle}(W)$) in U. We often write $a(x, \xi)$ for $a(x, \xi; \gamma)$ dropping γ.

It is clear $S^{(s)}(W) \subset S_{\rho,\delta}^{\langle 1,s \rangle}(W)$ if $1 - \rho \geq \delta/2$. It is also clear that one may replace $(|\alpha + \beta| + |\alpha + \beta|^s \langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}$ by $|\alpha + \beta|!(1 + |\alpha + \beta|^{s-1} \langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}$ in (4.2), still defining the same symbol class.

Since $p(x, \xi)$ is a polynomial in ξ of degree m it is clear that $p(x, \xi) \in S^{(s)}(\langle \xi \rangle_\gamma^m)$. Since $b_j(x, \xi')$ are defined only in a conic neighborhood of $\rho' = (\bar{x}, \bar{\xi}')$ we extend such symbols to $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$. Let $\chi(t) \in \gamma^{(s)}(\mathbb{R})$ be 1 for $|t| < c/2$ and 0 for $|t| > c$ with small $0 < c < 1/2$ and set

$$\begin{cases} y(x) = \chi(|x - \bar{x}|)(x - \bar{x}) + \bar{x}, \\ \eta'(\xi) = \chi(|\xi'\langle \xi \rangle_\gamma^{-1} - \bar{\xi}'|)(\xi' - \langle \xi \rangle_\gamma \bar{\xi}') + \langle \xi \rangle_\gamma \bar{\xi}'. \end{cases}$$

Then it is easy to see $\eta', b_j(y, \eta') \in S^{(s)}(\langle \xi \rangle_\gamma)$ and $\langle \xi \rangle_\gamma/C \leq |\eta'| \leq C\langle \xi \rangle_\gamma$ with some $C > 0$. In what follows we denote $b_j(y, \eta')$ by $b_j(x, \xi)$.

We now define $w(x, \xi)$, $\omega(x, \xi)$ by

$$\begin{cases} w(x, \xi) = (\sum_{j=1}^k b_j(x, \xi)^2 \langle \xi \rangle_\gamma^{-2} + \langle \xi \rangle_\gamma^{-2\delta})^{1/2}, \\ \omega(x, \xi) = (\phi(x, \xi)^2 + \langle \xi \rangle_\gamma^{-2\delta})^{1/2}, \quad \phi(x, \xi) = \sum_{j=1}^k \alpha_j b_j(x, \xi)\langle \xi \rangle_\gamma^{-1}. \end{cases}$$

Here we recall (3.1), that is

$$(4.3) \qquad\qquad\qquad H_\phi(\rho) \in \Gamma_\rho.$$

In what follows we always assume that

(4.4)
$$
\begin{cases}
0 < \delta < \rho < 1, \quad \rho + \delta = 1, \\
0 < s - 1 < (1 - \rho)/2\rho.
\end{cases}
$$

Lemma 4.1. *There exist $C, A > 0$ such that*

$$
|\partial_x^\beta \partial_\xi^\alpha w| \le C A^{|\alpha+\beta|}(|\alpha + \beta| + |\alpha + \beta|^s \langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha+\beta|} w \langle \xi \rangle_\gamma^{-\rho|\alpha|+\delta|\beta|}
$$

that is we have $w \in S_{\rho,\delta}^{\langle 1,s \rangle}(w)$ and $\omega^{\pm 1} \in S_{\rho,\delta}^{\langle 1,s \rangle}(\omega^{\pm 1})$.

We first remark an easy lemma.

Lemma 4.2. *Let $M > 0$ be such that $2(1 + 4 \sum_{j=0}^{\infty}(j+1)^{-2})M \le 1/2$ and $\Gamma_1(k) = Mk!/k^3$, $k \in \mathbb{N}$ where $\Gamma(0) = M$. Then we have*

$$
\sum_{\alpha'+\alpha''=\alpha} \binom{\alpha}{\alpha'} \Gamma_1(|\alpha'|)\Gamma_1(|\alpha''|) \le \Gamma_1(|\alpha|)/2.
$$

Proof of Lemma 4.1. It suffices to prove the assertion for ϵw with small $\epsilon > 0$ so that one can assume $|w| \le 1$. Thus with $w^2 = F$ there is $A_1 > 0$ such that

$$
|\partial_x^\alpha \partial_\xi^\beta F| \le A_1^{|\alpha+\beta|} \Gamma_1(|\alpha + \beta|)|\alpha + \beta|^{(s-1)|\alpha+\beta|}\langle \xi \rangle_\gamma^{-|\beta|}
$$

holds for any α, β. Noting $|\partial_x^\alpha \partial_\xi^\beta w| \le C_{\alpha\beta} w^{1-|\alpha+\beta|}\langle \xi \rangle_\gamma^{-|\beta|}$ for any α, β we choose $A \ge 2A_1$ so that $C_{\alpha\beta} \le A^{|\alpha+\beta|}\Gamma_1(|\alpha + \beta|)$ for $|\alpha + \beta| \le 4$ then we have

(4.5)
$$
\begin{aligned}
|\partial_x^\alpha \partial_\xi^\beta w| &\le A^{|\alpha+\beta|}\Gamma_1(|\alpha + \beta|)w\langle \xi \rangle_\gamma^{-\rho|\beta|+\delta|\alpha|} \\
&\times (w^{-1}\langle \xi \rangle_\gamma^{-\delta} + |\alpha + \beta|^{s-1}\langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}.
\end{aligned}
$$

Suppose that (4.5) holds for $|\alpha + \beta| \le k$, $4 \le k$ and let $|\alpha + \beta| = k + 1 \ge 4$. Noting

$$
2w\partial_x^\alpha \partial_\xi^\beta w = - \sum_{1 \le |\alpha'+\beta'| \le k} \binom{\alpha}{\alpha'}\binom{\beta}{\beta'} \partial_x^{\alpha'} \partial_\xi^{\beta'} w \partial_x^{\alpha-\alpha'} \partial_\xi^{\beta-\beta'} w + \partial_x^\alpha \partial_\xi^\beta F
$$

and $w^{-1} \ge 1$, applying Lemma 4.2 we see that $w|\partial_x^\alpha \partial_\xi^\beta w|$ is bounded by

$$
\begin{aligned}
&\frac{1}{2}A^{|\alpha+\beta|}\Gamma_1(|\alpha + \beta|)w^2\langle \xi \rangle_\gamma^{-\rho|\beta|+\delta|\alpha|}(w^{-1}\langle \xi \rangle_\gamma^{-\delta} + |\alpha + \beta|^{s-1}\langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha+\beta|} \\
&+ A_1^{|\alpha+\beta|}\Gamma_1(|\alpha + \beta|)w^2|\alpha + \beta|^{(s-1)|\alpha+\beta|}(w^{-2}\langle \xi \rangle_\gamma^{-\delta|\alpha+\beta|})\langle \xi \rangle_\gamma^{-\rho|\beta|+\delta|\alpha|}.
\end{aligned}
$$

Since we have $w^{-2}\langle \xi \rangle_\gamma^{-\delta|\alpha+\beta|} \le \langle \xi \rangle_\gamma^{-\delta(|\alpha+\beta|-2)} \le \langle \xi \rangle_\gamma^{-\delta|\alpha+\beta|/2}$ if $|\alpha + \beta| \ge 4$ then taking $A^{|\alpha+\beta|}/2 + A_1^{|\alpha+\beta|} \le A^{|\alpha+\beta|}$ into account we conclude that (4.5) holds for $|\alpha + \beta| = k + 1$. Therefore noting $w^{-1}\langle \xi \rangle_\gamma^{-\delta} \le 1$ we get

$$
|\partial_x^\alpha \partial_\xi^\beta w| \le A^{|\alpha+\beta|}\Gamma_1(|\alpha + \beta|)w\langle \xi \rangle_\gamma^{-\rho|\beta|+\delta|\alpha|}(1 + |\alpha + \beta|^{s-1}\langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}.
$$

The assertion for ω is proved similarly. As for ω^{-1}, using

$$
\begin{aligned}
\omega|\partial_x^\alpha \partial_\xi^\beta \omega^{-1}| &\le \sum \binom{\alpha}{\alpha'}\binom{\beta}{\beta'} A^{|\alpha+\beta|}\Gamma_1(|\alpha' + \beta'|)\Gamma_1(|\alpha + \beta| - |\alpha' + \beta'|) \\
&\times \langle \xi \rangle_\gamma^{-\rho|\beta|+\delta|\alpha|}(w^{-1}\langle \xi \rangle_\gamma^{-\delta} + |\alpha + \beta|^{s-1}\langle \xi \rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}
\end{aligned}
$$

the proof follows from induction on $|\alpha + \beta|$. $\square$

We now introduce a basic weight symbol which plays a key role in obtaining energy estimates:

$$(4.6) \qquad\qquad \psi = \langle\xi\rangle_\gamma^\kappa \log(\phi+\omega), \quad \kappa = \rho - \delta.$$

Lemma 4.3. *We have $(\phi+\omega)^{\pm 1} \in S_{\rho,\delta}^{\langle 1,s\rangle}((\phi+\omega)^{\pm 1})$. We have also $\psi \in S_{\rho,\delta}^{\langle 1,s\rangle}(\langle\xi\rangle_\gamma^\kappa \log\langle\xi\rangle_\gamma)$. Moreover $\partial_x^\beta \partial_\xi^\alpha \psi \in S_{\rho,\delta}^{\langle 1,s\rangle}(\omega^{-1}\langle\xi\rangle_\gamma^{\kappa-|\alpha|})$ for $|\alpha+\beta| = 1$.*

Proof. With $W = \phi + \omega$ we put for $|\alpha+\beta| = 1$

$$(4.7) \qquad\qquad \partial_x^\beta \partial_\xi^\alpha W = \frac{\partial_x^\beta \partial_\xi^\alpha \phi}{\omega} W + \frac{\partial_x^\beta \partial_\xi^\alpha \langle\xi\rangle_\gamma^{-2\delta}}{2\omega} = \Phi_\beta^\alpha W + \Psi_\beta^\alpha.$$

We examine $\partial_x^\beta \partial_\xi^\alpha \phi \in S_{\rho,\delta}^{\langle 1,s\rangle}(\omega\langle\xi\rangle_\gamma^{-\rho|\alpha|+\delta|\beta|})$ for $|\alpha+\beta| = 1$. Indeed noting $\omega^{-1}\langle\xi\rangle_\gamma^{-\delta} \le 1$ we have

$$|\partial_x^{\nu+\beta} \partial_\xi^{\mu+\alpha} \phi| \le CA^{|\mu+\nu|}\omega|\mu+\nu|^{s|\mu+\nu|}\langle\xi\rangle_\gamma^{-\delta|\mu+\nu|}\langle\xi\rangle_\gamma^{-\rho|\mu+\alpha|+\delta|\nu+\beta|}$$
$$\le CA^{|\mu+\nu|}\omega\langle\xi\rangle_\gamma^{-\rho|\alpha|+\delta|\beta|}(|\mu+\nu|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\mu+\nu|}\langle\xi\rangle_\gamma^{-\rho|\mu|+\delta|\nu|}.$$

Since $\omega^{-1} \in S_{\rho,\delta}^{\langle 1,s\rangle}(\omega^{-1})$ one can find $A_1 > 0$ such that

$$|\partial_x^\nu \partial_\xi^\mu \Phi_\beta^\alpha| \le A_1^{|\mu+\nu|+1}\langle\xi\rangle_\gamma^{-\rho|\alpha+\mu|+\delta|\beta+\nu|}|\mu+\nu|!(1+|\mu+\nu|^{s-1}\langle\xi\rangle_\gamma^{-\delta/2})^{|\mu+\nu|}, \quad \forall\mu,\nu$$

holds for $|\alpha+\beta| = 1$. Since $\langle\xi\rangle_\gamma^{-2\delta} \le W$ similar arguments prove

$$(4.8) \qquad |\partial_x^\nu \partial_\xi^\mu \Psi_\beta^\alpha| \le A_1^{|\mu+\nu|+1}W\langle\xi\rangle_\gamma^{-\rho|\alpha+\mu|+\delta|\beta+\nu|}|\mu+\nu|!(1+|\mu+\nu|^{s-1}\langle\xi\rangle_\gamma^{-\delta/2})^{|\mu+\nu|}, \quad \forall\mu,\nu.$$

