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DECOMPOSITION OF COMPLEX HYPERBOLIC ISOMETRIES BY TWO COMPLEX SYMMETRIES

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Abstract

Let $\mathbf{PU}(2,1)$ denote the holomorphic isometry group of the 2-dimensional complex hyperbolic space $\mathbf{H}^2_{\mathbb{C}}$, and the group $\mathbf{SU}(2,1)$ is a 3-fold covering of $\mathbf{PU}(2,1)$: $\mathbf{PU}(2,1) = \mathbf{SU}(2,1)/\{\omega I : \omega^3 = 1\}$. We study how to decompose a given pair of isometries $(A, B) \in \mathbf{SU}(2,1)^2$ under the form $A = I_1 I_2$ and $B = I_3 I_2$, where the I_k's are complex symmetries about complex lines. If (A, B) can be written as above, we call it is $\mathbb{C}$-decomposable. The main results are decomposability criteria, which improve and supplement the result of [17].

1. Introduction

Let $\mathbf{H}^2_{\mathbb{C}}$ denote the 2-dimensional complex hyperbolic space, and $\mathrm{Iso}(\mathbf{H}^2_{\mathbb{C}})$ denote the full isometry group which consists of holomorphic, as well as anti-holomorphic isometries. The projective unitary group $\mathbf{PU}(2,1) = \mathbf{SU}(2,1)/\{\omega I : \omega^3 = 1\}$ which is an index 2 subgroup of $\mathrm{Iso}(\mathbf{H}^2_{\mathbb{C}})$ denotes the holomorphic isometry group of $\mathbf{H}^2_{\mathbb{C}}$. There are two types of totally geodesic 2-dimensional submanifolds in $\mathbf{H}^2_{\mathbb{C}}$: complex lines and the $\mathbb{R}$-planes. These correspond to two kinds of isometric involutions of $\mathbf{H}^2_{\mathbb{C}}$. A complex line $C \subset \mathbf{H}^2_{\mathbb{C}}$ is fixed by a unique involutive holomorphic isometry. We call this isometry the complex symmetry about C, which is represented by an element $I_C \in \mathbf{SU}(2,1)$ that is given by

$$I_C(z) = -z + 2\frac{\langle z, \mathbf{c}\rangle}{\langle \mathbf{c}, \mathbf{c}\rangle}\mathbf{c}, \tag{1.1}$$

where $\mathbf{c}$ is a polar vector of C. Any $\mathbb{R}$-plane P is fixed pointwise by a unique anti-holomorphic isometry of order 2: the Lagrangian reflection about P. There is another involution in $\mathrm{Iso}(\mathbf{H}^2_{\mathbb{C}})$: the complex reflection about a point in $\mathbf{H}^2_{\mathbb{C}}$.

An element T in G is called reversible if T is conjugate to T^{-1}. Furthermore, if T is a product of two involutions, it is called strongly reversible. Reversible elements and strongly reversible elements have been extensive studied in several contexts (see [2], [3], [10], [11], [12], [15], [18]). In particular, when $G = \mathrm{Iso}(\mathbf{H}^2_{\mathbb{C}})$ there are three kinds involutive elements as mentioned above. In [4], Falbel and Zocca proved that every element in $\mathbf{PU}(2,1)$ is strongly reversible in $\mathrm{Iso}(\mathbf{H}^2_{\mathbb{C}})$, since it can be expressed as a product of two Lagrangian reflections. Gongopadhyay and Parker [9] classified reversible and strongly reversible element in $\mathbf{PU}(2,1)$ and shown that $T \in \mathbf{SU}(2,1)$ is reversible if and only if it is strongly reversible.

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For simplicity of presentation, we call an element $A \in G$ $\mathbb{C}$-strongly reversible, if A is a product of two complex symmetries about complex lines. The $\mathbb{C}$-strong reversibility of loxodromic elements has been considered in [17] (Theorem 1). In this paper, we give the $\mathbb{C}$-strong reversibility criteria for parabolic and elliptic elements (Theorem 2 and Theorem 3).

Theorem 1 (Proposition 4 of [17])**.** *Let A be a loxodromic element of* $\mathbf{PU}(2,1)$*. A is $\mathbb{C}$-strongly reversible if and only if A admits a lift to* $\mathbf{SU}(2,1)$ *with real trace greater than* 3.

Theorem 2. *Let A be a parabolic element of* $\mathbf{SU}(2,1)$*. Then A is $\mathbb{C}$-strongly reversible if and only if A is a* 3*-step unipotent parabolic. In other words, A is $\mathbb{C}$-strongly reversible if and only if A is strongly reversible.*

Theorem 3. *Let A be an elliptic element of* $\mathbf{SU}(2,1)$*. A is $\mathbb{C}$-strongly reversible if and only if A is strongly reversible and A is not a complex symmetry.*

A pair of elements $(A, B) \in \mathbf{SU}(2,1)^2$ or $\mathbf{PU}(2,1)^2$ is said to be $\mathbb{C}$-decomposable (resp. $\mathbb{R}$-decomposable) if there exist three complex symmetries (resp. three Lagrangian reflections) I_1, I_2 and I_3 such that $A = I_1 I_2$ and $B = I_3 I_2$ holds. Note that when writing the two elements A and B as products of complex symmetries (or Lagrangian reflections), the order in which the involutions appear is not important. $\mathbb{C}$-decomposability (resp. $\mathbb{R}$-decomposability) is very closely related to triangle groups (groups generated by three involutions). In the setting of $\mathbf{H}^2_{\mathbb{C}}$, many of the examples known of discrete groups are related to triangle groups, see for instance [6] and [16]. It also turns out that since the group $\langle A, B\rangle$ has index two in $\Gamma = \langle I_1, I_2, I_3\rangle$, then $\langle A, B\rangle$ is discrete if and only if Γ is. This can lead to considerable simplification in the study of the discreteness of 2-generator subgroups of $\mathbf{PU}(2,1)$. For example, Gilman has presented a new sufficient condition for a subgroup of $\mathbf{PSL}(2,\mathbb{C})$ to be discrete by using this idea in [5]. For these reasons, we wish to decompose a pair of elements (A, B) of $\mathbf{SU}(2,1)^2$ or $\mathbf{PU}(2,1)^2$ such that $\langle A, B\rangle$ contained with index 2 in a triangle group.

Will [17] gave $\mathbb{C}$-decomposability criterion and $\mathbb{R}$-decomposability criterion for a pair of loxodromic isometries (A, B) of $\mathbf{H}^2_{\mathbb{C}}$, which are expressed in terms of traces of elements of the group $\langle A, B\rangle$. Since an element of $\mathbf{PU}(2,1)$ admits 3 lifts to $\mathbf{SU}(2,1)$, the trace of an isometry is well defined up to this indetermination. We will say that an isometry has real trace if and only if it admits a lift to $\mathbf{SU}(2,1)$ which has real trace.

Theorem 4 (Theorem 1 of [17])**.** *Let A and B be two loxodromic isometries of* $\mathbf{H}^2_{\mathbb{C}}$ *and $G = \langle A, B\rangle$. Assume that G does not preserve a totally geodesic subspace. Then*

(1). *The following two propositions are equivalent:*

(i) *The isometry $[A, B]$ has real trace.*

(ii) *The pair (A, B) is $\mathbb{R}$-decomposable.*

(2). *The following two propositions are equivalent:*

(i) *The isometries A, B, AB and $A^{-1}B$ all have real trace.*

(ii) *Either the pair (A, B) is $\mathbb{C}$-decomposable, or the pair (A^2, B^2) is $\mathbb{C}$-decomposable.*

In 2013, Paupert and Will [14] provided a criterion to determine whether any two given elements of $\mathbf{PU}(2,1)$ is $\mathbb{R}$-decomposable, which completed the $\mathbb{R}$-decomposability criterion of elements in $\mathbf{PU}(2,1)$.

