



Title	ON THE MORSE-NOVIKOV NUMBER FOR 2-KNOTS
Author(s)	Endo, Hisaaki; Pajitnov, Andrei
Citation	Osaka Journal of Mathematics. 2017, 54(4), p. 723-734
Version Type	VoR
URL	<a href="https://doi.org/10.18910/67010">https://doi.org/10.18910/67010</a>
rights	
Note	

*The University of Osaka Institutional Knowledge Archive : OUKA*

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

# ON THE MORSE-NOVIKOV NUMBER FOR 2-KNOTS

HISAAKI ENDO and ANDREI PAJITNOV

(Received June 20, 2016, revised September 20, 2016)

## Abstract

Let  $K \subset S^4$  be a 2-knot. The Morse-Novikov number  $\mathcal{MN}(K)$  is the minimal possible number of critical points of a Morse map  $S^4 \setminus K \rightarrow S^1$  belonging to the canonical class in  $H^1(S^4 \setminus K)$ . We prove that for a classical knot  $K \subset S^3$  the Morse-Novikov number of the spun knot  $S(K)$  is  $\leq 2\mathcal{MN}(K)$ . This enables us to compute  $\mathcal{MN}(S(K))$  for every classical knot  $K$  with tunnel number 1.

## Contents

1. Introduction . . . . .	723
2. Lower bounds from Novikov homology . . . . .	725
3. Motion pictures and saddle numbers . . . . .	726
4. Circle-valued Morse maps for spun knots . . . . .	727
5. Morse-Novikov numbers for spun knots . . . . .	730
6. On fibering of high-dimensional spun knots . . . . .	732
7. Open questions . . . . .	733
References . . . . .	733

## 1. Introduction

**1.1. Overview of the article.** Let  $K \subset S^4$  be a 2-knot, that is, a  $C^\infty$  embedding of  $S^2$  into  $S^4$ . We say that  $K$  is fibred, if the complement  $C_K = S^4 \setminus K$  admits a fibration over  $S^1$ , which is standard nearby  $K$  (see Definition 1.1). In general a Morse map  $C_K \rightarrow S^1$  has critical points, the minimal number of these critical points will be called *the Morse-Novikov number of  $K$*  and denoted  $\mathcal{MN}(K)$ . The aim of this paper is to study this invariant of 2-knots and to compute it for several families of knots,

The Novikov homology provides lower bounds for the number  $\mathcal{MN}(K)$ , see Section 2. In Section 3 we introduce *the saddle number  $sd(K)$*  of a 2-knot  $K$ ; it is defined as a minimal possible number of critical points of index 1 of a generic projection of  $K$  to a line in  $\mathbb{R}^4$ . This number can be considered as a 2-dimensional analogue of the bridge number of a classical knot. We prove that  $\mathcal{MN}(K) \leq 2sd(K)$ . Using the results of Section 2 we deduce the following homological lower bound for the saddle number in terms of the Novikov torsion numbers:

$$\widehat{q}_1(C_K) + \widehat{q}_2(C_K) \leq sd(K).$$

In Sections 4 and 5 we study the Morse-Novikov numbers of spun knots. We prove in particular that for a classical knot  $K \subset S^3$  the Morse-Novikov number of the spun knot  $S(K)$  satisfies

$$\mathcal{MN}(S(K)) \leq 2\mathcal{MN}(K).$$

In [17] the second author proved that if  $K \subset S^3$  is a classical knot, then  $\mathcal{MN}(K) \leq 2t(K)$  where  $t(K)$  is the tunnel number of  $K$ . Using this inequality we prove that  $\mathcal{MN}(S(K)) \leq 4$  for any classical knot  $K$  of tunnel number 1 (Section 5).