Now suppose

$$(4.9) \qquad |\partial_x^\beta \partial_\xi^\alpha W| \le CA_2^{|\alpha+\beta|}W\langle\xi\rangle_\gamma^{-\rho|\alpha|+\delta|\beta|}|\alpha+\beta|!(1+|\alpha+\beta|^{s-1}\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}$$

holds for $|\alpha+\beta| \le \ell$ and letting $|\alpha+\beta+e_1+e_2| = \ell + 1$ we see

$$|\partial_x^{\beta+e_2} \partial_\xi^{\alpha+e_1} W| \le C\sum \binom{\alpha}{\alpha'}\binom{\beta}{\beta'}A_1^{|\alpha-\alpha'+\beta-\beta'|+1}A_2^{|\alpha'+\beta'|}$$
$$\times |\alpha-\alpha'+\beta-\beta'|!|\alpha'+\beta'|!W\langle\xi\rangle_\gamma^{-\rho|\alpha+e_1|+\delta|\beta+e_2|}(1+|\alpha+\beta|^{s-1}\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}$$
$$+ A_1^{|\alpha+\beta|+1}|\alpha+\beta|!W(1+|\alpha+\beta|^{s-1}\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}\langle\xi\rangle_\gamma^{-\rho|\alpha+e_1|+\delta|\beta+e_2|}$$
$$\le (CA_2^{|\alpha+\beta|+1}A_1(A_2-A_1)^{-1} + A_1^{|\alpha+\beta|+1})$$
$$\times |\alpha+\beta|!W(1+|\alpha+\beta|^{s-1}\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}\langle\xi\rangle_\gamma^{-\rho|\alpha+e_1|+\delta|\beta+e_2|}.$$

Thus it suffices to choose A_2 so that $A_1(A_2-A_1)^{-1} + C^{-1}(A_1A_2^{-1}) \le 1$ to conclude $\phi+\omega \in S_{\rho,\delta}^{\langle 1,s\rangle}(\phi+\omega)$. As for $(\phi+\omega)^{-1}$ it suffices to repeat the proof of Lemma 4.1.

We turn to the next assertion. From $\langle\xi\rangle_\gamma^{-2\delta}/C \le \phi+\omega \le C$ it is clear $|\psi| \le \langle\xi\rangle_\gamma^\kappa \log\langle\xi\rangle_\gamma$. Since $\partial_x^\beta \partial_\xi^\alpha \log(\phi+\omega) = \partial_x^\beta \partial_\xi^\alpha(\phi+\omega)/(\phi+\omega)$ for $|\alpha+\beta| = 1$ and $(\phi+\omega)^{\pm 1} \in S_{\rho,\delta}^{\langle 1,s\rangle}((\phi+\omega)^{\pm 1})$ we see $\psi \in S_{\rho,\delta}^{\langle 1,s\rangle}(\langle\xi\rangle_\gamma^\kappa \log\langle\xi\rangle_\gamma)$. Since $\partial_x^\beta \partial_\xi^\alpha \phi \in S_{\rho,\delta}^{\langle 1,s\rangle}(\langle\xi\rangle_\gamma^{-|\alpha|})$ for $|\alpha+\beta| = 1$ and $\omega^{-1} \in S_{\rho,\delta}^{\langle 1,s\rangle}(\omega^{-1})$ it follows from (4.7) and (4.9) that

$$|\partial_x^\nu \partial_\xi^\mu(\partial_x^\beta \partial_\xi^\alpha W)| \le CA^{|\mu+\nu|}\omega^{-1}W\langle\xi\rangle_\gamma^{-|\alpha|}|\mu+\nu|!$$
$$\times (1+|\mu+\nu|^{s-1}\langle\xi\rangle_\gamma^{-\delta/2})^{\mu+\nu}\langle\xi\rangle_\gamma^{-\rho|\mu|+\delta|\nu|}$$

which proves the second assertion.   □

5. Composition formula (energy estimates)

In studying $\text{Op}(e^\psi)P\text{Op}(e^{-\psi}) = \text{Op}(e^\psi \# P \# e^{-\psi})$, if $\psi \in S_{\rho,\delta}^{\langle 1,s\rangle}(\langle\xi\rangle_\gamma^{\kappa'})$ with $\kappa' < \rho - \delta$ one can apply the calculus obtained in [19] to get an asymptotic formula of $e^\psi \# P \# e^{-\psi}$, where the proof is based on the almost analytic extension of symbols and the Stokes' formula using a space $\rho - \delta - \kappa' > 0$. In the present case $\psi \in S_{\rho,\delta}^{\langle 1,s\rangle}(\langle\xi\rangle_\gamma^\kappa)$ there is no space between κ and $\rho - \delta$ and then, introducing a small parameter $\epsilon > 0$, we carefully estimate $e^{\epsilon\psi} \# p \# e^{-\epsilon\psi}$ directly to obtain the composition formula in Theorem 5.1 below.

We denote $a(x, \xi; \gamma, \epsilon) \in \epsilon^{\kappa_1} S_{\rho,\delta}^{\langle 1,s\rangle}(W)$ if $\epsilon^{-\kappa_1} a \in S_{\rho,\delta}^{\langle 1,s\rangle}(W)$ uniformly in $0 < \epsilon \ll 1$. Our aim in this section is to give a sketch of the proof of

Theorem 5.1. *Let* $p(x,\xi) \in S^{(s)}(\langle\xi\rangle_\gamma^m)$. *Then there exists* $\epsilon_0 > 0$ *such that one can find* $K = 1 + r$, $r \in \sqrt{\epsilon} \, S_{\rho,\delta}(1)$ *and* $\gamma_0(\epsilon) > 0$ *for* $0 < \epsilon \le \epsilon_0$ *so that we have for* $\gamma \ge \gamma_0(\epsilon)$

$$
\begin{aligned}
e^{\epsilon\psi} \# p \# e^{-\epsilon\psi} \# K &= \sum_{|\alpha+\beta|\le m} \frac{\epsilon^{|\alpha+\beta|}}{\alpha!\beta!} p_{(\alpha)}^{(\beta)}(-i\nabla_\xi\psi)^\alpha(i\nabla_x\psi)^\beta \\
&\quad + \sum_{1\le|\alpha+\beta|\le m} \epsilon^{|\alpha+\beta|+1/2} c_\beta^\alpha p_{(\alpha)}^{(\beta)} + R
\end{aligned}
$$

(5.1)

where $p_{(\alpha)}^{(\beta)} = \partial_\xi^\beta \partial_x^\alpha p$ *and* $c_\beta^\alpha \in S_{\rho,\delta}(\langle\xi\rangle_\gamma^{\rho|\beta|-\delta|\alpha|})$, $R \in S_{\rho,\delta}(\langle\xi\rangle_\gamma^{m-\delta(m+1)})$. *Moreover we have* $c_\beta^\alpha \in S_{\rho,\delta}(\omega^{-1}\langle\xi\rangle_\gamma^{\kappa-|\alpha|})$ *for* $|\alpha+\beta| = 1$. *In particular* $e^{\epsilon\psi} \# p \# e^{-\epsilon\psi} \in S_{\rho,\delta}(\langle\xi\rangle_\gamma^m)$.

Denote

$$
(5.2) \qquad p_\psi(x,\xi) = \sum_{|\alpha+\beta|\le m} \frac{\epsilon^{|\alpha+\beta|}}{\alpha!\beta!} p_{(\alpha)}^{(\beta)}(-i\nabla_\xi\psi)^\alpha(i\nabla_x\psi)^\beta
$$

which will be the principal part of $e^{\epsilon\psi} \# p \# e^{-\epsilon\psi} \# K$ and differs from the second term on the right-hand side of (5.1) by multiplicative factor $O(\epsilon^{1/2})$.

5.1. Estimates of symbol $e^{\epsilon\psi}$**.** Let $H = H(x,\xi;\gamma) > 0$ be a positive function. Assume that f satisfies

$$
\begin{aligned}
\left|\partial_x^\nu \partial_\xi^\mu f\right| &\le C_0 A_0^{|\mu+\nu|}(|\mu+\nu|-1)! \\
&\quad \times (1 + (|\mu+\nu|-1)^{s-1}\langle\xi\rangle_\gamma^{-\delta/2})^{|\mu+\nu|-1} H\langle\xi\rangle_\gamma^{\delta|\nu|-\rho|\mu|}
\end{aligned}
$$

for $|\mu+\nu| \ge 1$. Set $\Omega_\beta^\alpha = e^{-f}\partial_x^\beta \partial_\xi^\alpha e^f$ then we have

Lemma 5.1. *Notations being as above. There exist* $A_i, C > 0$ *such that the following estimate holds for* $|\alpha+\beta| \ge 1$;

$$
\begin{aligned}
\left|\partial_x^\nu \partial_\xi^\mu \Omega_\beta^\alpha\right| &\le C A_1^{|\nu+\mu|} A_2^{|\alpha+\beta|}\langle\xi\rangle_\gamma^{\delta|\beta+\nu|-\rho|\alpha+\mu|} \\
&\quad \times \sum_{j=1}^{|\alpha+\beta|} H^{|\alpha+\beta|-j+1}(|\mu+\nu|+j)!(1 + (|\mu+\nu|+j)^{s-1}\langle\xi\rangle_\gamma^{-\delta/2})^{|\mu+\nu|+j}.
\end{aligned}
$$

Corollary 5.1. *We have with some* $A, C > 0$

$$
|\partial_x^\beta \partial_\xi^\alpha e^f| \le C A^{|\alpha+\beta|}\langle\xi\rangle_\gamma^{\delta|\beta|-\rho|\alpha|}(H + |\alpha+\beta| + |\alpha+\beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|} e^f.
$$

Moreover for $|\alpha + \beta| \geq 1$, $|\partial_x^\beta \partial_\xi^\alpha e^f|$ is bounded by

$$CHA^{|\alpha+\beta|}\langle\xi\rangle_\gamma^{\delta|\beta|-\rho|\alpha|}(H + |\alpha + \beta| + |\alpha + \beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|-1}e^f.$$

Corollary 5.2. *Notations being as above. We have for $|\alpha + \beta| \geq 1$*

$$\Omega_\beta^\alpha \in S_{\rho,\delta}^{\langle 1,s\rangle}(H(H + |\alpha + \beta| + |\alpha + \beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|-1}\langle\xi\rangle_\gamma^{-\rho|\alpha|+\delta|\beta|}).$$

Corollary 5.3. *Let $\omega_\beta^\alpha = e^{-\epsilon\psi}\partial_x^\beta\partial_\xi^\alpha e^{\epsilon\psi}$. Then there exists $\gamma_0(\epsilon) > 0$ such that $\omega_\beta^\alpha \in \epsilon^{|\alpha+\beta|}S_{\rho,\delta}^{\langle 1,s\rangle}(\langle\xi\rangle_\gamma^{\rho|\beta|-\delta|\alpha|})$ for $\gamma \geq \gamma_0(\epsilon)$.*

5.2. Estimates of $(be^{\epsilon\psi})\#e^{-\epsilon\psi}$. Let $\chi(r) \in \gamma^{(s)}(\mathbb{R})$ be 1 in $|r| \leq 1/4$ and 0 outside $|r| \leq 1/2$. Let $b \in S_{\rho,\delta}^{\langle 1,s\rangle}(\omega^t\langle\xi\rangle_\gamma^m)$ and consider

$$(be^{\epsilon\psi})\#e^{-\epsilon\psi} = \int e^{-2i(\eta z - y\zeta)}b(X + Y)e^{\epsilon\psi(X+Y)-\epsilon\psi(X+Z)}dYdZ$$

$$= b(X) + \int e^{-2i(\eta z - y\zeta)}b(X + Y)(e^{\epsilon\psi(X+Y)-\epsilon\psi(X+Z)} - 1)dYdZ$$

where $Y = (y, \eta)$, $Z = (z, \zeta)$. Denoting $\hat{\chi} = \chi(\langle\eta\rangle\langle\xi\rangle_\gamma^{-1})\chi(\langle\zeta\rangle\langle\xi\rangle_\gamma^{-1})$, $\tilde{\chi} = \chi(|y|/4)\chi(|z|/4)$ we write

$$\int e^{-2i(\eta z - y\zeta)}b(X + Y)(e^{\epsilon\psi(X+Y)-\epsilon\psi(X+Z)} - 1)\{\tilde{\chi}\hat{\chi} + (1 - \tilde{\chi})\hat{\chi}\}dYdZ$$

$$+ \int e^{-2i(\eta z - y\zeta)}b(X + Y)(e^{\epsilon\psi(X+Y)-\epsilon\psi(X+Z)} - 1)(1 - \hat{\chi})dYdZ.$$

After the change of variables $Z \to Z + Y$ the first integral turns to

$$(5.3) \qquad \int e^{-2i(\eta z - y\zeta)}b(X + Y)(e^{\epsilon\psi(X+Y)-\epsilon\psi(X+Y+Z)} - 1)\hat{\chi}_0 dYdZ$$

where we have set $\hat{\chi}_0 = \tilde{\chi}(y, z)\chi(\langle\eta\rangle\langle\xi\rangle_\gamma^{-1})\chi(\langle\eta + \zeta\rangle\langle\xi\rangle_\gamma^{-1})$.