Theorem 5 (Theorem 4.1 of [14])**.** *Let $A, B \in \mathbf{PU}(2,1)$ be two isometries not fixing a common point in $\overline{\mathbf{H}^2_{\mathbb{C}}}$. Then: the pair (A, B) is $\mathbb{R}$-decomposable if and only if the commutator $[A, B]$ has a fixed point in $\overline{\mathbf{H}^2_{\mathbb{C}}}$ whose associated eigenvalue is real and positive.*

It should be pointed out that a number of issues related to $\mathbb{C}$-decomposability are still unclear. For example, a $\mathbb{C}$-decomposability criterion for a pair of parabolic or elliptic elements has never been considered. In this paper, we are concerned with how to decompose a pair elements $(A, B) \in \mathbf{SU}(2,1)^2$ under the form $A = I_1I_2$ and $B = I_3I_2$, where I_k's are complex symmetries, and we investigate criteria to determine whether two given elements of $\mathbf{SU}(2,1)$ can be $\mathbb{C}$-decomposable. Moreover, we also obtain the necessary and sufficient condition of $\mathbb{C}$-decomposability when one is a loxodromic element and the other one is a parabolic element. Our main results are the followings:

Theorem 6. *Let $A, B \in \mathbf{SU}(2,1)$ be two elements of the same type not fixing a common point in $\overline{\mathbf{H}^2_{\mathbb{C}}}$. Then, the pair (A, B) is $\mathbb{C}$-decomposable if and only if A, B are both $\mathbb{C}$-strongly reversible, and* $\mathrm{tr}(AB) \in \mathbb{R}$, $\mathrm{tr}(BA^{-1}) \in \mathbb{R}$.

Proposition 1.1. *If $A, B \in \mathbf{SU}(2,1)$ have a common fixed point in $\mathbf{H}^2_{\mathbb{C}}$, then (A, B) is $\mathbb{C}$-decomposable if and only if A, B are both $\mathbb{C}$-strongly reversible.*

Proposition 1.2. *Let $A, B \in \mathbf{SU}(2,1)$ have a common fixed point on $\partial\mathbf{H}^2_{\mathbb{C}}$.*

(i) *If A and B are both loxodromic elements, then (A, B) is $\mathbb{C}$-decomposable if and only if A, B are both $\mathbb{C}$-strongly reversible and* $\mathrm{fix}(A) = \mathrm{fix}(B)$.

(ii) *If A or B is a loxodromic element and the other one is a 3-step unipotent parabolic element, then (A, B) is not $\mathbb{C}$-decomposable.*

(iii) *If A and B are both 3-step unipotent parabolic elements, then (A, B) is $\mathbb{C}$-decomposable if and only if A, B don't commute or A, B have the same invariant fan.*

Theorem 7. *Let (A, B) be a pair of elements of $\mathbf{SU}(2,1)$, where A is a loxodromic element and B is a parabolic element. Then (A, B) is $\mathbb{C}$-decomposable if and only if A, B are both $\mathbb{C}$-strongly reversible,* $\mathrm{tr}(AB) \in \mathbb{R}$, $\mathrm{tr}(BA^{-1}) \in \mathbb{R}$, *and A, B have distinct fixed points.*

Our Theorem 6 contains the result of Will's Theorem 4 (2). Propositions 1.1 and 1.2 complement the conclusion of Theorem 4. Theorem 7 shows the $\mathbb{C}$-decomposability criterion for one element is loxodromic and the other one is parabolic, which hasn't been considered in [17].

This paper is organized as follows. We start with some geometric preliminaries is Section 2. The definition of invariant fan of a parabolic element in Proposition 1.2 is also in Section 2. The proofs of Theorems 2 and 3 will be given in Section 3. Finally the proofs of our main results are presented in Section 4.

2. Preliminaries

2.1. Complex hyperbolic space and isometries. We begin with some background material on complex hyperbolic geometry. Much of this is found in Goldman's book [7].

Let $\mathbb{C}^{2,1}$ be a complex vector space of dimension 3 with a Hermitian form of signature $(2, 1)$. Consider the subspaces

$$V_- = \{\mathbf{z} \in \mathbf{C}^{2,1} : \langle \mathbf{z}, \mathbf{z} \rangle < 0\},$$

$$V_0 = \{\mathbf{z} \in \mathbf{C}^{2,1} : \langle \mathbf{z}, \mathbf{z} \rangle = 0\},$$
$$V_+ = \{\mathbf{z} \in \mathbf{C}^{2,1} : \langle \mathbf{z}, \mathbf{z} \rangle > 0\}.$$

where $\mathbf{z}$ is the column vector $\begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^T$. Let $\mathbf{P} : \mathbb{C}^{2,1} \setminus \{0\} \longrightarrow \mathbb{C}\mathbf{P}^2$ be the canonical projection onto complex projective space. The complex hyperbolic space is defined to be $\mathbf{H}^2_{\mathbb{C}} = \mathbf{P}(V_-)$, and $\partial\mathbf{H}^2_{\mathbb{C}} = \mathbf{P}(V_0)$ is its boundary.

For the projective model the metric on $\mathbf{H}^2_{\mathbb{C}}$, called the Bergman metric is given by the distance function $\rho(\cdot,\cdot)$ defined by the formula

$$\cosh^2\left(\frac{\rho(\mathbf{P}(\mathbf{z}),\mathbf{P}(\mathbf{w}))}{2}\right) = \frac{\langle \mathbf{z}, \mathbf{w} \rangle\langle \mathbf{w}, \mathbf{z} \rangle}{\langle \mathbf{z}, \mathbf{z} \rangle\langle \mathbf{w}, \mathbf{w} \rangle}. \tag{2.2}$$

There are two standard models of $\mathbf{H}^2_{\mathbb{C}}$. The first one is called the ball model of $\mathbf{H}^2_{\mathbb{C}}$, when the Hermitian form is given by $\langle \mathbf{z}, \mathbf{z} \rangle = -|z_1|^2+|z_2|^2+|z_3|^2$. The second one is called the Siegel domain model of $\mathbf{H}^2_{\mathbb{C}}$, when the Hermitian form is given by $\langle \mathbf{z}, \mathbf{z} \rangle = z_1\overline{w}_3+z_2\overline{w}_2+z_3\overline{w}_1$. From (2.2) it is easy to show that the projective unitary group $\mathbf{PU}(2,1)$ acts by isometries on $\mathbf{H}^2_{\mathbb{C}}$, which we identify with the holomorphic isometry group of $\mathbf{H}^2_{\mathbb{C}}$. The group $\mathbf{SU}(2,1)$ is a 3-fold covering of $\mathbf{PU}(2,1)$:

$$\mathbf{PU}(2,1) = \mathbf{SU}(2,1)/\{I, \omega I, \omega^2 I\},$$

where $\omega = (-1+\sqrt{3}i)/2$ is a cube root of unity.

The familiar trichotomy from real hyperbolic geometry applies in the complex hyperbolic setting as well: $A \in \mathbf{PU}(2,1)$ is said to be:

- loxodromic if it fixes exactly two points of $\partial\mathbf{H}^2_{\mathbb{C}}$;
- parabolic if it fixes exactly one point of $\partial\mathbf{H}^2_{\mathbb{C}}$;
- elliptic if it fixes at least one point of $\mathbf{H}^2_{\mathbb{C}}$.

It is clear that a fixed point of an isometry A lying in $\mathbf{H}^2_{\mathbb{C}}$ or its boundary corresponds to an eigenvector of the corresponding matrix lying in V_- or V_0 respectively. So we have the following theorem.