The case of high-dimensional knots is different from that of 1- and 2-knots. We gathered some results on  $\mathcal{MN}(K)$  of knots of dimension  $\geq 6$  in Section 6. The classical spinning construction of E. Artin has natural generalizations to higher dimensions. Namely, for a knot  $K^n \subset S^{n+2}$  and a natural number  $p \geq 1$  the  $p$ -spun knot  $S_p(K^n) \subset S^{n+p+2}$  is defined (see [1], Section III, and [3]). We prove that for a given knot  $K^n$ ,  $n \geq 4$  and two numbers  $p, q \geq 1$  of the same parity the fibredness of  $S_p(K^n)$  is equivalent to the fibredness of  $S_q(K^n)$ .

**1.2. Definition and first properties of the Morse-Novikov numbers for 2-knots.** Let  $K \subset S^4$  be a 2-knot. Choose a  $C^\infty$  trivialisation

$$\Phi : N(K) \rightarrow K \times B^2(0, \epsilon)$$

of a tubular neighbourhood  $N(K)$ . We denote the complement to  $K$  by  $C_K$ . This space is non-compact, and to develop the Morse theory on it, we will assume that the functions and vector field have standard behaviour nearby  $K$ . For the Morse maps and their gradients we use the terminology of [18].

**DEFINITION 1.1.** A Morse map  $f : C_K \rightarrow S^1$  is said to be *regular* if there is a  $C^\infty$  trivialisation

$$(1) \quad \Phi : N(K) \rightarrow K \times B^2(0, \epsilon)$$

of a tubular neighbourhood of  $K$  such that the restriction  $f|(N(K) \setminus K)$  satisfies  $f \circ \Phi^{-1}(x, z) = z/|z|$ .

An  $f$ -gradient  $v$  of a regular Morse map  $f : C_K \rightarrow S^1$  will be called *regular* if there is a  $C^\infty$  trivialisation (1) such that  $\Phi^*(v)$  equals  $(0, v_0)$  where  $v_0$  is the Riemannian gradient of the function  $z \mapsto z/|z|$ .

A pair  $(f, v)$ , where  $f$  is a regular Morse map and  $v$  is its regular gradient will be called a *Morse pair*.

If  $f$  is a Morse map of a manifold to  $\mathbb{R}$  or to  $S^1$ , then we denote by  $m_p(f)$  the number of critical points of  $f$  of index  $p$ . The number of all critical points of  $f$  is denoted by  $m(f)$ .

**DEFINITION 1.2.** The minimal number  $m(f)$  where  $f : C_K \rightarrow S^1$  is a regular Morse map is called *the Morse-Novikov number of  $K$*  and denoted by  $\mathcal{MN}(K)$ .

A regular Morse map  $f : C_K \rightarrow S^1$  is called *minimal* if the number  $m(f)$  is minimal on the class of all regular Morse functions.

The following proposition allows to get rid of local maxima and minima for regular func-

tions. Its proof repeats the proof of Lemma 3.2 in [19] and will be omitted.

**Proposition 1.3.** *Let  $f : C_K \rightarrow S^1$  be a regular Morse map. Then there is a regular Morse map  $g : C_K \rightarrow S^1$  such that  $m_i(g) \leq m_i(f)$  for every  $i$  and  $m_0(g) = m_4(g) = 0$ .*

The next Corollary is immediate.

**Corollary 1.4.** *There is a minimal Morse function  $f : C_K \rightarrow S^1$  without critical points of indices 0 and 4.*

**DEFINITION 1.5.** A Morse map  $f : C_K \rightarrow S^1$  is called strongly minimal if for every  $i, 0 \leq i \leq 4$  the number  $m_i(f)$  is minimal in the class of all regular functions  $C_K \rightarrow S^1$ .

**REMARK 1.6.** For all the 2-knots where we are able to compute the Morse-Novikov number, the strongly minimal Morse functions exist (see Section 5). However it is not clear whether strongly minimal functions exist for every  $K$ . The case of 2-knots is different here from the case of classical knots, for which the concept of the strongly minimal Morse map is the same as that of minimal map (by [19]).

**REMARK 1.7.** The main object of study in the present paper are 2-knots. However the definitions above generalize immediately to the knots of any dimension, and we will use corresponding terminology throughout the paper.