Lemma 5.2. *Let $\Psi(X, Y, Z) = \psi(X + Y) - \psi(X + Y + Z)$ then on the support of $\hat{\chi}_0$ one has*

$$|\Psi(X, Y, Z)| \leq C\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z)$$

where $g_{(x,\xi)}(y, \eta) = \langle\xi\rangle_\gamma^{2\delta}|y|^2 + \langle\xi\rangle_\gamma^{-2\rho}|\eta|^2$ and for $|\alpha + \beta| \geq 1$

$$|\partial_{(x,y)}^\beta \partial_{(\xi,\eta)}^\alpha e^{\epsilon\Psi}| \leq \epsilon CA^{|\alpha+\beta|}\langle\xi\rangle_\gamma^{-\rho|\alpha|+\delta|\beta|}\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z)$$

$$\times (\epsilon\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z) + |\alpha + \beta| + |\alpha + \beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|-1}e^{\epsilon\Psi}.$$

Proof. The assertions follow from Lemma 4.3 and Corollary 5.2. □

Introducing the following differential operators and symbols

$$\begin{cases} L = 1 + 4^{-1}\langle\xi\rangle_\gamma^{2\rho}|D_\eta|^2 + 4^{-1}\langle\xi\rangle_\gamma^{-2\delta}|D_y|^2, \\ M = 1 + 4^{-1}\langle\xi\rangle_\gamma^{2\delta}|D_\zeta|^2 + 4^{-1}\langle\xi\rangle_\gamma^{-2\rho}|D_z|^2, \\ \Phi = 1 + \langle\xi\rangle_\gamma^{2\rho}|z|^2 + \langle\xi\rangle_\gamma^{-2\delta}|\zeta|^2 = 1 + \langle\xi\rangle_\gamma^{2\kappa}g_X(Z), \\ \Theta = 1 + \langle\xi\rangle_\gamma^{2\delta}|y|^2 + \langle\xi\rangle_\gamma^{-2\rho}|\eta|^2 = 1 + g_X(Y) \end{cases}$$

so that $\Phi^{-N}L^N e^{-2i(\eta z - y\zeta)} = e^{-2i(\eta z - y\zeta)}$, $\Theta^{-\ell}M^\ell e^{-2i(\eta z - y\zeta)} = e^{-2i(\eta z - y\zeta)}$ we make integration

by parts in (5.3). Let $F = b(X + Y)(e^{\epsilon\Psi} - 1)$, $\chi^* = \chi(\epsilon\Phi)$, $\chi_* = 1 - \chi^*$ and note $|(\langle\xi\rangle_\gamma^{-\rho}\partial_z)^\alpha(\langle\xi\rangle_\gamma^\delta\partial_\zeta)^\beta\chi^*| \leq C_{\alpha\beta}$ with $C_{\alpha\beta}$ independent of ϵ. Consider

(5.4)
$$\int e^{-2i(\eta z - y\zeta)}\chi^*\partial_x^\beta\partial_\xi^\alpha F\hat\chi_0 dYdZ$$
$$= \int e^{-2i(\eta z - y\zeta)}\Phi^{-N}L^N\Theta^{-\ell}M^\ell(\chi^*\partial_x^\beta\partial_\xi^\alpha F\hat\chi_0)dYdZ.$$

Applying Corollary 5.1 we can estimate the integrand of the right-hand side of (5.4);

(5.5)
$$|\Phi^{-N}L^N\Theta^{-\ell}M^\ell(\chi^*\partial_x^\beta\partial_\xi^\alpha F\hat\chi_0)| \leq C_\ell A^{2N+|\alpha+\beta|+\ell}\Phi^{-N}\Theta^{-\ell}\Big\{\epsilon\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z)$$
$$\times (\epsilon\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z) + 2N + |\alpha + \beta| + (2N + |\alpha + \beta|)^s\langle\xi\rangle_\gamma^{-\delta/2})^{2N+|\alpha+\beta|-1}e^{\epsilon\Psi}$$
$$+ |e^{\epsilon\Psi} - 1|(2N + |\alpha + \beta| + (2N + |\alpha + \beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{2N+|\alpha+\beta|}\Big\}\omega^t(X + Y)\langle\xi\rangle_\gamma^m.$$

Here we remark the following easy lemma.

Lemma 5.3. *Let $A \geq 0$, $B \geq 0$. Then there exists $C > 0$ independent of $n, m \in \mathbb{N}$, A, B such that*

$$(A + n + m + (n + m)^s B)^{n+m} \leq C^{n+m}(A + n + n^s B)^n(A + m + m^s B)^m.$$

Since $|e^{\epsilon\Psi} - 1| \leq C|\epsilon\Psi| \leq C\epsilon\Phi^{1/2} \leq C\sqrt{\epsilon}$ on the support of χ^*, the right-hand side of (5.5) can be estimated by

$$C_\ell A_1^{2N+|\alpha+\beta|}\Phi^{-N}\Theta^{-\ell}\Big\{\epsilon\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z)(\epsilon\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z) + 2N - 1$$
$$+ (2N - 1)^s\langle\xi\rangle_\gamma^{-\delta/2})^{2N-1}(\epsilon\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z) + |\alpha + \beta|$$
$$+ |\alpha + \beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|-1} + \sqrt{\epsilon}(\epsilon\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z) + 2N + (2N)^s\langle\xi\rangle_\gamma^{-\delta/2})^{2N}$$
$$\times (\epsilon\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z) + |\alpha + \beta| + |\alpha + \beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}\Big\}\omega^t(X + Y)\langle\xi\rangle_\gamma^m$$

where we remark $\omega^{\pm1}(X + Y) \leq C\omega^{\pm1}(X)(1 + g_X^{1/2}(Y))$ on the support of $\hat\chi_0$ and hence we have $\omega^t(X + Y) \leq C\omega^t(X)\Theta^{t'}$ with some t'. Noting

$$A_1^{2N}\Phi^{-N}(\epsilon\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z) + 2N + (2N)^s\langle\xi\rangle_\gamma^{-\delta/2})^{2N}$$
$$= \Big(\epsilon\frac{A_1\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z)}{\Phi^{1/2}} + \frac{2A_1N}{\Phi^{1/2}} + \frac{A_1(2N)^s\langle\xi\rangle_\gamma^{-\delta/2}}{\Phi^{1/2}}\Big)^{2N}$$

and

$$A_1^{2N}\Phi^{-N}\epsilon\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z)(\epsilon\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z) + 2N - 1 + (2N - 1)^s\langle\xi\rangle_\gamma^{-\delta/2})^{2N-1}$$
$$\leq \epsilon A_1\Big(\epsilon\frac{A_1\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z)}{\Phi^{1/2}} + \frac{2A_1N}{\Phi^{1/2}} + \frac{A_1(2N)^s\langle\xi\rangle_\gamma^{-\delta/2}}{\Phi^{1/2}}\Big)^{2N-1}$$

we choose $N = N(z, \zeta, \xi)$ so that $2A_1N = \bar c\Phi^{1/2}$ with a small $\bar c > 0$. Then noting that $\Phi \leq 1 + 2\langle\xi\rangle_\gamma^{2\rho}|z|^2 + 2\langle\xi\rangle_\gamma^{-2\delta}|\zeta|^2 \leq C\langle\xi\rangle_\gamma^{2\rho}$ on the support of $\hat\chi_0$ and therefore $\Phi^{(s-1)/2}\langle\xi\rangle_\gamma^{-\delta/2} \leq C\langle\xi\rangle_\gamma^{-\epsilon_1} \leq C\gamma^{-\epsilon_1}$ with some $\epsilon_1 > 0$ we have

$$\Big(\epsilon\frac{A_1\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z)}{\Phi^{1/2}} + \frac{2A_1N}{\Phi^{1/2}} + \frac{A_1(2N)^s\langle\xi\rangle_\gamma^{-\delta/2}}{\Phi^{1/2}}\Big)^{2N-1}$$
$$\leq (A_1\epsilon + \bar c + \bar c^s\Phi^{(s-1)/2}\langle\xi\rangle_\gamma^{-\delta/2})^{2N-1} \leq Ce^{-c_1\Phi^{1/2}}$$

choosing $\bar{c}$ small and $\gamma \geq \gamma_0(\epsilon)$ large. On the other hand since $\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z) \leq \Phi^{1/2}$ it is clear

$$(\epsilon\langle\xi\rangle_\gamma^\kappa g_X^{1/2}(Z) + |\alpha+\beta| + |\alpha+\beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}e^{-c\,\Phi^{1/2}}$$
$$\leq CA^{|\alpha+\beta|}(|\alpha+\beta| + |\alpha+\beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}e^{-c'\Phi^{1/2}}.$$

Set $\ell' = \ell - t'$. Then noting $e^{-c'\Phi^{1/2}} \leq C\Phi^{-\ell'}$ we have

(5.6)
$$|\Phi^{-N}L^N\Theta^{-\ell}M^\ell(\chi^*\partial_x^\beta\partial_\xi^\alpha F\hat{\chi}_0)|$$
$$\leq \sqrt{\epsilon}\,CA^{|\alpha+\beta|}(|\alpha+\beta| + |\alpha+\beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}\omega^t(X)\langle\xi\rangle_\gamma^{m-\rho|\alpha|+\delta|\beta|}\Theta^{-\ell'}\Phi^{-\ell'}.$$

Finally choosing $\ell > t' + (n+1)/2$ and recalling $\int \Theta^{-\ell'}\Phi^{-\ell'}dYdZ = C$ we conclude

$$\left|\int e^{-2i(\eta z-y\zeta)}\chi^*\partial_x^\beta\partial_\xi^\alpha F\hat{\chi}_0 dYdZ\right|$$
$$\leq \sqrt{\epsilon}\,CA^{|\alpha+\beta|}(|\alpha+\beta| + |\alpha+\beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}\omega^t(X)\langle\xi\rangle_\gamma^{m-\rho|\alpha|+\delta|\beta|}.$$

We next consider

$$\int e^{-2i(\eta z-y\zeta)}\chi_*\partial_x^\beta\partial_\xi^\alpha F\hat{\chi}_0 dYdZ$$
$$= \int e^{-2i(\eta z-y\zeta)}L^N\Phi^{-N}M^\ell\Theta^{-\ell}\chi_*\partial_x^\beta\partial_\xi^\alpha F\hat{\chi}_0 dYdZ.$$