Theorem 8 ([13])**.** *Let A be a matrix in* $\mathbf{SU}(2,1)$*. Then one of the following possibilities occurs:*

(i) *A has two null eigenvectors with eigenvalues λ and $\overline{\lambda}^{-1}$ where* $|\lambda| \neq 1$*, in which case A is loxodromic;*

(ii) *A has a repeated eigenvalue of unit modulus whose eigenspace is spanned by a null vector, in which case A is parabolic;*

(iii) *A has a negative eigenvector, in which case A is elliptic.*

An eigenvalue λ of $A \in \mathbf{SU}(2,1)$ is said to be of negative type, positive type or null if every eigenvector of λ is in V_-, V_+ or V_0 respectively. The eigenvalue λ is said to be of indefinite type if there are some eigenvectors of λ in V_- and some in V_+.

A parabolic element in $\mathbf{SU}(2,1)$ is called unipotent if it is a unipotent matrix. Unipotent parabolic elements are either 2-step or 3-step, according to whether the minimal polynomial of the matrix is $(x-1)^2$ or $(x-1)^3$. If a parabolic element is not unipotent, we call it screw-parabolic. It can be decomposed as $A = PE = EP$, where P is a unipotent parabolic element and E is an elliptic element.

An elliptic element in $\mathbf{SU}(2,1)$ is called regular if it has three distinct eigenvalues. A non-regular elliptic element is called special. Special elliptic elements have two kinds: An elliptic element is a complex reflection about complex line if it has 2 equal eigenvalues, and one of which has eigenvectors in V_-; An elliptic element is a complex reflection in a point if it has 2 equal eigenvalues, and the remaining one has eigenvectors in V_-. These reflections may not have order 2, and not even finite order.

Also, we can use the trace of $A \in \mathbf{SU}(2,1)$ to decide whether it is elliptic, parabolic or loxodromic.

Lemma 9 ([7]). *Let f be the polynomial $f(z) = |z|^4 - 8\Re(z^3) + 18|z|^2 - 27$, where $z \in \mathbb{C}$. Denote by C_3 is the set of cube roots of unity in $\mathbb{C}$. Let $A \in \mathbf{SU}(2,1)$. Then:*

(1) *A is regular elliptic $\Leftrightarrow f(\mathrm{tr}(A)) < 0$;*
(2) *A is loxodromic $\Leftrightarrow f(\mathrm{tr}(A)) > 0$;*
(3) *A is screw parabolic or special elliptic $\Leftrightarrow f(\mathrm{tr}(A)) = 0$ and $\mathrm{tr}(A) \notin 3C_3$;*
(4) *A is unipotent or the identity $\Leftrightarrow \mathrm{tr}(A) \in 3C_3$.*

2.2. The ball model of $\mathbf{H}^2_{\mathbb{C}}$. The ball model of $\mathbf{H}^2_{\mathbb{C}}$ arises from the choice of Hermitian form

$$H = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.3}$$

The vector $\mathbf{z} = \begin{bmatrix} 1 & z_1 & z_2 \end{bmatrix}^T$ is the standard lift of $z \in \mathbf{H}^2_{\mathbb{C}}$ to V_-. Furthermore, we see that $z \in \mathbf{H}^2_{\mathbb{C}}$ provided

$$\langle \mathbf{z}, \mathbf{z} \rangle = -1 + |z_1|^2 + |z_2|^2 < 0.$$

It is obviously that any elliptic element of $\mathbf{H}^2_{\mathbb{C}}$ is conjugate to one given by the diagonal matrix

$$E_{(\alpha,\beta)} = \begin{bmatrix} e^{-i(\alpha+\beta)/3} & 0 & 0 \\ 0 & e^{i(2\alpha-\beta)/3} & 0 \\ 0 & 0 & e^{i(2\beta-\alpha)/3} \end{bmatrix}. \tag{2.4}$$

Projectively, the associated isometry is given by

$$(z_1, z_2) \mapsto (e^{i\alpha} z_1, e^{i\beta} z_2).$$

Sometimes it is more convenient to work with the lift to $\mathbf{U}(2,1)$ given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha} & 0 \\ 0 & 0 & e^{i\beta} \end{bmatrix}. \tag{2.5}$$

2.3. The Siegel domain model of $\mathbf{H}^2_{\mathbb{C}}$. The Siegel domain model of complex hyperbolic space $\mathbf{H}^2_{\mathbb{C}}$ corresponds to the Hermitian form given by the matrix :

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The Siegel domain model of $\mathbf{H}^2_{\mathbb{C}}$ with horospherical coordinates is

$$\mathbf{H}^2_{\mathbb{C}} = \{(z,t,u) : z \in \mathbb{C}, t \in \mathbb{R}, u \in \mathbb{R}_+\}.$$

The boundary of the Siegel domain is

$$\partial\mathbf{H}^2_{\mathbb{C}} = \{(z,t,0) : z \in \mathbb{C}, t \in \mathbb{R}\} \cup \{\infty\}.$$

Points in $\mathbf{H}^2_{\mathbb{C}}$ may be identified with negative vectors in $\mathbb{C}^{2,1}$ and points of $\partial\mathbf{H}^2_{\mathbb{C}}$ may be identified with null vectors in $\mathbb{C}^{2,1}$ by the map $\psi : \overline{\mathbf{H}^2_{\mathbb{C}}} \to \mathbb{C}^{2,1}$ given by

$$\psi : (z,t,u) \mapsto \begin{bmatrix} (-|z|^2 - u + it)/2 \\ z \\ 1 \end{bmatrix}, \quad \psi : \infty \mapsto \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{2.6}$$

The boundary $\partial\mathbf{H}^2_{\mathbb{C}}\backslash\{\infty\}$ is a copy of the Heisenberg group $\mathfrak{N}$ of dimension 3, with group law given in (z,t) coordinates by:

$$(z_1,t_1) * (z_2,t_2) = (z_1 + z_2, t_1 + t_2 + 2\Im(z_1\overline{z}_2)).$$

We conclude this subsection by considering the subgroup of $\mathbf{PU}(2,1)$ stabilising the point at infinity. Such maps will be called Heisenberg similarities. The corresponding elements in $\mathbf{SU}(2,1)$ are generated by the following 3 types: Heisenberg translations $T_{(z,t)}$ ($(z,t) \in \mathbb{C}\times\mathbb{R}$), Heisenberg rotations R_θ ($\theta \in \mathbb{R}/2\pi\mathbb{Z}$) and Heisenberg dilations D_r ($r > 1$), where:

$$T_{(z,t)} = \begin{bmatrix} 1 & -\overline{z} & -(|z|^2 - it)/2 \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}, \; R_\theta = \begin{bmatrix} e^{-i\theta/3} & 0 & 0 \\ 0 & e^{2i\theta/3} & 0 \\ 0 & 0 & e^{-i\theta/3} \end{bmatrix}, \; D_r = \begin{bmatrix} r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/r \end{bmatrix}. \tag{2.7}$$

2.4. The invariant fan of a 3-step unipotent parabolic. The standard reference for the invariant fan of a 3-unipotent parabolic element is [8] (see also Section 2.3 of [14]).

For any $z \in \mathbb{C}$ there exists a unique complex line which contains ∞ and the point $(z,0)$. This induces a projection $\widetilde{\Pi} : \overline{\mathbf{H}^2_{\mathbb{C}}} \setminus \{\infty\} \longmapsto \mathbb{C}$ whose fibers are the complex lines through ∞. In restriction to the boundary, this projection is just the vertical projection $\Pi : (z,t) \longmapsto (z,0)$, which is given in Heisenberg coordinates.