**2. Lower bounds from Novikov homology**

Let  $L = \mathbb{Z}[t^\pm]$ ; denote by  $\widehat{L} = \mathbb{Z}((t))$  and  $\widehat{L}_\mathbb{Q} = \mathbb{Q}((t))$  the rings of all series in one variable  $t$  with integer (respectively rational) coefficients and finite negative part. Recall that  $\widehat{L}$  is a PID, and  $\widehat{L}_\mathbb{Q}$  is a field.

Consider the infinite cyclic covering  $\overline{C}_K \rightarrow C_K$ ; the Novikov homology of  $C_K$  is defined as follows:

$$\widehat{H}_*(C_K) = H_*(\overline{C}_K) \otimes_{\widehat{L}} \widehat{L}.$$

The rank and torsion number of the  $\widehat{L}$ -module  $\widehat{H}_*(C_K)$  will be denoted by  $\widehat{b}_k(C_K)$ , respectively  $\widehat{q}_k(C_K)$ . For any regular Morse function  $f : C_K \rightarrow S^1$  there is a Novikov complex  $\mathcal{N}_*(f)$  over  $\widehat{L}$  generated in degree  $k$  by critical points of  $f$  of index  $k$  and such that  $H_*(\mathcal{N}_*(f)) \approx \widehat{H}_*(C_K)$ . Therefore we have the Novikov inequalities

$$\sum_k (\widehat{b}_k(C_K) + \widehat{q}_k(C_K) + \widehat{q}_{k-1}(C_K)) \leq \mathcal{MN}(K).$$

The numbers  $\widehat{b}_k(C_K)$ ,  $\widehat{q}_k(C_K)$  satisfy certain relations. The homology of  $C_K$  is the same as that of  $S^1$ , therefore (by [14]) the  $\mathbb{Q}$ -vector space  $H_*(\overline{C}_K, \mathbb{Q})$  has finite dimension. This implies  $H_*(\overline{C}_K) \otimes_{\widehat{L}} \widehat{L}_\mathbb{Q} = 0$ , and  $\widehat{b}_i(C_K) = 0$  for all  $i$ . It is clear that  $\widehat{q}_0 = \widehat{q}_4 = 0$ . Furthermore, since there is always a regular Morse map without critical points of index 4, the Novikov homology has no torsion in degree 3, therefore  $\widehat{q}_3 = 0$ . Thus the Novikov inequalities boil down to the following:

(2) 
$$2(\widehat{q}_1(C_K) + \widehat{q}_2(C_K)) \leq \mathcal{MN}(K).$$

Observe that both  $\widehat{q}_1, \widehat{q}_2$  can be non-zero, as the example of the spun knot of the  $5_2$ -knot shows (see Remark 5.6).

In the sequel we will use the universal Novikov complex as well. Denote by  $G$  the group  $\pi_1(C_K)$ , put  $\Lambda = \mathbb{Z}G$ . Let  $\xi \in H^1(M, \mathbb{Z})$  be the generator of the group  $H^1(M, \mathbb{Z}) \approx \mathbb{Z}$ , positive on every meridian of  $K$ , we will call it *the canonical generator*. It can be considered as a homomorphism  $G \rightarrow \mathbb{Z}$ . Recall the Novikov ring

$$\widehat{\Lambda}_\xi = \{\lambda = \sum_{k \in \mathbb{N}} n_k g_k \mid n_k \in \mathbb{Z}, g_k \in G \text{ and } \xi(g_k) \rightarrow -\infty\}.$$

Let  $(f, v)$  be a regular Morse pair on  $C_K$ . Observe that the homotopy class  $[f] \in [C_K, S^1] \approx H^1(C_K, \mathbb{Z})$  equals  $\xi$ . We have a chain complex  $\widetilde{\mathcal{N}}_*(f)$  of free  $\widehat{\Lambda}_\xi$ -modules generated in degree  $k$  by  $Crit_k(f)$ , and such that

$$C_*(\widetilde{C}_K) \otimes_{\Lambda} \widehat{\Lambda}_\xi \sim \widetilde{\mathcal{N}}_*(f)$$

(see [16]). This chain complex is defined via counting flow lines of an  $f$ -gradient  $v$ , so we will denote it by  $\widetilde{\mathcal{N}}_*(f, v)$  when the dependence on  $v$  is important (as for example in Subsection 4.4).