Similar arguments obtaining (5.6) show that

$$|\Phi^{-N}L^N\Theta^{-\ell}M^\ell(\chi_*\partial_x^\beta\partial_\xi^\alpha F\hat{\chi}_0)| \leq CA^{|\alpha+\beta|}(|\alpha+\beta| + |\alpha+\beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}$$
$$\times\omega^t(X)\langle\xi\rangle_\gamma^{m-\rho|\alpha|+\delta|\beta|}\Theta^{-\ell'}\Phi^{-\ell'}e^{-c\,\Phi^{1/2}}.$$

Since $\Phi^{1/2} \geq \epsilon^{-1/2}$ on the support of χ_* we see $e^{-c\,\Phi^{1/2}} \leq e^{-c\,\epsilon^{-1/2}} \leq C\sqrt{\epsilon}$ and this proves

$$\left|\partial_x^\beta\partial_\xi^\alpha\int e^{-2i(\eta z-y\zeta)}F\hat{\chi}_0 dYdZ\right|$$
$$\leq \sqrt{\epsilon}\,CA^{|\alpha+\beta|}(|\alpha+\beta| + |\alpha+\beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}\omega^t(X)\langle\xi\rangle_\gamma^{m-\rho|\alpha|+\delta|\beta|}.$$

We then consider

$$\int e^{-2i(\eta z-y\zeta)}(|y|^2 + |z|^2)^{-N}(|D_\zeta|^2 + |D_\eta|^2)^N F\hat{\chi}_1 dYdZ$$

where $F = b(X + Y)(e^{\epsilon\psi(X+Y)-\epsilon\psi(X+Z)} - 1)$ and $\hat{\chi}_1 = (1 - \tilde{\chi})\hat{\chi}$. Let $\kappa < \kappa_1 < \rho$ then since $|\psi(X+Y)| + |\psi(X+Z)|$ is bounded by $C\langle\xi\rangle_\gamma^{\kappa_1}$ and $C^{-1} \leq \langle\xi+\eta\rangle_\gamma/\langle\xi\rangle_\gamma, \langle\xi+\zeta\rangle_\gamma/\langle\xi\rangle_\gamma \leq C$ with some $C > 0$ on the support of $\hat{\chi}_1$ thanks to Corollary 5.1 it is not difficult to show

$$\left|(|D_\zeta|^2 + |D_\eta|^2)^N\partial_x^\beta\partial_\xi^\alpha F\hat{\chi}_1\right| \leq CA^{2N+|\alpha+\beta|}\omega^t(X + Y)\langle\xi\rangle_\gamma^{m-\rho|\alpha|+\delta|\beta|}$$
$$\times (\langle\xi\rangle_\gamma^{\kappa_1} + |\alpha+\beta| + |\alpha+\beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}$$
$$\times \langle\xi\rangle_\gamma^{-2\rho N}(\langle\xi\rangle_\gamma^{\kappa_1} + 2N + (2N)^s\langle\xi\rangle_\gamma^{-\delta/2})^{2N}e^{c\langle\xi\rangle_\gamma^{\kappa_1}}.$$

Choose $N = c_1\langle\xi\rangle_\gamma^\rho$ with small $c_1 > 0$ so that

$$A^{2N}\langle\xi\rangle_\gamma^{-2\rho N}(\langle\xi\rangle_\gamma^{\kappa_1} + 2N + (2N)^s\langle\xi\rangle_\gamma^{-\delta/2})^{2N}$$

is bounded by $Ce^{-c\langle\xi\rangle_\gamma^\rho}$ and $\langle\xi\rangle_\gamma^{\kappa_1|\alpha+\beta|}e^{-c\langle\xi\rangle_\gamma^\rho}$ is bounded by $CA^{|\alpha+\beta|}e^{-c'\langle\xi\rangle_\gamma^\rho}$. Then noting $\omega^t(X +$

$Y) \leq C\omega^t(X)\langle\xi\rangle_\gamma^{t'}$ and $e^{-c'\langle\xi\rangle_\gamma^\rho} \leq \sqrt{\epsilon}\,C\langle\xi\rangle_\gamma^{-2(n+1)-t'}$ for $\gamma \geq \gamma_0(\epsilon)$ and that $\langle\xi\rangle_\gamma^{-2(n+1)}\int(|y|^2 + |z|^2)^{-N}\hat{\chi}_1 dYdZ \leq C$ we conclude

Lemma 5.4. *Let* $\hat{\chi} = \chi(\langle\eta\rangle\langle\xi\rangle_\gamma^{-1})\chi(\langle\zeta\rangle\langle\xi\rangle_\gamma^{-1})$. *Then we have for* $\gamma \geq \gamma_0(\epsilon)$

$$\left|\partial_x^\beta\partial_\xi^\alpha\int e^{-2i(z\eta-y\zeta)}b(X+Y)(e^{\epsilon\psi(X+Y)-\epsilon\psi(X+Z)}-1)\hat{\chi}dYdZ\right|$$
$$\leq \sqrt{\epsilon}\,CA^{|\alpha+\beta|}(|\alpha+\beta|+|\alpha+\beta|^s\langle\xi\rangle_\gamma^{-\delta/2})^{|\alpha+\beta|}\omega^t(X)\langle\xi\rangle_\gamma^{m-\rho|\alpha|+\delta|\beta|}.$$

Let us write

$$1 - \hat{\chi} = (1 - \chi(\langle\eta\rangle\langle\xi\rangle_\gamma^{-1}))(1 - \chi(\langle\zeta\rangle\langle\xi\rangle_\gamma^{-1})) + (1 - \chi(\langle\eta\rangle\langle\xi\rangle_\gamma^{-1}))\chi(\langle\zeta\rangle\langle\xi\rangle_\gamma^{-1})$$
$$+ (1 - \chi(\langle\zeta\rangle\langle\xi\rangle_\gamma^{-1}))\chi(\langle\eta\rangle\langle\xi\rangle_\gamma^{-1}) = \hat{\chi}_2 + \hat{\chi}_3 + \hat{\chi}_4.$$

Denoting $F = b(X+Y)(e^{\epsilon\psi(X+Y)-\epsilon\psi(X+Z)}-1)$ again we consider

$$\int e^{-2i(\eta z-y\zeta)}\langle\eta\rangle^{-2N_2}\langle\zeta\rangle^{-2N_1}\langle D_z\rangle^{2N_2}\langle D_y\rangle^{2N_1}\langle y\rangle^{-2\ell}\langle z\rangle^{-2\ell}\langle D_\zeta\rangle^{2\ell}\langle D_\eta\rangle^{2\ell}\partial_x^\beta\partial_\xi^\alpha F\hat{\chi}_2\chi^* dYdZ$$

where χ^* is either $\chi(\langle\zeta\rangle\langle\eta\rangle^{-1}/4)$ or $1 - \chi(\langle\zeta\rangle\langle\eta\rangle^{-1}/4)$. If $\chi^* = \chi(\langle\zeta\rangle\langle\eta\rangle^{-1}/4)$ we choose $N_1 = \ell, N_2 = N$ and noting $\omega^t(X+Y) \leq C\langle\eta\rangle^{t'}$ with some $t' \geq 0$ and $|\psi(X+Y)|+|\psi(X+Z)| \leq C\langle\eta\rangle^{\kappa_1}$ with $\kappa < \kappa_1 < \rho$ on the support of $\hat{\chi}_2\chi^*$ it is not difficult to see that

$$|\langle\eta\rangle^{-2N}\langle\zeta\rangle^{-2\ell}\langle D_z\rangle^{2N}\langle D_y\rangle^{2\ell}\langle y\rangle^{-2\ell}\langle z\rangle^{-2\ell}\langle D_\zeta\rangle^{2\ell}\langle D_\eta\rangle^{2\ell}\partial_x^\beta\partial_\xi^\alpha F\hat{\chi}_2\chi^*|$$

is bounded by

$$C_\ell A^{2N+|\alpha+\beta|}\langle\eta\rangle^{-2N}\langle\zeta\rangle^{-2\ell}\langle y\rangle^{-2\ell}\langle z\rangle^{-2\ell}\langle\eta\rangle^{m+t'+2\delta\ell}\langle\eta\rangle^{6\ell\rho}$$

(5.7)
$$\times(\langle\eta\rangle^{\kappa_1} + 2N\langle\eta\rangle^\delta + N^s\langle\eta\rangle^{\delta/2})^{2N}$$
$$\times(\langle\eta\rangle^{\kappa_1} + \langle\eta\rangle^\delta|\alpha+\beta| + \langle\eta\rangle^{\delta/2}|\alpha+\beta|^s)^{|\alpha+\beta|}e^{C\langle\eta\rangle^{\kappa_1}}.$$

Here writing

$$A^{2N}\langle\eta\rangle^{-2N}(\langle\eta\rangle^{\kappa_1} + 2N\langle\eta\rangle^\delta + N^s\langle\eta\rangle^{\delta/2})^{2N}$$
$$= \left(\frac{A\langle\eta\rangle^{\kappa_1}}{\langle\eta\rangle} + \frac{2AN}{\langle\eta\rangle^\rho} + \frac{AN^s\langle\eta\rangle^{\delta/2}}{\langle\eta\rangle}\right)^{2N}$$

we take $2N = c_1\langle\eta\rangle^\rho$ with small $c_1 > 0$ so that the right-hand side is bounded by $Ce^{-c\langle\eta\rangle^\rho}$. Noting $\langle\eta\rangle^{\delta|\alpha+\beta|}e^{-c\langle\eta\rangle^\rho} \leq CA_1^{|\alpha+\beta|}|\alpha+\beta|^{\delta|\alpha+\beta|/\rho}e^{-c_1\langle\eta\rangle^\rho}$ and $\langle\eta\rangle^{\kappa_1|\alpha+\beta|}e^{-c\langle\eta\rangle^\rho} \leq CA_1^{|\alpha+\beta|}|\alpha+\beta|^{|\alpha+\beta|}e^{-c_1\langle\eta\rangle^\rho}$ one sees that (5.7) is bounded by

$$C_\ell A_1^{|\alpha+\beta|}\langle\zeta\rangle^{-2\ell}\langle y\rangle^{-2\ell}\langle z\rangle^{-2\ell}(|\alpha+\beta|^{1+\delta/\rho} + |\alpha+\beta|^{s+\delta/2\rho})^{|\alpha+\beta|}e^{-c_1\langle\eta\rangle^\rho}.$$

Similarly if $\chi^* = 1 - \chi(\langle\zeta\rangle\langle\eta\rangle^{-1}/4)$ choosing $N_1 = N, N_2 = \ell$ it is proved that (5.7) is estimated by

$$C_\ell A_1^{|\alpha+\beta|}\langle\eta\rangle^{-2\ell}\langle y\rangle^{-2\ell}\langle z\rangle^{-2\ell}(|\alpha+\beta|^{1+\delta/\rho} + |\alpha+\beta|^{s+\delta/2\rho})^{|\alpha+\beta|}e^{-c_1\langle\zeta\rangle^\rho}.$$

Thus taking $1 + \delta/\rho = 1/\rho$ and $s + \delta/2\rho \leq 1/\rho$ into account and recalling that $\langle\xi\rangle_\gamma \leq \langle\eta\rangle$, $\langle\xi\rangle_\gamma \leq \langle\zeta\rangle$ on the support of $\hat{\chi}_2$ we get

Lemma 5.5. *We have*

$$\left| \partial_x^\beta \partial_\xi^\alpha \int e^{-2i(z\eta - y\zeta)} b(X+Y)(e^{\epsilon\psi(X+Y) - \epsilon\psi(X+Z)} - 1)\hat{\chi}_2 dY dZ \right|$$

$$\leq C A^{|\alpha+\beta|} |\alpha+\beta|^{|\alpha+\beta|/\rho} e^{-c_1\langle\xi\rangle_\gamma^\rho}.$$

Repeating similar arguments we can prove

$$\left| \partial_x^\beta \partial_\xi^\alpha \int e^{-2i(\eta z - y\zeta)} b(X+Y)(e^{\epsilon\psi(X+Y) - \epsilon\psi(X+Z)} - 1)\hat{\chi}_i dY dZ \right|$$

$$\leq C A^{|\alpha+\beta|} |\alpha+\beta|^{|\alpha+\beta|/\rho} e^{-c_1\langle\xi\rangle_\gamma^\rho}$$

for $i = 3, 4$. We summarize what we have proved in

Proposition 5.1. *Let $b \in S_{\rho,\delta}^{\langle 1,s \rangle}(\omega^t \langle\xi\rangle_\gamma^m)$ then we have*

$$(be^{\epsilon\psi}) \# e^{-\epsilon\psi} = b + \omega^t \hat{b} + R$$

where $\hat{b} \in \sqrt{\epsilon} S_{\rho,\delta}^{\langle 1,s \rangle}(\langle\xi\rangle_\gamma^m)$ and $R \in S_{0,0}^{(1/\rho)}(e^{-c_1\langle\xi\rangle_\gamma^\rho})$, that is

$$|\partial_x^\beta \partial_\xi^\alpha R| \leq C A^{|\alpha+\beta|} |\alpha+\beta|^{|\alpha+\beta|/\rho} e^{-c_1\langle\xi\rangle_\gamma^\rho}.$$

5.3. Proo of Theorem 5.1. We start with the next lemma which is proved repeating similar arguments in the preceding subsection.