A fan through ∞ is the preimage of any affine line in $\mathbb{C}$ under the projection $\widetilde{\Pi}$. A general fan is the image of a fan through ∞ by an element of $\mathbf{PU}(2,1)$. As stated in [8], fans enjoy a double foliation, by $\mathbb{R}$-planes and complex lines. In [14], the authors make the foliation of fan explicit. See the following:

Lemma 10 (Lemma 2.2 of [14])**.** *Let $L_{w,k}$ be the affine line in $\mathbb{C}$ parameterized by $L_{w,k} = \{w(s+ik), s \in \mathbb{R}\}$, for some unit modulus w and $k \geq 0$. Then the boundary foliation of the fan above $L_{w,k}$ is given by the lines parameterized in Heisenberg coordinates by $L_{t_0} =$*

$\{(w(s + ik), t_0 + 2sk), s \in \mathbb{R}\}$.

If T is a 3-step unipotent parabolic element of $\mathbf{PU}(2,1)$, there exists a unique fan F_T through the fixed point of T such that it is stable under T and every leaf of the foliation of F_T by real planes is stable under T. We call this fan F_T the invariant fan of T.

Remark 2.1. When $T = T_{(z,t)}$ with $z \neq 0$, the fan F_T is the one above the affine line $L_{w,k}$, where $w = z/|z|$ and $k = t/(4|z|)$.

For future reference, let us state the following proposition.

Proposition 2.1 (Lemma 2.3 of [14]). *Let $T_{(z_1,t_1)}$ and $T_{(z_2,t_2)}$ be two 3-step unipotent parabolic elements. Then these two translations commute if and only if $\bar{z}_1 z_2 \in \mathbb{R}$, which is equivalent to saying that their invariant fans are parallel.*

2.5. $\mathbb{C}$-strong reversibility and $\mathbb{C}$-decomposability. In [9], the authors described necessary and sufficient conditions of reversibility or strong reversibility of $A \in \mathbf{SU}(2,1)$, which is written in terms of trace and eigenvalue of A. Since reversibility is equivalent to strong reversibility for $A \in \mathbf{SU}(2,1)$ (see Theorem 4.2 of [9]), we have the following theorem:

Theorem 11 (Corollary 4.10 of [9]). *Let A be an element in $\mathbf{SU}(2,1)$.*

(1) *A is a loxodromic element. A is strongly reversible in $\mathbf{SU}(2,1)$ if and only if $\operatorname{tr}(A) \in \mathbb{R}$.*

(2) *$A = PE$ is a parabolic element. A is strongly reversible in $\mathbf{SU}(2,1)$ if and only if the trace of A is real, the null eigenvalue of A is 1 or -1 and the minimum polynomial of P is $(x-1)^3$.*

(3) *A is an elliptic element. A is strongly reversible in $\mathbf{SU}(2,1)$ if and only if the trace of A is real and the eigenvalue of negative type or indefinite type of A is 1 or -1.*

We define $A \in \mathbf{SU}(2,1)$ is $\mathbb{C}$-strongly reversible, if $A = I_1 I_2$. A pair of elements $(A, B) \in \mathbf{SU}(2,1)^2$, if $A = I_1 I_2$ and $B = I_3 I_2$, we call (A, B) is $\mathbb{C}$-decomposable. The above I_1, I_2, I_3 are both elements of $\mathbf{SU}(2,1)$, which represent three complex symmetries about complex lines as (1.1). It is apparent that if A is $\mathbb{C}$-strongly reversible, then A is strongly reversible. Generally speaking, the converse implication is not true.

Lemma 12. *Let $A \in \mathbf{SU}(2,1)$ be $\mathbb{C}$-strongly reversible, then A has real trace.*

Proof. If A is $\mathbb{C}$-strongly reversible, then it may be written as $A = I_1 I_2$, where I_1, I_2 are two matrices in $\mathbf{SU}(2,1)$ corresponding two complex symmetries. Hence $A^{-1} = I_2 I_1 = (I_1)^{-1} A (I_1) = (I_2) A (I_2)^{-1}$. In particular, A is conjugate to A^{-1}, so they have the same trace. Since in $\mathbf{SU}(2,1)$ we have $\operatorname{tr}(A^{-1}) = \overline{\operatorname{tr}(A)}$, we see $\operatorname{tr}(A) = \overline{\operatorname{tr}(A)}$, so $\operatorname{tr}(A)$ is real. □

The following proposition will be needed in the Section 4.

Proposition 2.2 (Proposition 4 of [17]). *$A \in \mathbf{SU}(2,1)$ is a loxodromic element, if I_1 and I_2 are two complex symmetries such that $A = I_1 I_2$, both I_1 and I_2 permute the fixed points of A.*

If $(A, B) \in \mathbf{SU}(2,1)^2$ is $\mathbb{C}$-decomposable, that is $A = I_1 I_2$ and $B = I_3 I_2$, where $I_1, I_2, I_3 \in \mathbf{SU}(2,1)$ given by (1.1) which represent three complex symmetries. It follows that $AB = I_1(I_2 I_3 I_2)$ and $BA^{-1} = I_3 I_1$ are both $\mathbb{C}$-strongly reversible. According to Lemma 12, we obtain the following proposition:

Proposition 2.3. *If* $(A, B) \in \mathbf{SU}(2,1)^2$ *is* $\mathbb{C}$*-decomposable. Then* A, B, AB *and* BA^{-1} *all have real trace.*

From the above we know that elements of $\mathbf{SU}(2,1)$ with real trace are very important for us.

Proposition 2.4 (Proposition 2.3 of [14])**.** *Let* $A \in \mathbf{SU}(2,1)$ *satisfy* $\mathrm{tr}(A) \in \mathbb{R}$. *Then* A *has an eigenvalue equal to* 1. *More precisely:*
- *If* A *is loxodromic then* A *has eigenvalues* $\{1, r, 1/r\}$ *for some* $r > 1$ *or* $r < -1$.
- *If* A *is elliptic then* A *has eigenvalues* $\{1, e^{i\theta}, e^{-i\theta}\}$ *for some* $\theta \in (0, \pi]$.
- *If* A *is parabolic then* A *has eigenvalues* $\{1, 1, 1\}$ *or* $\{1, -1, -1\}$.

The main purpose of this paper is to discuss the $\mathbb{C}$-strong reversibility and $\mathbb{C}$-decomposability of elements in $\mathbf{SU}(2,1)$. It is simple to show that the $\mathbb{C}$-strong reversibility for one element and the $\mathbb{C}$-decomposability for a pair elements of $\mathbf{SU}(2,1)$ are both invariant under conjugation, which make things a little easier.

3. $\mathbb{C}$-strong reversibility

In this section, we study the $\mathbb{C}$-strong reversibility of parabolic and elliptic elements. We have known the results about strong reversibility of elements of $\mathbf{SU}(2,1)$ from Theorem 11, then to investigate $\mathbb{C}$-strong reversibility one needs to rule out the case where at least one of I_1 and I_2 fixes a point.

Lemma 13. (1) *Suppose that* $A = I_1 I_2$ *where* I_1 *and* I_2 *are complex involutions in* $\mathbf{SU}(2,1)$ *with unique fixed points* p_1 *and* p_2 *respectively. Then*

$$\mathrm{tr}(A) = 2\cosh\left(\rho(p_1, p_2)\right) + 1.$$

In particular, if A *is not the identity map then* $\mathrm{tr}(A) > 3$, *so* A *is hyperbolic.*

(2) *Suppose that* $A = I_1 I_2$ *where* I_1 *and* I_2 *are complex involutions in* $\mathbf{SU}(2,1)$, I_1 *has a unique fixed points* p_1 *and* I_2 *fixes the complex line* L_2. *Then*

$$\mathrm{tr}(A) = -2\cosh\left(\rho(p_1, L_2)\right) + 1.$$

In particular, $\mathrm{tr}(A) \leq -1$. *If* $p_1 \notin L_2$ *then* $\mathrm{tr}(A) \leq -1$ *and* A *is hyperbolic. If* $p_1 \in L_2$ *then* A *is a complex symmetry fixing a complex line through* p_1 *orthogonal to* L_2.