### 3. Motion pictures and saddle numbers

Let  $K \subset \mathbb{R}^4$  be a 2-knot. Choose a projection  $p$  of  $\mathbb{R}^4$  onto a line. Assume that the critical points of the function  $p|_K$  are non-degenerate.

DEFINITION 3.1. The minimal number of saddle points of the function  $p|_K : K \rightarrow \mathbb{R}$  (the minimum is taken over all embeddings of  $S^2$  in  $\mathbb{R}^4$  ambient isotopic to  $K$ , and projections  $p$ ), will be called *saddle number of  $K$* , and denoted by  $sd(K)$ .

It is clear that  $sd(K) = 0$  implies that  $K$  is trivial. It turns out that  $sd(K) = 1$  also implies that  $K$  is trivial. Indeed, if for  $K \subset \mathbb{R}^4$  a projection  $p|_K$  has only one critical point of index 1, then  $m_0(p|_K) + m_2(p|_K) = 3$ , and the function  $p|_K$  has on the total 4 non-degenerate critical points. A deep theorem of M. Scharlemann [21] implies that such knot  $K$  is trivial.

It is not difficult to prove that for any classical knot  $K$  we have

$$sd(S(K)) \leq 2(b(K) - 1),$$

where  $b(K)$  denotes the bridge number of  $K$ , and  $S(K)$  is the spun knot of  $K$ . The invariant  $sd(K)$  is closely related to the *ch-index* of  $K$ , introduced and studied by K. Yoshikawa in [22]. In particular, we have  $sd(K) \leq ch(K)$ . In order to relate the number  $sd(K)$  to  $\mathcal{MN}(K)$  we will reformulate the definition of the saddle number.

Let  $K \subset S^4$  be a 2-knot. The equatorial 3-sphere  $\Sigma^3$  of the standard Euclidean sphere  $S^4$  divides  $S^4$  into two parts:

$$S^4 = D_+^4 \cup D_-^4, \text{ with } D_+^4 \cap D_-^4 = \Sigma^3.$$

We assume that  $K$  is included in  $\text{Int}(D_-^4)$  and  $K$  does not include the centre of  $D_-^4$ . Perturbing the embedding  $K \subset D_-^4$  if necessary, we can assume that the restriction  $\rho = r|_K$  of the radius function  $r : D_-^4 \rightarrow [0, 1]$  is a Morse function. The family  $\{(r^{-1}(t), \rho^{-1}(t))\}_{t \in [0, 1]}$  of possibly

singular knots can be drawn as a *motion picture* (see [10], Chapter 8). Each singularity of a knot in the family corresponds to a critical point of  $\rho$ . A critical point of  $\rho$  of index 0 (1, 2, respectively) is called *minimal point* (*saddle point*, *maximal point*, respectively) of  $\rho$ , which is represented by a *minimal band* (*saddle band*, *maximal band*, respectively) in (a modification of) the motion picture.

It is clear that the minimal number of the saddle points for all such Morse functions  $\rho$  is equal to  $sd(K)$ .

**Proposition 3.2.**  $\mathcal{MN}(K) \leq 2 sd(K)$ .

Proof. Since  $\rho$  is a Morse function, the manifold  $D_-^4 \setminus \text{Int } N(K)$  admits a handle decomposition with one 0-handle and  $m_i(\rho)$  ( $i + 1$ )-handles for  $i \in \{0, 1, 2\}$  (see [8], and also [7], Proposition 6.2.1).