Lemma 5.6. *Let $a \in S^{(s)}(\langle\xi\rangle_\gamma^d)$ and $b \in S_{\rho,\delta}^{\langle 1,s \rangle}(\langle\xi\rangle_\gamma^h)$. Then we have*

$$(be^{\epsilon\psi}) \# a = \sum_{|\alpha+\beta| < N} \frac{(-1)^{|\beta|}}{(2i)^{|\alpha+\beta|} \alpha! \beta!} a_{(\alpha)}^{(\beta)} (be^{\epsilon\psi})_{(\beta)}^{(\alpha)} + b_N e^{\epsilon\psi} + R$$

where $b_N \in S_{\rho,\delta}^{\langle 1,s \rangle}(\langle\xi\rangle_\gamma^{d+h-\delta N})$, $R \in S_{0,0}^{(1/\rho)}(e^{-c\langle\xi\rangle_\gamma^\rho})$. For $a \# (be^{\epsilon\psi})$ similar assertion holds, where $(-1)^{|\beta|}$ is replaced by $(-1)^{|\alpha|}$.

We can also prove

Lemma 5.7. *Let $R \in S_{0,0}^{(1/\rho)}(e^{-c\langle\xi\rangle_\gamma^\rho})$. Then we have*

$$R \# e^{\pm\epsilon\psi}, \quad e^{\pm\epsilon\psi} \# R \in S_{0,0}^{(1/\rho)}(e^{-c'\langle\xi\rangle_\gamma^\rho}).$$

Corollary 5.4. *Let $R \in S_{0,0}^{(1/\rho)}(e^{-c\langle\xi\rangle_\gamma^\rho})$. Then for any $t \in \mathbb{R}$ we have*

$$e^{\pm\epsilon\psi} \# R \# e^{\mp\epsilon\psi} \in S(\langle\xi\rangle_\gamma^t).$$

Lemma 5.8. *Let $p \in S^{(s)}(\langle\xi\rangle_\gamma^d)$. Then one can write*

$$(e^{\epsilon\psi}) \# p = \sum_{|\alpha+\beta| < N} \frac{(-1)^{|\beta|}}{i^{|\alpha+\beta|} \alpha! \beta!} p_{(\alpha)}^{(\beta)} \# (\omega_\beta^\alpha e^{\epsilon\psi}) + r_N e^{\epsilon\psi} + R$$

where $r_N \in S_{\rho,\delta}^{\langle 1,s \rangle}(\langle\xi\rangle_\gamma^{d-\delta N})$, $R \in S_{0,0}^{(1/\rho)}(e^{-c\langle\xi\rangle_\gamma^\rho})$ and $\omega_\beta^\alpha = e^{-\epsilon\psi} \partial_x^\beta \partial_\xi^\alpha e^{\epsilon\psi}$.

Proof. We first examine

$$(5.8) \qquad p_{(\alpha)}^{(\beta)} \omega_\beta^\alpha e^{\epsilon\psi} - \sum_{|\gamma+\delta| < N} \frac{(-1)^{|\gamma|}}{(2i)^{|\gamma+\delta|} \gamma! \delta!} p_{(\alpha+\delta)}^{(\beta+\gamma)} \# (\omega_{\beta+\gamma}^{\alpha+\delta} e^{\epsilon\psi}) = r_{N,|\alpha+\beta|} e^{\epsilon\psi} + R$$

with $r_{N,|\alpha+\beta|} \in S_{\rho,\delta}^{\langle 1,s \rangle}(\langle \xi \rangle_\gamma^{d-\delta N})$. Indeed since $\partial_x^\nu \partial_\xi^\mu (\omega_\beta^\alpha e^{\epsilon \psi}) = \omega_{\beta+\nu}^{\alpha+\mu} e^{\epsilon \psi}$ thanks to Lemma 5.6 one can write

$$\sum_{|\gamma+\delta|<N} \frac{(-1)^{|\gamma|}}{(2i)^{|\gamma+\delta|}\gamma!\delta!} p_{(\alpha+\delta)}^{(\beta+\gamma)} \# (\omega_{\beta+\gamma}^{\alpha+\delta} e^{\epsilon \psi})$$

$$= \sum_{|\gamma'+\delta'|<2N} \frac{(-1)^{\gamma'|}}{(2i)^{|\gamma'+\delta'|}\gamma'!\delta'!} \left(\sum \binom{\gamma'}{\mu}\binom{\delta'}{\nu}(-1)^{|\mu+\nu|} \right) p_{(\alpha+\delta')}^{(\beta+\gamma')} \omega_{\beta+\gamma'}^{\alpha+\delta'} e^{\epsilon \psi}$$

$$+ r_{N,|\alpha+\beta|} e^{\epsilon \psi} + R$$

where $\sum \binom{\gamma'}{\mu}\binom{\delta'}{\nu}(-1)^{|\mu+\nu|} = 0$ if $|\gamma' + \delta'| > 0$ so that the right-hand side is

$$p_{(\alpha)}^{(\beta)} \omega_\beta^\alpha e^{\epsilon \psi} + r_{N,|\alpha+\beta|} e^{\epsilon \psi} + R, \quad r_{N,|\alpha+\beta|} \in S_{\rho,\delta}^{\langle 1,s \rangle}(\langle \xi \rangle_\gamma^{d-\delta N})$$

which proves (5.8). Now insert the expression of $p_{(\alpha)}^{(\beta)} \omega_\beta^\alpha e^{\epsilon \psi}$ in (5.8) into

$$(e^{\epsilon \psi}) \# p = \sum_{|\alpha+\beta|<N} \frac{(-1)^{|\beta|}}{(2i)^{|\alpha+\beta|}\alpha!\beta!} p_{(\alpha)}^{(\beta)} \omega_\beta^\alpha e^{\epsilon \psi} + r_N e^{\epsilon \psi} + R$$

which follows from Lemma 5.6 to get

$$\sum_{|\alpha'+\beta'|<2N} \frac{(-1)^{|\beta'|}}{(2i)^{|\alpha'+\beta'|}\alpha'!\beta'!} \left(\sum \binom{\alpha'}{\delta}\binom{\beta'}{\gamma} \right) p_{(\alpha')}^{(\beta')} \# (\omega_{\beta'}^{\alpha'} e^{\epsilon \psi}) + \tilde{r}_N e^{\epsilon \psi} + R$$

where $\tilde{r}_N \in S_{\rho,\delta}^{\langle 1,s \rangle}(\langle \xi \rangle_\gamma^{d-\delta N})$. Here we note $\sum \binom{\alpha'}{\delta}\binom{\beta'}{\gamma} = 2^{|\alpha'+\beta'|}$. It is clear that $p_{(\alpha')}^{(\beta')} \# (\omega_{\beta'}^{\alpha'} e^{\epsilon \psi}) = r' e^{\epsilon \psi} + R$ with $r' \in S_{\rho,\delta}^{\langle 1,s \rangle}(\langle \xi \rangle_\gamma^{d-\delta N})$ for $|\alpha' + \beta'| \geq N$ and hence we get the assertion. $\square$

Proof of Theorem 5.1. From Lemma 5.8 we see

$$(e^{\epsilon \psi}) \# p \# e^{-\epsilon \psi} = \sum_{|\alpha+\beta| \leq m} \frac{(-1)^{|\beta|}}{i^{|\alpha+\beta|}\alpha!\beta!} p_{(\alpha)}^{(\beta)} \# ((\omega_\beta^\alpha e^{\epsilon \psi}) \# e^{-\epsilon \psi}) + (r_m e^{\epsilon \psi} + R) \# e^{-\epsilon \psi}$$

where $(r_m e^{\epsilon \psi} + R) \# e^{-\epsilon \psi} \in S_{\rho,\delta}(\langle \xi \rangle_\gamma^{m-\delta(m+1)})$ which follows from Propositions 5.1 and Lemma 5.7. Therefore Proposition 5.1 together with Corollary 5.3 gives

$$(e^{\epsilon \psi}) \# p \# e^{-\epsilon \psi} = \sum_{|\alpha+\beta| \leq m} \frac{(-1)^{|\beta|}}{i^{|\alpha+\beta|}\alpha!\beta!} p_{(\alpha)}^{(\beta)} \# (\omega_\beta^\alpha + \bar{\omega}_\beta^\alpha) + R$$

where $\bar{\omega}_\beta^\alpha \in \epsilon^{|\alpha+\beta|+1/2} S_{\rho,\delta}^{\langle 1,s \rangle}(\langle \xi \rangle_\gamma^{\rho|\beta|-\delta|\alpha|})$, $R \in S_{\rho,\delta}(\langle \xi \rangle_\gamma^{m-\delta(m+1)})$ and

$$\bar{\omega}_\beta^\alpha \in \epsilon^{3/2} S_{\rho,\delta}^{\langle 1,s \rangle}(\omega^{-1} \langle \xi \rangle_\gamma^{\kappa-|\alpha|}), \quad |\alpha+\beta| = 1.$$

Since $e^{\epsilon \psi} \# e^{-\epsilon \psi} = 1 - r_1$, $r_1 \in \sqrt{\epsilon} S_{\rho,\delta}(1)$ by Proposition 5.1 there exists $K = 1 + r$, $r \in \sqrt{\epsilon} S_{\rho,\delta}(1)$ such that $e^{\epsilon \psi} \# e^{-\epsilon \psi} \# K = 1$ if $0 < \epsilon \leq \epsilon_0$ is small ([1, Theorem 3.2] and [15, Theorem 2.6.27] for example). Thus we have

$$(e^{\epsilon \psi}) \# p \# e^{-\epsilon \psi} \# K = p + \sum_{1 \leq |\alpha+\beta| \leq m} \frac{(-1)^{|\beta|}}{i^{|\alpha+\beta|}\alpha!\beta!} p_{(\alpha)}^{(\beta)} \# (\omega_\beta^\alpha + \bar{\omega}_\beta^\alpha) \# K + R.$$

On the other hand it is clear $(\omega_\beta^\alpha + \bar{\omega}_\beta^\alpha) \# (1 + r) = \omega_\beta^\alpha + \tilde{\omega}_\beta^\alpha$ with $\tilde{\omega}_\beta^\alpha \in \epsilon^{|\alpha+\beta|+1/2} S_{\rho,\delta}(\langle \xi \rangle_\gamma^{\rho|\beta|-\delta|\alpha|})$.