The above result is easy to verify, so the proof is omitted.

3.1. $\mathbb{C}$-strong reversibility of parabolic elements. Owing to Proposition 2.4, in the Siegel domain model of $\mathbf{H}^2_{\mathbb{C}}$, any parabolic element of $\mathbf{SU}(2,1)$ which has real trace is conjugate in $\mathbf{SU}(2,1)$ to exactly one of the following:

- If it is 3-step unipotent parabolic: $\begin{bmatrix} 1 & -1 & -1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$;
- If it is 2-step unipotent parabolic: $\begin{bmatrix} 1 & 0 & i/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

• If it is screw parabolic: $\begin{bmatrix} -1 & 0 & -i/2 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

From Theorem 11, any 2-step unipotent parabolic elements and screw parabolic elements are not $\mathbb{C}$-strongly reversible. Combined with Lemma 13, we can get the following result immediately.

Theorem 14. *Let A be a parabolic element of* $\mathbf{SU}(2,1)$*. Then A is $\mathbb{C}$-strongly reversible if and only if A is a 3-step unipotent parabolic. In other words, A is $\mathbb{C}$-strongly reversible if and only if A is strongly reversible.*

As stated above, if A is a 3-step unipotent parabolic, we can assume $A = T_{(1,0)} \in \mathbf{SU}(2,1)$, and the null eigenvalue of A is 1. We can decompose A as following:

$$A = T_{(1,0)} = \begin{bmatrix} -1 & -1 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{3.8}$$

The first (resp. second) matrix in the right hand side product corresponds to the complex symmetry about the complex line polar to $\begin{bmatrix} 1/2 & -1 & 0 \end{bmatrix}^T$ (resp. $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$).

More generally,

$$T_{(z,0)} = \begin{bmatrix} -1 & -\bar{z} & |z|^2/2 \\ 0 & 1 & -z \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{3.9}$$

where $z \neq 0$. The first (resp. second) matrix in the right hand side product corresponds to the complex symmetry about the complex line polar to $\begin{bmatrix} \bar{z}/2 & -1 & 0 \end{bmatrix}^T$ (resp. $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$).

Proposition 3.1. *Let $A \in \mathbf{SU}(2,1)$ be a 3-step unipotent parabolic element fixing $p \in \partial\mathbf{H}^2_{\mathbb{C}}$, and $A = I_1 I_2$, where I_1 and I_2 are both complex symmetries. Then I_1, I_2 both fix the point p. Especially, the fixed lines of I_1 and I_2 lie in the invariant fan of A.*

Proof. Normalise $A = T_{(1,0)}$, suppose $I_2(\infty) = q \neq \infty$, where $q \in \partial\mathbf{H}^2_{\mathbb{C}}$. Since $A(\infty) = I_1 I_2(\infty) = \infty$, then $I_1(q) = \infty$. Because $I_1^2 = I_2^2 = \mathrm{Id}$, we get $A(q) = q$ which is a contradiction. Thus, I_2 fixes ∞. Similarly, I_1 also fixes ∞. Let L_1 and L_2 be two complex lines fixed pointwise by I_1 and I_2 respectively. Then L_1 and L_2 both through ∞, we can obtain

$$I_k = \begin{bmatrix} -1 & -2\bar{z}_k & 2|z_k|^2 \\ 0 & 1 & -2z_k \\ 0 & 0 & -1 \end{bmatrix},$$

where $z_k \in \mathbb{C}, k = 1, 2$. As $A = I_1 I_2$, we have $z_1, z_2 \in \mathbb{R}$. Thus, L_1 and L_2 both lie in the invariant fan of A. □

3.2. $\mathbb{C}$-strong reversibility of elliptic elements. In this subsection, we use the unit ball model of $\mathbf{H}^2_{\mathbb{C}}$ with the Hermitian form H in (2.3). Let A be an elliptic element with real trace. Combining Proposition 2.4 and Theorem 11, we know that if A is $\mathbb{C}$-strongly reversible, A may be conjugate in $\mathbf{SU}(2,1)$ to exactly one of the following:

- If it is regular elliptic: $E_{(\theta,-\theta)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{-i\theta} \end{bmatrix}$, where $\theta \in (0, \pi)$;
- If it is a complex reflection about a complex line (or boundary elliptic): $E_l = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;
- If it is a complex reflection in a point: $E_{(\pi,-\pi)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

From the above analysis and Lemma 13, we obtain the following theorem, which states the $\mathbb{C}$-strong reversibility of elliptic elements in $\mathbf{SU}(2,1)$.

Theorem 15. *Let A be an elliptic element of* $\mathbf{SU}(2,1)$. *A is $\mathbb{C}$-strongly reversible if and only if A is a regular elliptic or a complex reflection in a point which is conjugate to one given by the matrix $E_{(\theta,-\theta)}$ $(\theta \in (0, \pi])$. In other words, A is $\mathbb{C}$-strongly reversible if and only if A is strongly reversible and A is not a complex symmetry.*

We can decompose $E_{(\theta,-\theta)}$ as following:

$$E_{(\theta,-\theta)} = I_1 I_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & e^{i\theta} \\ 0 & e^{-i\theta} & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \theta \in (0, \pi],$$

where I_1 represents the complex symmetry about the complex line polar to $\begin{bmatrix} 0 & \sqrt{2}e^{i\theta}/2 & \sqrt{2}/2 \end{bmatrix}^T$, I_2 represents the complex symmetry about the complex line polar to $\begin{bmatrix} 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T$.

Remark 3.1. Any pair of elliptic elements which conjugates to $(E_{(\theta,-\theta)}, E_{(\alpha,-\alpha)})$ $(\theta, \alpha \in (0, \pi])$ is $\mathbb{C}$-decomposable.

We have known that if an elliptic element A is $\mathbb{C}$-strongly reversible, either it is a regular elliptic element which conjugates to $E_{(\theta,-\theta)}$ $(\theta \in (0, \pi))$, or it is a complex reflection of order 2 about a point in $\mathbf{H}^2_{\mathbb{C}}$. Then the unique fixed point of A is in $\mathbf{H}^2_{\mathbb{C}}$. The following proposition is well known.

Proposition 3.2. *Let $A \in$ $\mathbf{SU}(2,1)$ be an elliptic element fixing $p \in \mathbf{H}^2_{\mathbb{C}}$, and $A = I_1 I_2$, where I_1 and I_2 are both complex symmetries. Then I_1, I_2 both fix the point p.*

Let E be any $\mathbb{C}$-strongly reversible regular elliptic element fixing the point 0 (or it is a complex reflection in the point 0). As E is conjugate in $\mathbf{SU}(2,1)$ to $E_{(\theta,-\theta)}$ $(\theta \in (0, \pi])$, we can represent such E by:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta \pm i\sqrt{1-|b|^2}\sin\theta & i\bar{b}\sin\theta \\ 0 & ib\sin\theta & \cos\theta \mp i\sqrt{1-|b|^2}\sin\theta \end{bmatrix}, \tag{3.10}$$

where $\theta \in (0, \pi]$, $b \in \mathbb{C}$ and $0 \le |b| \le 1$.

4. $\mathbb{C}$-decomposability

4.1. Main results. In this section, we give the $\mathbb{C}$-decomposability of two elements of the same type of $\mathbf{SU}(2,1)$ and get the necessary and sufficient condition of $\mathbb{C}$-decomposability when one is a loxodromic element and the other one is a parabolic element. Recall that a pair of elements $(A,B) \in \mathbf{SU}(2,1)^2$ is said to be $\mathbb{C}$-decomposable if there exist three complex symmetries I_1, I_2, I_3 such that $A = I_1 I_2$ and $B = I_3 I_2$. Now we are ready to prove our main result.