The exterior  $E(K) = S^4 \setminus \text{Int } N(K)$  of  $K$  is obtained by attaching a 4-handle  $D_+^4$  to  $D_-^4 \setminus \text{Int } N(K)$ . Since  $D_-^4 \setminus \text{Int } N(K)$  is connected, there is a 3-handle in  $D_-^4 \setminus \text{Int } N(K)$  which connects  $\partial N(K)$  with  $\partial D_-^4$ . Thus the 3-handle cancels the 4-handle  $D_+^4$  (see [13], Section 5). Turning the handlebody upside down, we obtain a dual decomposition of  $E(K)$  and a corresponding Morse function  $f : E(K) \rightarrow \mathbb{R}$  which is constant on  $\partial E(K)$  and the following Morse numbers:  $m_1(f) = m_2(\rho) - 1$ ,  $m_2(f) = m_1(\rho)$ ,  $m_3(f) = m_0(\rho)$ ,  $m_4(f) = 1$ .

Using the argument from [17], p. 629, we can deform the real-valued Morse function  $f$  to a circle-valued regular function  $\phi : E(K) \rightarrow S^1$ , such that  $m_k(f) = m_k(\phi)$  for every  $k$ . Consider the function  $-\phi$ , which has one critical point of index 0. Applying the cancellation of this local minimum, we obtain a Morse function  $\psi : E(K) \rightarrow S^1$  belonging to the class  $-\xi$ , and such that  $m_0(\psi) = 0$ ,  $m_1(\psi) = m_3(f) - 1$ ,  $m_2(\psi) = m_2(f)$ ,  $m_3(\psi) = m_1(f)$ ,  $m_4(\psi) = 0$ . Put  $g = -\psi$ . Then we have

$$m_0(g) = m_4(g) = 0, \quad m_1(g) = m_2(\rho) - 1,$$

$$m_2(g) = m_1(\rho), \quad m_3(g) = m_0(\rho) - 1.$$

Observe that  $m_0(\rho) - m_1(\rho) + m_2(\rho) = \chi(S^2) = 2$ , therefore the total number of critical points of  $g$  equals  $2m_1(\rho)$ . Choosing the function  $\rho$  with  $m_1(\rho) = sd(K)$  we accomplish the proof.  $\square$

Taking into account the inequality (2) we obtain the following.

**Corollary 3.3.**

$$\widehat{q}_1(C_K) + \widehat{q}_2(C_K) \leq sd(K) \leq ch(K). \quad \square$$

**4. Circle-valued Morse maps for spun knots**

Let  $K \subset S^3$  be a classical knot, and  $(\phi, v)$  be a regular Morse pair on  $C_K$ . Denote by  $S(K)$  the spun knot of  $K$  (see Subsection 4.1 for definition). In this section we associate to  $(\phi, v)$  a regular Morse pair  $(F, w)$  on  $C_{S(K)}$ . We compute the Novikov complex of  $(F, w)$  in terms of the Novikov complex of  $(\phi, v)$  (Propositions 4.1 and 4.3).

**4.1. Spun knots: the definition.** Let us recall the classical construction (due to Artin [2]) of the spun knot for a classical knot  $K \subset S^3$ . The equatorial 2-sphere  $\Sigma$  of the standard Euclidean sphere  $S^3$  divides  $S^3$  into two parts:

$$S^3 = D_+^3 \cup D_-^3, \text{ with } D_+^3 \cap D_-^3 = \Sigma.$$

We can assume that  $K \cap D_-^3$  is a half-circle of the sphere  $S^3$ . Let  $K_+ = D_+^3 \cap K$ , observe that  $\text{Int}(D_+^3 \setminus K_+)$  is diffeomorphic to  $S^3 \setminus K$ . The sphere  $S^4$  can be considered as an open book with binding  $\Sigma = \partial D_+^3$  and pages diffeomorphic to  $D_+^3$ . In other words,  $S^4$  is obtained by rotating of  $D_+^3$  around  $\partial D_+^3 = \Sigma$ . The result of rotation of  $K_+ \subset D_+^3$  is an embedded 2-sphere in  $S^4$ , thus a 2-knot, which is called *the spun knot of K*; we denote it by  $S(K)$ .