Since $\omega_\beta^\alpha \in \epsilon S_{\rho,\delta}(\omega^{-1}\langle\xi\rangle_\gamma^{\kappa-|\alpha|})$ for $|\alpha+\beta| = 1$ it is also clear that $\tilde\omega_\beta^\alpha \in \epsilon^{3/2} S_{\rho,\delta}(\omega^{-1}\langle\xi\rangle_\gamma^{\kappa-|\alpha|})$ for $|\alpha+\beta| = 1$. Note

$$p_{(\alpha)}^{(\beta)}\#(\omega_\beta^\alpha + \bar\omega_\beta^\alpha) - \sum_{|\mu+\nu|\leq m-|\alpha+\beta|} \frac{(-1)^{|\nu|}}{i^{|\mu+\nu|}\mu!\nu!} p_{(\alpha+\nu)}^{(\beta+\mu)}(\omega_\beta^\alpha + \bar\omega_\beta^\alpha)_{(\mu)}^{(\nu)}$$

$$\in S_{\rho,\delta}(\langle\xi\rangle_\gamma^{m-\delta|\alpha+\beta|-\rho(m+1-|\alpha+\beta|)}) \subset S_{\rho,\delta}(\langle\xi\rangle_\gamma^{m-\delta(m+1)})$$

and $(\omega_\beta^\alpha + \bar\omega_\beta^\alpha)_{(\mu)}^{(\nu)} \in \gamma^{-\kappa|\mu+\nu|} S_{\rho,\delta}^{\langle 1,s\rangle}(\langle\xi\rangle_\gamma^{\rho|\alpha+\mu|-\delta|\beta+\nu|})$ which is contained in $\epsilon^{|\alpha+\beta|+|\mu+\nu|+1/2} S_{\rho,\delta}^{\langle 1,s\rangle}(\langle\xi\rangle_\gamma^{\rho|\alpha+\mu|-\delta|\beta+\nu|})$ if $|\mu+\nu| \geq 1, \gamma \geq \gamma_0(\epsilon)$ so that

$$(e^{\epsilon\psi})\#p\#e^{-\epsilon\psi}\#K = p + \sum_{1\leq|\alpha+\beta|\leq m} \frac{(-1)^{|\beta|}}{i^{|\alpha+\beta|}\alpha!\beta!} p_{(\alpha)}^{(\beta)}(\omega_\beta^\alpha + \hat\omega_\beta^\alpha) + R$$

where $\hat\omega_\beta^\alpha \in \epsilon^{|\alpha+\beta|+1/2} S_{\rho,\delta}(\langle\xi\rangle_\gamma^{\rho|\beta|-\delta|\alpha|})$ and $\hat\omega_\beta^\alpha \in \epsilon^{3/2} S_{\rho,\delta}(\omega^{-1}\langle\xi\rangle_\gamma^{\kappa-|\alpha|})$ for $|\alpha+\beta| = 1$. Now check ω_β^α. For $|\alpha+\beta| = 1$ we have $\omega_\beta^\alpha = \epsilon(-i\nabla_\xi\psi)^\alpha(i\nabla_x\psi)^\beta$. Let $|\alpha+\beta| \geq 2$ then ω_β^α is a linear combination of terms $(\epsilon\psi)_{(\beta_1)}^{(\alpha_1)}\cdots(\epsilon\psi)_{(\beta_s)}^{(\alpha_s)}$ with $\alpha_1 + \cdots + \alpha_s = \alpha$, $\beta_1 + \cdots + \beta_s = \beta$, $|\alpha_j+\beta_j| \geq 1$. If $|\alpha_j+\beta_j| = 1$ for all j it is clear $\omega_\beta^\alpha = \epsilon^{|\alpha+\beta|}(-i\nabla_\xi\psi)^\alpha(i\nabla_x\psi)^\beta$. If $|\alpha_j+\beta_j| \geq 2$ for some j so that $s \leq |\alpha+\beta| - 2$ then one has

$$(\epsilon\psi)_{(\beta_1)}^{(\alpha_1)}\cdots(\epsilon\psi)_{(\beta_s)}^{(\alpha_s)} \in S_{\rho,\delta}(\langle\xi\rangle_\gamma^{-\kappa+\rho|\beta|-\delta|\alpha|}) \subset \gamma^{-\kappa} S_{\rho,\delta}(\langle\xi\rangle_\gamma^{\rho|\alpha|-\delta|\beta|}).$$

Since we can assume $\gamma^{-\kappa} \leq \epsilon^{|\alpha+\beta|+1/2}$ for $\gamma \geq \gamma_0(\epsilon)$ we get the assertion. $\qquad\square$

6. Energy estimates

To obtain energy estimates we follow [13] where the main point is to derive microlocal energy estimates. We sketch how to get microlocal energy estimates. Let us denote

$$P_\psi = \mathrm{Op}(e^{\epsilon\psi})P\mathrm{Op}(e^{-\epsilon\psi})\mathrm{Op}(K)$$

of which principal symbol is given by $p_\psi = e^{\epsilon\psi}\#p\#e^{-\epsilon\psi}\#K$. In this section we say $a(x,\xi;\gamma,\epsilon) \in \tilde S_{\rho,\delta}(W)$ if $a \in S_{\rho,\delta}(W)$ for each fixed $0 < \epsilon \ll 1$. Let $a \in \tilde S_{\rho,\delta}(W)$ and let $N \in \mathbb{N}$ be given. Then with a fixed small $0 < 2\tau < \rho - \delta$ we have

$$|\partial_x^\beta\partial_\xi^\alpha a| \leq C_{\alpha\beta}(\epsilon)W\langle\xi\rangle_\gamma^{-\rho|\alpha|+\delta|\beta|} \leq C_{\alpha\beta}\gamma^{-2\tau|\alpha+\beta|}W\langle\xi\rangle_\gamma^{-(\rho-\tau)|\alpha|+(\delta+\tau)|\beta|}$$

where one can assume that $C_{\alpha\beta}(\epsilon)\gamma^{-2\tau|\alpha+\beta|}$ are arbitrarily small for $1 \leq |\alpha+\beta| \leq N$ taking γ large.

6.1. Symbol of P_ψ. Define $h_j(x,\xi)$ by

$$h_j(x,\xi) = \sum_{1\leq\ell_1<\ell_2<\cdots<\ell_j\leq m} |q_{\ell_1}|^2\cdots|q_{\ell_j}|^2, \quad q_j = \xi_0 - \lambda_j(x,\xi').$$

Lemma 6.1. *There exists $c > 0$ such that*

$$h_{m-k}(x,\xi - i\epsilon\omega^{-1}\langle\xi\rangle_\gamma^\kappa\theta) \geq c(\epsilon\omega)^{2(j-k)}\langle\xi\rangle_\gamma^{2(j-k)}h_{m-j}(x,\xi - i\epsilon\omega^{-1}\langle\xi\rangle_\gamma^\kappa\theta)$$

for $j = k,\ldots,m$ where $h_0 = 1$ and $1 \leq k \leq m$.

Proof. We show the case $k = 1$. By definition $h_{m-1}(x,\xi - i\epsilon\omega^{-1}\langle\xi\rangle_\gamma^\kappa\theta)$ is bounded from below by

$$2^{-1}(|q_i(x,\xi)|^2 + |q_j(x,\xi)|^2 + \epsilon^2\omega^{-2}\langle\xi\rangle_\gamma^{2\kappa})\prod_{k\neq i,j}|q_k(x,\xi - i\epsilon\omega^{-1}\langle\xi\rangle_\gamma^\kappa\theta)|^2.$$

From Lemma 3.2 we have $|q_i(x,\xi)|^2 + |q_j(x,\xi)|^2 \geq c\,|b'(x,\xi)|^2$. Since

$$|b'(x,\xi)|^2 + \epsilon^2\omega^{-2}\langle\xi\rangle_\gamma^{2\kappa}$$
$$= \epsilon^2\omega^{-2}\langle\xi\rangle_\gamma^2(\epsilon^{-2}|b'(x,\xi)|^2\omega^2\langle\xi\rangle_\gamma^{-2} + \langle\xi\rangle_\gamma^{-4\delta}) \geq c\,\epsilon^2\omega^2\langle\xi\rangle_\gamma^2$$

with some $c > 0$ because $C|b'(x,\xi)|^2\langle\xi\rangle_\gamma^{-2} \geq \phi^2$ and $\omega^2 \geq \phi^2$ then it is clear that $h_{m-1}(x,\xi - i\epsilon\omega^{-1}\langle\xi\rangle_\gamma^\kappa\theta)$ is bounded from below by

$$c\epsilon^2\omega^2\langle\xi\rangle_\gamma^2\prod_{k\neq i,j}|q_k(x,\xi - i\epsilon\omega^{-1}\langle\xi\rangle_\gamma^\kappa\theta)|^2.$$

Summing up over all pair i, j ($i \neq j$) we get the assertion for the case $j = 2$. Continuing this argument one can prove the case $j \geq 3$.   □

Let us put

$$h(x,\xi) = h_{m-1}(x,\xi - i\epsilon\omega^{-1}\langle\xi\rangle_\gamma^\kappa\theta)^{1/2}.$$

Lemma 6.2. *There exists $C > 0$ such that we have*

$$\begin{cases} |p_{(\beta)}^{(\alpha)}| \leq C(\epsilon\omega)^{1-|\alpha+\beta|}\langle\xi\rangle_\gamma^{1-|\alpha|}h, & 1 \leq |\alpha+\beta| \leq m, \\ |p\,p_{(\beta)}^{(\alpha)}| \leq C(\epsilon\omega)^{2-|\alpha+\beta|}\langle\xi\rangle_\gamma^{2-|\alpha|}h^2, & 2 \leq |\alpha+\beta| \leq m. \end{cases}$$

Proof. From [4, Proposition 3] one has

$$|p_{(\beta)}^{(\alpha)}(x,\xi)| \leq Ch_{m-|\alpha+\beta|}(x,\xi)^{1/2}|\xi|^{|\beta|}$$

for $|\alpha+\beta| \leq m$ which is bounded by $Ch_{m-|\alpha+\beta|}(x,\xi - i\epsilon\omega^{-1}\langle\xi\rangle_\gamma^\kappa\theta)^{1/2}|\xi|^{|\beta|}$ clearly. On the other hand it follows from Lemma 6.1

$$Ch(x,\xi) \geq (\epsilon\omega)^{|\alpha+\beta|-1}\langle\xi\rangle_\gamma^{|\alpha+\beta|-1}h_{m-|\alpha+\beta|}(x,\xi - i\epsilon\omega^{-1}\langle\xi\rangle_\gamma^\kappa\theta)^{1/2}$$

for $1 \leq |\alpha+\beta| \leq m$ which proves the assertion. The proof of the second assertion is similar.   □

Lemma 6.3. *Assume that $c_\beta^\alpha \in S_{\rho,\delta}(\langle\xi\rangle_\gamma^{\rho|\beta|-\delta|\alpha|})$ and $c_\beta^\alpha \in S_{\rho,\delta}(\omega^{-1}\langle\xi\rangle_\gamma^{\kappa-|\alpha|})$ for $|\alpha+\beta| = 1$. Then for $1 \leq |\alpha+\beta| \leq m$ we have $p_{(\beta)}^{(\alpha)}c_\alpha^\beta \in \tilde{S}_{\rho,\delta}(\omega^{-1}\langle\xi\rangle_\gamma^\kappa h)$ and $\epsilon^{|\alpha+\beta|}|p_{(\beta)}^{(\alpha)}c_\alpha^\beta| \leq C\epsilon\omega^{-1}\langle\xi\rangle_\gamma^\kappa h$ with $C > 0$ independent of ϵ.*

Proof. Let $2 \leq |\alpha+\beta| \leq m$ then since $1 = \kappa + 2\delta$ by (4.4) and (4.6) we see by Lemma 6.2

$$\epsilon^{|\alpha+\beta|}|p_{(\beta)}^{(\alpha)}c_\alpha^\beta| \leq C\epsilon\omega^{1-|\alpha+\beta|}\langle\xi\rangle_\gamma^{1-|\alpha|+\rho|\alpha|-\delta|\beta|}h$$
$$\leq C\epsilon\omega^{-1}\langle\xi\rangle_\gamma^\kappa(\omega^{-1}\langle\xi\rangle_\gamma^{-\delta})^{|\alpha+\beta|-2}h \leq C\epsilon\omega^{-1}\langle\xi\rangle_\gamma^\kappa h.$$