Theorem 16. *Let $A, B \in \mathbf{SU}(2,1)$ be two elements of the same type not fixing a common point in $\overline{\mathbf{H}^2_{\mathbb{C}}}$. Then, the pair (A,B) is $\mathbb{C}$-decomposable if and only if A, B are both $\mathbb{C}$-strongly reversible, and* $\mathrm{tr}(AB) \in \mathbb{R}$, $\mathrm{tr}(BA^{-1}) \in \mathbb{R}$.

Proof. (1). Let (A,B) be a pair of loxodromic elements of $\mathbf{SU}(2,1)$ and A, B have distinct fixed points. From Theorems 1 and 4, we get the result.

(2). Let $(A,B) \in \mathbf{SU}(2,1)^2$ be a pair of parabolic elements and $\mathrm{fix}(A) \cap \mathrm{fix}(B) = \emptyset$.

($\Rightarrow$) Assume (A,B) is $\mathbb{C}$-decomposable, then A and B must be $\mathbb{C}$-strongly reversible and A, B are both 3-step unipotent parabolic by Theorem 14. Thus, $\mathrm{tr}(AB) \in \mathbb{R}$ and $\mathrm{tr}(BA^{-1}) \in \mathbb{R}$ by Proposition 2.3.

($\Leftarrow$) Now that A and B are both $\mathbb{C}$-strongly reversible, it follows that A, B are both 3-step unipotent parabolic. For simplicity, we may take $A = T_{(z,t)}$, B is a 3-step unipotent parabolic element fixing 0, where B has the form:

$$\begin{bmatrix} 1 & 0 & 0 \\ \zeta & 1 & 0 \\ \frac{-|\zeta|^2+iv}{2} & -\overline{\zeta} & 1 \end{bmatrix}, \ (\zeta, v) \in \{\mathbb{C} \setminus 0\} \times \mathbb{R}.$$

Notice that AB and BA^{-1} both have real trace, we get:

$$\mathrm{tr}(AB) = 3 - 2\mathfrak{R}(\overline{z}\zeta) + \frac{|z|^2|\zeta|^2 - tv}{4} - \frac{i(|z|^2 v + |\zeta|^2 t)}{4} \in \mathbb{R}$$

and

$$\mathrm{tr}(BA^{-1}) = 3 + 2\mathfrak{R}(\overline{z}\zeta) + \frac{|z|^2|\zeta|^2 + tv}{4} - \frac{i(|z|^2 v - |\zeta|^2 t)}{4} \in \mathbb{R},$$

Due to $z \neq 0$ and $\zeta \neq 0$, it follows that $t = v = 0$. Thus we derived that

$$B = \begin{bmatrix} 1 & 0 & 0 \\ \zeta & 1 & 0 \\ \frac{-|\zeta|^2}{2} & -\overline{\zeta} & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ \zeta & 1 & 0 \\ |\zeta|^2/2 & \overline{\zeta} & -1 \end{bmatrix}. \tag{4.11}$$

The first(resp. second) matrix in the right hand side product corresponds to the complex symmetry about the complex line polar to $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$ (resp. $\begin{bmatrix} 0 & 1 & \overline{\zeta}/2 \end{bmatrix}^T$).

Consequently, (A,B) is $\mathbb{C}$-decomposable from (3.9) and (4.11).

(3). Let A be a $\mathbb{C}$-strongly reversible elliptic element in $\mathbf{SU}(2,1)$ fixing the origin in the ball model. Then the origin corresponds to a 1-eigenvector of A, and so A has the following form:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & -\overline{\mu}_1 \\ 0 & \mu_1 & \overline{\lambda}_1 \end{bmatrix},$$

where $\lambda_1 + \overline{\lambda}_1 = 2\cos(\theta_1)$ for some $\theta_1 \in (0, 2\pi)$. Without loss of generality, the fixed point of B is $p = (\tanh(t), 0) \in \mathbb{B}^2$. A map in $\mathbf{SU}(2,1)$ sending the origin to p is

$$\begin{bmatrix} \cosh(t) & \sinh(t) & 0 \\ \sinh(t) & \cosh(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore we may suppose that B has the form

$$\begin{aligned} B &= \begin{bmatrix} \cosh(t) & \sinh(t) & 0 \\ \sinh(t) & \cosh(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda_2 & -\overline{\mu}_2 \\ 0 & \mu_2 & \overline{\lambda}_2 \end{bmatrix} \begin{bmatrix} \cosh(t) & -\sinh(t) & 0 \\ -\sinh(t) & \cosh(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1-(\lambda_2-1)\sinh^2(t) & (\lambda_2-1)\cosh(t)\sinh(t) & -\overline{\mu}_2\sinh(t) \\ -(\lambda_2-1)\cosh(t)\sinh(t) & \lambda_2+(\lambda_2-1)\sinh^2(t) & -\overline{\mu}_2\cosh(t) \\ -\mu_2\sinh(t) & \mu_2\cosh(t) & \overline{\lambda}_2 \end{bmatrix}. \end{aligned}$$

Note that if (A, B) is $\mathbb{C}$-decomposable as $A = I_1 I_2$ and $B = I_3 I_2$, then I_2 must fix the complex line passing through the fixed points of A and B from Proposition 3.2. In the case above,

$$I_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, (A, B) is $\mathbb{C}$-decomposable if and only if $\mathrm{tr}(AI_2) = \mathrm{tr}(BI_2) = -1$.

$$\mathrm{tr}(AI_2) = -1 + \overline{\lambda}_1 - \lambda_1, \quad \mathrm{tr}(BI_2) = -1 + \overline{\lambda}_2 - \lambda_2,$$

so (A, B) is $\mathbb{C}$-decomposable if and only if λ_1 and λ_2 are both real.

The necessity is obvious. Now if $\mathrm{tr}(AB)$ and $\mathrm{tr}(BA^{-1})$ are both real. A simple calculation shows that

$$\mathrm{tr}(AB) = 1 + \lambda_1\lambda_2 + \overline{\lambda}_1\overline{\lambda}_2 + (\lambda_1 - 1)(\lambda_2 - 1)\sinh^2(t) - (\overline{\mu}_1\mu_2 + \mu_1\overline{\mu}_2)\cosh(t),$$

$$\mathrm{tr}(BA^{-1}) = 1 + \overline{\lambda}_1\lambda_2 + \lambda_1\overline{\lambda}_2 + (\overline{\lambda}_1 - 1)(\lambda_2 - 1)\sinh^2(t) + (\overline{\mu}_1\mu_2 + \mu_1\overline{\mu}_2)\cosh(t).$$

Therefore

$$2i\Im(\mathrm{tr}(AB) + \mathrm{tr}(BA^{-1})) = (\lambda_1 + \overline{\lambda}_1 - 2)(\lambda_2 - \overline{\lambda}_2)\sinh^2(t),$$

$$2i\Im(\mathrm{tr}(AB) - \mathrm{tr}(BA^{-1})) = (\lambda_1 - \overline{\lambda}_1)(\lambda_2 + \overline{\lambda}_2 - 2)\sinh^2(t).$$

Since $\lambda_j + \overline{\lambda}_j = 2\cos(\theta_j) < 2$ $(j = 1, 2)$, we see that λ_1 and λ_2 must be real as required. Thus, (A, B) is $\mathbb{C}$-decomposable. □

4.2. Groups fixing a point. In this subsection, we think about the case when A and B have a common fixed point in $\overline{\mathbf{H}^2_{\mathbb{C}}}$.

Proposition 4.1. *If $A, B \in \mathbf{SU}(2,1)$ have a common fixed point in $\mathbf{H}^2_{\mathbb{C}}$, then (A, B) is $\mathbb{C}$-decomposable if and only if A, B are both $\mathbb{C}$-strongly reversible.*

Proof. Let $(A, B) \in \mathbf{SU}(2,1)^2$ be a pair of elliptic elements and have a common point $p \in \mathbf{H}^2_{\mathbb{C}}$. The necessity is trivial.

Now suppose A and B are both $\mathbb{C}$-strongly reversible, thus A and B are both regular elliptic elements, or both complex symmetries in a point, or one of them is regular elliptic and the other one is complex reflection in a point by Theorem 15.

(i). If A and B are both complex symmetries in a point p, then $A = B$ and (A, B) is $\mathbb{C}$-decomposable.

(ii). If A and B are both regular elliptic elements, because $A(p) = B(p) = p$, we may assume $A = E_{(\theta,-\theta)}$ $(\theta \in (0,\pi))$, and B has the form (3.10) with parameters α, b, where $\alpha \in (0,\pi), b \in \mathbb{C}$ and $0 \le |b| \le 1$.

When $b \neq 0$, we put $z_1 = -i\overline{b}e^{i\theta}$, $z_2 = i/\overline{b}$ and $z_3 = b(\pm\sqrt{1-|b|^2}\sin\alpha + i\cos\alpha)$.

Therefore,

$$A = I_1 I_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & z_1 \\ 0 & \overline{z}_1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & \overline{z}_2 \\ 0 & z_2 & 0 \end{bmatrix},$$

and

$$B = I_3 I_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\sqrt{1-|z_3|^2} & \overline{z}_3 \\ 0 & z_3 & \sqrt{1-|z_3|^2} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & \overline{z}_2 \\ 0 & z_2 & 0 \end{bmatrix}.$$

The polar vectors to the complex lines corresponding to I_k are $\mathbf{n}_k$, where

$$\mathbf{n}_1 = \begin{bmatrix} 0 \\ \frac{\overline{b}}{\sqrt{2}} \\ \frac{ie^{-i\theta}}{\sqrt{2}} \end{bmatrix}, \mathbf{n}_2 = \begin{bmatrix} 0 \\ \frac{-i}{\sqrt{2}} \\ \frac{1}{\overline{b}\sqrt{2}} \end{bmatrix}, \mathbf{n}_3 = \begin{bmatrix} 0 \\ \sqrt{\frac{1-|b|\sin\alpha}{2}} \\ \frac{b(\pm\sqrt{1-|b|^2}\sin\alpha + i\cos\alpha)}{\sqrt{2(1-|b|\sin\alpha)}} \end{bmatrix}.$$

It is clear that (A, B) is $\mathbb{C}$-decomposable.

When $b = 0$, it is apparent from Remark 3.1 that (A, B) is $\mathbb{C}$-decomposable.

(iii). If one is a regular elliptic element and the other one is a complex reflection in a point, without loss of generality, we suppose A is a regular elliptic and B is a complex reflection in a point. Since A, B have the same fix point in $\mathbf{H}^2_{\mathbb{C}}$, we set $A = E_{(\theta,-\theta)}$ $(\theta \in (0,\pi))$, $B = E_{(\pi,-\pi)}$. From Remark 3.1, then (A, B) is $\mathbb{C}$-decomposable. □

Proposition 4.2. *Let $A, B \in \mathbf{SU}(2,1)$ have a common fixed point on $\partial\mathbf{H}^2_{\mathbb{C}}$.*

(i) *If A and B are both loxodromic elements, then (A, B) is $\mathbb{C}$-decomposable if and only if A, B are both $\mathbb{C}$-strongly reversible and* $\mathrm{fix}(A) = \mathrm{fix}(B)$.

(ii) *If A or B is a loxodromic element and the other one is a 3-step unipotent parabolic element, then (A, B) is not $\mathbb{C}$-decomposable.*

(iii) *If A and B are both 3-step unipotent parabolic elements, then (A, B) is $\mathbb{C}$-decomposable if and only if A, B don't commute or A, B have the same invariant fan.*

Note that the 3 parts of Proposition 4.2 cover all cases where A and B have a common fixed point on $\partial\mathbf{H}^2_{\mathbb{C}}$, because an elliptic element which has a fixed point on $\partial\mathbf{H}^2_{\mathbb{C}}$ is not $\mathbb{C}$-strongly reversible and a parabolic element which is not 3-step unipotent is not $\mathbb{C}$-strongly reversible too.

We are now turning to the proof of Proposition 4.2.

Proof. (i) ($\Rightarrow$) A and B are both loxodromic elements and (A, B) is $\mathbb{C}$-decomposable. Then $A = I_1 I_2, B = I_3 I_2$, where $I_k (k = 1, 2, 3)$ is complex symmetry. Suppose A fixes the points p, q and B fixes the points p, q'. From Proposition 2.2, we get $I_2(p) = q = q'$. Thus $\mathrm{fix}(A) = \mathrm{fix}(B)$.

($\Leftarrow$) A, B are both $\mathbb{C}$-strongly reversible and $\mathrm{fix}(A) = \mathrm{fix}(B)$. Without loss of generality, we set the two fixed points are 0 and ∞. By Theorem 1, A, B are conjugate to

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\lambda \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1/\lambda \\ 0 & -1 & 0 \\ \lambda & 0 & 0 \end{bmatrix} \ (\lambda > 1).$$

Therefore (A, B) is $\mathbb{C}$-decomposable.

(ii) Assume that A is a loxodromic element and B is a 3-step unipotent parabolic element. The fixed points of A are p, q, the fixed point of B is p. If there exist three complex symmetries I_1, I_2, I_3 such that $A = I_1 I_2$ and $B = I_3 I_2$. Then from Proposition 2.2 and 3.1, we get $p = q$. This is a contradiction to $p \neq q$. So (A, B) is not $\mathbb{C}$-decomposable.

(iii) Let (A, B) be a pair of 3-step unipotent parabolic elements of $\mathbf{SU}(2, 1)$ and A, B have the same fixed point.

($\Rightarrow$) If (A, B) is $\mathbb{C}$-decomposable, we can assume $A = I_1 I_2$ and $B = I_3 I_2$. From Proposition 3.1, we know that I_2 must fix a complex line in the invariant fan of A and one in the invariant fan of B. Hence, these two fans must intersect in (at least) a complex line. Therefore they are either the same or non-parallel.

($\Leftarrow$) If A and B either do not commute or have the same invariant fan, there exists a complex line L contained in both of their invariant fans. Writing I_2 for the complex symmetry fixing L, it is easy to check AI_2 and BI_2 are both complex symmetries. Thus (A, B) is $\mathbb{C}$-decomposable.

This completes the proof of Proposition 4.2. □

4.3. The $\mathbb{C}$-decomposability of one loxodromic and one parabolic. In this subsection, we consider the case that A is a loxodromic element and B is a parabolic element. Now we prove the following theorem.