When $|\alpha+\beta| = 1$ noting $c_\alpha^\beta \in S_{\rho,\delta}(\omega^{-1}\langle\xi\rangle_\gamma^{\kappa-|\beta|})$ we get the same assertion. We next estimate $\sum p_{(\beta+\nu')}^{(\alpha+\mu')}(c_\alpha^\beta)_{(\nu'')}^{(\mu'')}$. If $|\alpha+\mu'+\beta+\nu'| \geq m$ we have

$$\left| p_{(\beta+\nu')}^{(\alpha+\mu')} \omega_{\alpha(\nu'')}^{\beta(\mu'')} \right| \leq \langle \xi \rangle_\gamma^{m-|\alpha+\mu'|} \langle \xi \rangle_\gamma^{\rho|\alpha|-\delta|\beta|} \langle \xi \rangle_\gamma^{-\rho|\mu''|+\delta|\nu''|}$$

$$\leq C_\epsilon \omega^{-(m-1)} h \langle \xi \rangle_\gamma^{1-|\alpha+\mu'|} \langle \xi \rangle_\gamma^{\rho|\alpha|-\delta|\beta|+\rho|\mu'|-\delta|\nu'|} \langle \xi \rangle_\gamma^{-\rho|\mu|+\delta|\nu|}$$

$$\leq C_\epsilon \omega^{-1} \langle \xi \rangle_\gamma^\kappa (\omega^{-(m-2)} \langle \xi \rangle_\gamma^{-\delta(|\alpha+\mu'+\beta+\nu'|-2)}) \langle \xi \rangle_\gamma^{-\rho|\mu|+\delta|\nu|} h$$

where the right-hand side is bounded by $C_\epsilon \omega^{-1} \langle \xi \rangle_\gamma^{\kappa-\rho|\mu|+\delta|\nu|} h$. We turn to the case $|\alpha + \mu' + \beta + \nu'| \leq m$. From Lemma 6.2 it follows

$$|p_{(\beta+\nu')}^{(\alpha+\mu')}| \leq C_\epsilon \omega^{1-|\alpha+\mu'+\beta+\nu'|} \langle \xi \rangle_\gamma^{1-|\alpha+\mu'|} h$$

$$\leq C_\epsilon (\omega^{-1} \langle \xi \rangle_\gamma^{-\delta})^{|\mu'+\nu'|} \omega^{1-|\alpha+\beta|} \langle \xi \rangle_\gamma^{1-|\alpha|} h \langle \xi \rangle_\gamma^{-\rho|\mu'|+\delta|\nu'|}$$

therefore for $|\alpha + \beta| \geq 2$ we see easily

$$\left| p_{(\beta+\nu')}^{(\alpha+\mu')} c_{\alpha(\nu'')}^{\beta(\mu'')} \right| \leq C_\epsilon (\omega^{-1} \langle \xi \rangle_\gamma^{-\delta})^{|\alpha+\beta|-2} \omega^{-1} \langle \xi \rangle_\gamma^\kappa h \langle \xi \rangle_\gamma^{-\rho|\mu|+\delta|\nu|}$$

$$\leq C_\epsilon \omega^{-1} \langle \xi \rangle_\gamma^\kappa h \langle \xi \rangle_\gamma^{-\rho|\mu|+\delta|\nu|}$$

which also holds for $|\alpha + \beta| = 1$ because $c_\alpha^\beta \in S_{\rho,\delta}(\omega^{-1} \langle \xi \rangle_\gamma^{\kappa-|\beta|})$. $\qquad \square$

6.2. Definition of $Q(z)$ which separates $P_\psi(z)$. We follow the arguments in [13]. Let us define $\tilde{p}(x + iy, \xi + i\eta)$ by

$$\tilde{p}(x + iy, \xi + i\eta) = \sum_{|\alpha+\beta| \leq m} \frac{1}{\alpha! \beta!} \partial_x^\alpha \partial_\xi^\beta p(x,\xi)(iy)^\alpha (i\eta)^\beta.$$

Then p_ψ given in (5.2) is expressed as $\tilde{p}(z - i\epsilon H_\psi)$, which one can also write as

$$\tilde{p}(z - i\epsilon H_\psi) = \sum_{j=0}^m \left(i\frac{\partial}{\partial t} \right)^j p(z - \epsilon t H_\psi)/j! \Big|_{t=0}.$$

Using this expression we define $Q(z)$ which separates $\tilde{p}(z - i\epsilon H_\psi)$ by

$$Q(z) = \epsilon^{-1} |\tilde{H}_\psi|^{-1} \left(\frac{\partial}{\partial t} \right) \sum_{j=0}^m \left(i\frac{\partial}{\partial t} \right)^j p(z - \epsilon t H_\psi)/j! \Big|_{t=0}$$

where $\tilde{H}_\psi = (\langle \xi \rangle_\gamma \nabla_\xi \psi, -\nabla_x \psi)$. By the homogeneity it is clear that

$$\tilde{p}(z - i\epsilon H_\psi) = \langle \xi \rangle_\gamma^m \tilde{p}(\tilde{z} - i\lambda(z)\tilde{H}_\psi/|\tilde{H}_\psi|), \quad \lambda(x,\xi) = \epsilon |\tilde{H}_\psi| \langle \xi \rangle_\gamma^{-1}$$

where $\tilde{z} = (x, \xi \langle \xi \rangle_\gamma^{-1})$. It is not difficult to check $\tilde{p}(z - i\epsilon H_\psi) \in S_{\rho,\delta}(\langle \xi \rangle_\gamma^m)$ and $Q \in S_{\rho,\delta}(\langle \xi \rangle_\gamma^{m-1})$. We study $\tilde{p}(z - i\epsilon H_\psi)$ and $Q(z)$ in a conic neighborhood of ρ. We first recall

Proposition 6.1 ([14, Lemma 5.8]). *Let ρ be a characteristic of p of order m and let $K \subset \Gamma_\rho$ be a compact set. Then one can find a conic neighborhood V of ρ and positive $C > 0$ such that for any $(x,\xi) \in V$, $\zeta \in K$ and small $s \in \mathbb{R}$ one can write*

$$p(z - s\zeta) = e(z, \zeta, s) \prod_{j=1}^m (s - \mu_j(z, \zeta))$$

where $\mu_j(z, \zeta)$ are real valued and $e(z, \zeta, s) \neq 0$ for $(z, \zeta, s) \in V \times K \times \{|s| < s_0\}$. Moreover there exists $C > 0$ such that we have

(6.1) $|\mu_j(z,(0,\theta))|/C \le |\mu_j(z,\zeta)| \le C|\mu_j(z,(0,\theta))|, \quad j = 1,2,\ldots,m$

for any $(x,\xi) \in V$, $\zeta \in K$.

Writing $\prod_{j=1}^m (t - \mu_j) = \sum_{\ell=0}^m p_\ell t^\ell$ we see that $\tilde{p}(z - is\zeta)$ is written

$$\sum_{j=0}^m \frac{1}{j!}\left(is\frac{\partial}{\partial t}\right)^j \left(e \prod_{j=1}^m (t - \mu_j)\right)\bigg|_{t=0} = \sum_{\ell=0}^m \sum_{k=0}^{m-\ell} \frac{1}{k!}\left(is\frac{\partial}{\partial t}\right)^k e\bigg|_{t=0} p_\ell(is)^\ell$$

which is equal to

$$\sum_{\ell=0}^m \left(\sum_{k=0}^m \frac{1}{k!}\left(is\frac{\partial}{\partial t}\right)^k e\bigg|_{t=0} - \sum_{m-\ell+1 \le k \le m} \frac{1}{k!}\left(is\frac{\partial}{\partial t}\right)^k e\bigg|_{t=0}\right) p_\ell(is)^\ell$$

$$= \sum_{k=0}^m \frac{1}{k!}\left(is\frac{\partial}{\partial t}\right)^k e\bigg|_{t=0} \prod_{j=1}^m (is - \mu_j) + O(s^{m+1})$$

which proves

(6.2) $\tilde{p}(z - is\zeta) = e_0(z,\zeta,s) \prod_{k=1}^m (is - \mu_j(z,\zeta)) + O(s^{m+1}).$

Note that $e_0(z,\zeta,s) = \sum_{k=0}^m (is\partial/\partial t)^k e(z,\zeta,t)/k!\big|_{t=0}$ and hence we have $e_0(z,\zeta,0) = e(z,\zeta,0) \ne 0$.

Lemma 6.4. *There exist a conic neighborhood U of ρ and a compact convex set $K \subset \Gamma_\rho$ such that $\tilde{H}_\psi/|\tilde{H}_\psi| \in K$ for $(x,\xi) \in U$, $\gamma \ge \gamma_0$.*

Proof. From $\psi = \langle \xi \rangle_\gamma^\kappa \log(\phi + \omega)$ it is easy to see

$$\begin{cases} \nabla_x \psi = \omega^{-1}\langle \xi \rangle_\gamma^\kappa \nabla_x \phi, \\ \nabla_\xi \psi = \omega^{-1}\langle \xi \rangle_\gamma^\kappa \nabla_\xi \phi + O(\langle \xi \rangle_\gamma^{\kappa-1}) \log(\phi + \omega) + O(\langle \xi \rangle_\gamma^{\kappa-1}). \end{cases}$$

Therefore one has

$$\tilde{H}_\psi = \omega^{-1}\langle \xi \rangle_\gamma^\kappa (\tilde{H}_\phi + (O(1)\omega \log(\phi + \omega), 0)).$$

Since $|\phi + \omega| \le 2\omega$ we can assume $\omega \log(\phi + \omega)$ is enough small taking U small. In particular we have $\omega^{-1}\langle \xi \rangle_\gamma^\kappa/C \le |\tilde{H}_\psi| \le C\omega^{-1}\langle \xi \rangle_\gamma^\kappa$. Then noting $\tilde{H}_\phi(\rho)/|\tilde{H}_\phi(\rho)| \in \Gamma_\rho$ which follows from (4.3) we get the assertion.   □

We rewrite $Q(z)$ according to (6.2).

Lemma 6.5. *Let $\tilde{\omega} = \tilde{H}_\psi/|\tilde{H}_\psi|$. Then we have*

$$Q(z) = \langle \xi \rangle_\gamma^{m-1}\Big\{-i\partial e_0(\tilde{z},\tilde{\omega},\lambda)/\partial\lambda \prod_{j=1}^m (i\lambda - \mu_j(\tilde{z},\tilde{\omega}))$$

$$+ e_0(\tilde{z},\tilde{\omega},\lambda) \sum_{j=1}^m \prod_{k=1,k\ne j}^m (i\lambda - \mu_k(\tilde{z},\tilde{\omega})) + O(\lambda^m)\Big\}.$$

Proof. Noting $\lambda(z)|\tilde{H}_\psi|^{-1}\langle \xi \rangle_\gamma = \epsilon$ one can write

$$Q(z) = \epsilon^{-1}|\tilde{H}_\psi|^{-1}\langle\xi\rangle_\gamma^m \sum_{j=0}^m \left(\frac{\partial}{\partial t}\right)\left(i\frac{\partial}{\partial t}\right)^j p(\tilde{z} - t\lambda(z)\tilde{\omega})/j!\Big|_{t=0}$$

$$= \langle\xi\rangle_\gamma^{m-1} \sum_{j=0}^m \lambda(z)^j \left(\frac{\partial}{\partial t}\right)\left(i\frac{\partial}{\partial t}\right)^j p(\tilde{z} - t\tilde{\omega})/j!\Big|_{t=0}$$

which is equal to

$$\langle\xi\rangle_\gamma^{m-1}\frac{\partial}{\partial s}\frac{1}{i}\sum_{j=0}^m \frac{1}{(j+1)!}\left(is\frac{\partial}{\partial t}\right)^{j+1} p(\tilde{z} - t\tilde{\omega})\Big|_{t=0,s=\lambda(z)}$$

$$(6.3) \qquad = \frac{1}{i}\langle\xi\rangle_\gamma^{m-1}\frac{\partial}{\partial s}\{\tilde{p}(\tilde{z} - is\tilde{\omega}) - p(\tilde{z}) + O(s^{m+1})\}_{s=\lambda(z)}$$

$$= \frac{1}{i}\langle\xi\rangle_\gamma^{m-1}\left\{\frac{\partial}{\partial s}\tilde{p}(\tilde{z} - is\tilde{\omega})\Big|_{s=\lambda(z)} + O(\lambda^m)\right\}.$$