Theorem 17. *Let (A, B) be a pair of elements of $\mathbf{SU}(2, 1)$, where A is a loxodromic element and B is a parabolic element. Then (A, B) is $\mathbb{C}$-decomposable if and only if A, B are both $\mathbb{C}$-strongly reversible, $\mathrm{tr}(AB) \in \mathbb{R}$, $\mathrm{tr}(BA^{-1}) \in \mathbb{R}$, and A, B have distinct fixed points.*

Proof. ($\Rightarrow$) If the pair (A, B) is $\mathbb{C}$-decomposable, we can normalise the parabolic element $B = T_{(1,0)}$ and have the decomposition given in equation (3.8). Then

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

should conjugate A to its inverse. Hence A has the form

$$A = \begin{bmatrix} a & b & c \\ d & e & -\overline{b} \\ g & -\overline{d} & \overline{a} \end{bmatrix}$$

where $a, b, d \in \mathbb{C}$, $c, e, g \in \mathbb{R}$. This immediately implies

$$\operatorname{tr}(AB) = 2\Re(a) - 2\Re(d) + e - g/2, \quad \operatorname{tr}(B^{-1}A) = 2\Re(a) + 2\Re(d) + e - g/2$$

are real. Moreover, to show A and B have distinct fixed points, it suffices to show $g \neq 0$. If $g = 0$, since $2\Re(a)g + |d|^2 = 0$, then $d = 0$ and so $a^2 - bd + cg = 1$ implies $a^2 = 1$; $\det(A) = |a|^2 e = 1$ implies $e = 1$. Thus $\operatorname{tr}(A) = 3$ or -1, which contradicts the assumption A is loxodromic.

($\Leftarrow$) If A, B are both $\mathbb{C}$-strongly reversible and $\operatorname{fix}(A) \cap \operatorname{fix}(B) = \emptyset$, we may assume $A = D_r$ ($r > 1$) by Theorem 1. Without loss of generality, the fixed point of B is $q = (x, t, 0) \neq 0, \infty$ ($x, t \in \mathbb{R}$). The standard lift of q is $\mathbf{q} = \begin{bmatrix} (-x^2 + it)/2 & x & 1 \end{bmatrix}^T$. B is conjugate to $T_{(1,0)}$ by Theorem 14, then we can denote B by

$$\begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix}, \tag{4.12}$$

where $\lambda_{11} = 1 + \overline{f}g - (\frac{-x^2+it}{2})(d\overline{g} + \frac{1}{2}|g|^2)$, $\lambda_{12} = \overline{f}gx - (\frac{-x^2+it}{2})(e\overline{g} + \frac{1}{2}|g|^2 x)$, $\lambda_{13} = i\Im[(x^2+it)\overline{f}g] - \frac{x^4+t^2}{8}|g|^2$, $\lambda_{21} = \overline{e}g - \overline{g}dx - \frac{1}{2}|g|^2 x$, $\lambda_{22} = 1 - \frac{1}{2}|g|^2 x^2 + 2xi\Im(\overline{e}g)$, $\lambda_{23} = (\frac{-x^2-it}{2})(\overline{e}g - \frac{1}{2}|g|^2 x) - \overline{g}fx$, $\lambda_{31} = 2i\Im(\overline{d}g) - \frac{1}{2}|g|^2$, $\lambda_{32} = \overline{d}gx - e\overline{g} - \frac{1}{2}|g|^2 x$, $\lambda_{33} = 1 - f\overline{g} + (\frac{-x^2-it}{2})(\overline{d}g - \frac{1}{2}|g|^2)$, $d, e, f, g \in \mathbb{C}$, $g \neq 0$ and $2\Re(d\overline{f}) + |e|^2 = 1$, $e\overline{g}x = \frac{x^2-it}{2}d\overline{g} - f\overline{g}$.

As $\operatorname{tr}(AB) \in \mathbb{R}$ and $\operatorname{tr}(BA^{-1}) \in \mathbb{R}$, a simple manipulation yields

$$r\overline{f}g - r(\frac{-x^2+it}{2})(d\overline{g} + \frac{1}{2}|g|^2) + 2xi\Im(\overline{e}g) - \frac{1}{r}f\overline{g} - \frac{1}{r}(\frac{x^2+it}{2})(\overline{d}g - \frac{|g|^2}{2}) \in \mathbb{R}, \tag{4.13}$$

and

$$\frac{1}{r}\overline{f}g - \frac{1}{r}(\frac{-x^2+it}{2})(d\overline{g} + \frac{1}{2}|g|^2) + 2xi\Im(\overline{e}g) - rf\overline{g} - r(\frac{x^2+it}{2})(\overline{d}g - \frac{|g|^2}{2}) \in \mathbb{R}. \tag{4.14}$$

(4.13) minus (4.14), we assert $t = 0$, then $x \neq 0$. Substitute $t = 0$ into formula (4.13), we find

$$2xi\Im(\overline{e}g) + r\overline{f}g - \frac{f\overline{g}}{r} + \frac{x^2}{2}(rd\overline{g} - \frac{\overline{d}g}{r}) \in \mathbb{R},$$

then $\Im(\overline{e}g) = 0$.

Set L be a complex line spanned by 0 and ∞. Let L_2 be a complex line through q orthogonal to L, and I_2 is the complex symmetry fixing L_2. In the case above,

$$I_2 = \begin{bmatrix} 0 & 0 & \frac{x^2}{2} \\ 0 & -1 & 0 \\ \frac{2}{x^2} & 0 & 0 \end{bmatrix}.$$

By a simple calculation, we can derive that AI_2 and BI_2 are both complex symmetries. Therefore, we claim that (A, B) is $\mathbb{C}$-decomposable. $\square$

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References

[1] S. Chen and L. Greenberg: *Hyperbolic spaces*, in Contributions to Analysis. Academic Press, New York (1974), 49–87.
[2] D.Z. Djokovic: *The product of two involutions in the unitary group of a hermitian form*, Indiana Univ. Math. J. **21** (1971), 449–456.
[3] E.W. Ellers: *Cyclic decomposition of unitary spaces*, J. Geom. **21** (1983), 101–107.
[4] E. Falbel and V. Zocca: *A Poincaré's polyhedron theorem for complex hyperbolic geometry*, J. Reine Angew. Math. **516** (1999), 133–158.
[5] J. Gilman: *A discreteness condition for subgroups of* $\mathbf{PSL}(2, C)$, Contemp. Math. **211** (1997), 261–267.
[6] W.M. Goldman and J. Parker: *Complex hyperbolic ideal triangle groups*, Journal für die reine und angewandte Math. **425** (1992), 71–86.
[7] W.M. Goldman: Complex Hyperbolic Geometry, Oxford Mathematical Monographs, Oxford University Press (1999).
[8] W.M. Goldman and J. Parker: *Dirichlet Polyhedra for Dihedral Groups Acting on Complex Hyperbolic Space*, J. Geom. Anal. **91** (1999), 517–554.
[9] K. Gongopadhyay and J.R. Parker: *Reversible complex hyperbolic isometries*, Linear Algebra Appl. **438** (2013), 2728–2739.
[10] K. Gongopadhyay and C. Thomas: *Decomposition of complex hyperbolic isometries by involutions*, Linear Algebra Appl. **500** (2016), 63–76.
[11] F. Knüppel and K. Nielsen: *On products of two involutions in the orthogonal group of a vector space*, Linear Algebra Appl. **94** (1987), 209–216.
[12] C.P. Leo Jr: *Real elements in small small cancellation groups*, Math. Ann. **208** (1974), 279–293.
[13] J.R. Parker:*Notes on Complex Hyperbolic geometry*, Preliminary version, 2003.
[14] J. Paupert and P. Will: *Real reflections, commutators and cross-ratios in complex hyperbolic space*, Groups Geom. Dyn. **11** (2017), 311–352.
[15] P. H. Tiep and A. E. Zalesski: *Real conjugacy classes in algebraic groups and finite groups of Lie type*, J. Group Theory **8** (2005), 291–315.
[16] P. Will: *The punctured torus and Lagrangian triangle groups in* $\mathbf{PU}(2, 1)$, J. Reine Angew. Math. **602** (2007), 95–121.
[17] P. Will: *Traces, cross-ratios and 2-generator subgroups of* $\mathbf{SU}(2, 1)$, Canad. J. Math. **61** (2009), 1407–1436.
[18] M.J. Wonenburger: *Transformations which are products of two involutions*, J. Math. Mech. **16** (1966), 327–338.

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