From (6.2) the right-hand side of (6.3) turns to be

$$(6.4) \qquad \frac{1}{i}\langle\xi\rangle_\gamma^{m-1}\left\{\frac{\partial}{\partial s}(e_0(\tilde{z},\tilde{\omega},s)\prod_{j=1}^m(is - \mu_j(\tilde{z},\tilde{\omega})) + O(s^{m+1}))\Big|_{s=\lambda}\right\}$$

modulo $O(\lambda^m)\langle\xi\rangle_\gamma^{m-1}$ which proves the assertion. $\qquad\square$

6.3. Microlocal energy estimates. To derive microlocal energy estimates we study $\mathrm{Im}\,(P_\psi\chi u, Q\chi u)$ where χ is a cutoff symbol supported in a conic neighborhood of ρ. Thus we are led to consider $\mathrm{Im}\,(P_\psi\bar{Q})$ in a conic neighborhood of ρ. Recall that $P_\psi = p_\psi + \sum_{j=0}^{m-1}(P_j)_\psi$ where $P_j(x,D)$ is the homogeneous part of degree j of P and $(P_j)_\psi \in S_{\rho,\delta}(\langle\xi\rangle_\gamma^j)$ by Theorem 5.1. Take any small $0 < \epsilon^* \ll 1$ and we fix ϵ^* and put

$$\delta = (1 - \epsilon^*)/m, \quad \rho = (m - 1 + \epsilon^*)/m, \quad \kappa = \rho - \delta$$

where $\rho + \delta = 1$.

Lemma 6.6. *Let $S_0(z) = \mathrm{Im}(\tilde{p}(z - i\epsilon H_\psi)\overline{Q(z)})$. Then one can find a conic neighborhood V of ρ and $C > 0$ such that we have*

$$\epsilon\omega^{-1}\langle\xi\rangle_\gamma^\kappa h^2(z)/C \le S_0(z) \le C\epsilon\omega^{-1}\langle\xi\rangle_\gamma^\kappa h^2(z).$$

Proof. Write $e_0(\tilde{z},\tilde{\omega},\lambda) = e(\tilde{z},\tilde{\omega},0) + i\lambda(\partial e/\partial\lambda)(\tilde{z},\tilde{\omega},0) + O(\lambda^2)$ then it is clear $|e_0|^2 = |e(\tilde{z},\tilde{\omega},0)|^2 + O(\lambda^2)$ so that $\mathrm{Re}\,(\partial\bar{e}_0/\partial\lambda)e_0 = 2^{-1}\partial|e_0|^2/\partial\lambda = O(\lambda)$. Thus from Lemma 6.5 and (6.2) it follows that

$$\mathrm{Im}(\tilde{p}(z - i\epsilon H_\psi)\overline{Q(z)}) = \langle\xi\rangle_\gamma^{2m-1}|e_0|^2\lambda\left\{\sum_{j=1}^m \prod_{k=1,k\ne j}^m (\lambda^2 + \mu_k^2)\left(1 + O\left(\lambda + \sum_{j=1}^m |\mu_j|\right)\right)\right\}.$$

Since $\mu_j(\rho,\tilde{\omega}) = 0$, $j = 1, 2, \ldots, m$ one obtains

$$C^{-1} \le \left\{\langle\xi\rangle_\gamma^{2m-1}\lambda\sum_{j=1}^m \prod_{k=1,k\ne j}^m (\lambda^2 + \mu_k(\tilde{z},\tilde{\omega})^2)\right\}\Big/ S_0(z) \le C.$$

On the other hand noting $C^{-1} \le \lambda(z)/(\epsilon\omega^{-1}\langle\xi\rangle_\gamma^{\kappa-1}) \le C$ and

$$C^{-1} \le \left\{ \langle\xi\rangle_\gamma^{2m-2} \sum_{j=1}^{m} \prod_{k=1,k\ne j}^{m} (\epsilon^2\omega^{-2}\langle\xi\rangle_\gamma^{2\kappa-2} + \mu_k(\tilde{z},(0,\theta))^2) \right\}$$
$$\times h_{m-1}(x,\xi - i\epsilon\omega^{-1}\langle\xi\rangle_\gamma^\kappa\theta)^{-1} \le C$$

we conclude the assertion from Lemma 6.4 and Proposition 6.1. □

Lemma 6.7. *We have $Q \in \tilde{S}_{\rho,\delta}(h)$ and $S_0^{\pm 1} \in \tilde{S}_{\rho,\delta}((\omega^{-1}\langle\xi\rangle_\gamma^\kappa h^2)^{\pm 1})$ in U for $\gamma \ge \gamma_0(\epsilon)$. Moreover $|Q| \le Ch$ with $C > 0$ independent of ϵ.*

Proof. From the definition one can see easily that Q is a sum of terms, up to constant factor;

(6.5) $$\epsilon^{|\alpha+\beta|-1} p_{(\beta)}^{(\alpha)}(z)(\nabla_\xi\psi)^\beta(\nabla_x\psi)^\alpha/(\langle\xi\rangle_\gamma^2|\nabla_\xi\psi|^2 + |\nabla_x\psi|^2)^{1/2}$$

with $1 \le |\alpha + \beta| \le m + 1$. We also note that $\mathrm{Im}\,Q$ is a sum of such terms (6.5) with $2 \le |\alpha + \beta| \le m + 1$. From Lemma 4.3 it follows

$$\begin{cases} \nabla_\xi\psi \in S_{\rho,\delta}(\omega^{-1}\langle\xi\rangle_\gamma^{\kappa-1}), \\ \nabla_x\psi \in S_{\rho,\delta}(\omega^{-1}\langle\xi\rangle_\gamma^\kappa) \end{cases}$$

in U and hence $(\langle\xi\rangle_\gamma^2|\nabla_\xi\psi|^2 + |\nabla_x\psi|^2)^{-1/2} \in S_{\rho,\delta}((\omega^{-1}\langle\xi\rangle_\gamma^\kappa)^{-1})$ then we have

$$V_\alpha^\beta = (\nabla_\xi\psi)^\beta(\nabla_x\psi)^\alpha(\langle\xi\rangle_\gamma^2|\nabla_\xi\psi|^2 + |\nabla_x\psi|^2)^{-1/2} \in S_{\rho,\delta}(\omega\langle\xi\rangle_\gamma^{-\kappa}\langle\xi\rangle_\gamma^{\rho|\alpha|-\delta|\beta|}).$$

Noting $\omega^{-2}\langle\xi\rangle_\gamma^{\kappa-1} \le 1$ it suffices to repeat the proof of Lemma 6.3 to conclude $p_{(\beta)}^{(\alpha)}V_\alpha^\beta \in \tilde{S}_{\rho,\delta}(h)$ and $\epsilon^{|\alpha+\beta|-1}|p_{(\beta)}^{(\alpha)}V_\alpha^\beta| \le Ch$ with C independent of ϵ for $1 \le |\alpha + \beta| \le m + 1$. From Lemma 6.3 it follows that $\tilde{p}(z - i\epsilon H_\psi) - p(z) \in \tilde{S}_{\rho,\delta}(\omega^{-1}\langle\xi\rangle_\gamma^\kappa h)$. Since from Lemma 6.2 one can check that $p(z)\mathrm{Im}\,Q \in \tilde{S}_{\rho,\delta}(\omega^{-1}\langle\xi\rangle_\gamma^\kappa h^2)$ we get the assertion for $S_0(z)$. The assertion for $S_0^{-1}(z)$ follows from Lemma 6.6 and $S_0(z)S_0^{-1}(z) = 1$. □

From Theorem 5.1 and Lemma 6.3 one can write

$$p_\psi - \tilde{p}(z - i\epsilon H_\psi) = \sqrt{\epsilon}\,r + r_0$$

where $r \in \tilde{S}_{\rho,\delta}(\omega^{-1}\langle\xi\rangle_\gamma^\kappa h)$ with $|r| \le C\epsilon\omega^{-1}\langle\xi\rangle_\gamma^\kappa h$ and $r_0 \in S_{\rho,\delta}(\langle\xi\rangle_\gamma^{m-\delta(m+1)})$. Thus $r\#\bar{Q} \in \tilde{S}_{\rho,\delta}(\omega^{-1}\langle\xi\rangle_\gamma^\kappa h^2)$ and $|r\bar{Q}| \le C\epsilon\omega^{-1}\langle\xi\rangle_\gamma^\kappa h^2$. On the other hand Lemma 6.1 shows

$$\langle\xi\rangle_\gamma^{m-\delta(m+1)} = \langle\xi\rangle_\gamma^{m-1+\kappa-\delta(m-1)} \le C\epsilon^{1-m}\gamma^{-\delta}\omega^{-1}\langle\xi\rangle_\gamma^\kappa h$$

so that we see $r_0 \in S_{\rho,\delta}(\omega^{-1}\langle\xi\rangle_\gamma^\kappa h)$ and $|r_0| \le C\epsilon^{3/2}\omega^{-1}\langle\xi\rangle_\gamma^\kappa h$ for $\gamma \ge \gamma_0(\epsilon)$. Therefore in virtue of Lemma 6.7 there is $\tilde{r} \in \tilde{S}_{\rho,\delta}(1)$ with $|\tilde{r}| \le C\sqrt{\epsilon}$ such that

$$\mathrm{Im}\,(p_\psi\#\bar{Q}) = S_0(1 - \tilde{r})$$

in some conic neighborhood of ρ.

We turn to $R = \sum_{j=0}^{m-1}(P_j)_\psi \in S_{\rho,\delta}(\langle\xi\rangle_\gamma^{m-1})$. From Lemma 6.1 again we have

$$\langle\xi\rangle_\gamma^{m-1} \le C\epsilon^{1-m}\omega^{-(m-1)}h = C\epsilon^{1-m}\omega^{-1}\langle\xi\rangle_\gamma^\kappa h(\omega^{-(m-2)}\langle\xi\rangle_\gamma^{-\kappa})$$
$$\le C\epsilon^{1-m}\gamma^{-\kappa+\delta(m-2)}\omega^{-1}\langle\xi\rangle_\gamma^\kappa h.$$

Recalling that $\kappa - \delta(m - 2) = \epsilon^* > 0$ and hence we can assume $C\epsilon^{1-m}\gamma^{-\epsilon^*} \le C\epsilon^{3/2}$ for

$\gamma \geq \gamma_0(\epsilon)$ so that there exists $\hat{r} \in \tilde{S}_{\rho,\delta}(1)$ with $|\hat{r}| \leq C\sqrt{\epsilon}$ such that $\text{Im}(R\#\bar{Q}) = S_0(1 - \hat{r})$ in a conic neighborhood of ρ. Thus we conclude

$$\begin{cases} \text{Im}(P_\psi\#\bar{Q}) = E^2, \quad E \in \tilde{S}_{\rho,\delta}(\omega^{-1/2}\langle\xi\rangle_\gamma^{\kappa/2}h), \\ \epsilon^{1/2}\omega^{-1/2}\langle\xi\rangle_\gamma^{\kappa/2}h/C \leq |E| \leq C\epsilon^{1/2}\omega^{-1/2}\langle\xi\rangle_\gamma^{\kappa/2}h \end{cases}$$

in a conic neighborhood of ρ with C independent of ϵ. The rest of the proof of deriving microlocal energy estimates is just a repetition of the arguments in [13] and we conclude that the Cauchy problem for $p + P_{m-1} + \cdots$ is $\gamma^{(1/\kappa)}$ well-posed at the origin. Note that $1/\kappa = m/(m - 2 + 2\epsilon^*)$ and $\epsilon^* > 0$ is arbitrarily small so that $1/\kappa$ is as close to $m/(m - 2)$ as we please.

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