**4.2. Geometric set-up.**

*1. The embedding  $K_+ \subset D_+^3$ .*

The intersection  $K_+ \cap \Sigma$  consists of two points, denote them by  $s$  and  $n$ . We can assume that there is a collar  $h : \Sigma \times [0, \delta] \rightarrow D_+^3$  such that  $h^{-1}(K_+ \cap D_+^3) = \{s, n\} \times [0, \delta]$ . Let  $(\phi, v)$  be a Morse pair on  $C_K$ . Without changing the Novikov complex  $\mathcal{N}_*(\phi, v)$  we can make the following assumptions on the map  $\phi$  and its gradient  $v$ :

- 1) All the critical points of  $\phi$  are in  $D_+^3$ .
- 2)  $\phi \circ h$  is constant along each interval  $x \times [0, \delta]$ , where  $x \in \Sigma$ .
- 3) There is a local coordinate system  $Q_n : D^2(0, \delta) \rightarrow \Sigma$  around  $n$  such that  $(h \circ (Q_n \times \text{Id})^{-1})^*(v)$  equals the vector field  $(\frac{z}{|z|}, 0)$  in  $D^2(0, \delta) \times [0, \delta]$ . There is a similar coordinate system  $Q_s$  around  $s$ .
- 4) The trivialisation  $N(K_+) \xrightarrow{\Phi} D^2(0, \delta) \times K_+$  required by the definition of regular Morse pair is compatible with  $h$  in the neighbourhood  $h(D^2(0, \delta) \times [0, \delta])$ .

In the sequel we will need two auxiliary functions defined on  $D_+^3$ . Let  $\xi : [0, \delta] \rightarrow \mathbb{R}$  be a  $C^\infty$  function such that  $\xi(r) = 0$  for  $r \in [0, \delta/3]$ , and  $\xi'(r) > 0$  for  $r \in ]\delta/3, 2\delta/3[$ , and  $\xi(r) = 1$  for  $r \in [2\delta/3, \delta]$ .

A) Define a function  $\alpha : D_+^3 \rightarrow [0, 1]$  as follows. For  $x = h(y, r)$  with  $y \in \Sigma \times [0, \delta]$  put  $\alpha(x) = \xi(r)$ . For  $x$  outside  $\text{Im } h$  put  $\alpha(x) = 1$ . Then  $\alpha$  is a  $C^\infty$  function vanishing in a neighbourhood of  $\Sigma$ .

B) Define a function  $\beta : D_+^3 \rightarrow [0, 1]$  as follows. For  $x = \Phi(z, t)$  with  $z \in D^2(0, \delta), t \in K_+$  put  $\beta(x) = \xi(|z|)$ . Otherwise put  $\beta(x) = 1$ . Then  $\beta$  is a  $C^\infty$  function vanishing in a neighbourhood of  $K$ .

*2. The embedding  $S(K) \subset S^4$ .*

The neighbourhood of  $\Sigma$  in  $S^4$  can be parametrized by a map  $H : \Sigma \times [0, \delta] \times S^1 \rightarrow N(\Sigma)$  where  $H|_{\Sigma \times [0, \delta] \times \{1\}} = h$  and for a given  $a \in \Sigma$  the map  $(a, r, \theta) \mapsto H(a, r, \theta)$  gives the polar coordinates in the 2-disc normal to  $H(a, 0, 0) \in \Sigma$ . The coordinate  $\theta$  is defined as a  $C^\infty$  function on  $S^4 \setminus \Sigma$ . Denote  $D_+^3 \setminus \partial D_+^3$  by  $B_+^3$ . We have a diffeomorphism

$$S^4 \setminus \Sigma \approx B_+^3 \times S^1,$$

its second projection  $S^4 \setminus \Sigma \rightarrow S^1$  extends the angle coordinate  $\theta$  defined on  $N(\Sigma) \setminus \Sigma$ , its first projection  $S^4 \setminus \Sigma \approx B_+^3 \times S^1 \rightarrow B_+^3$  will be denoted by  $p_1$ .



**4.3. Construction of a Morse map**  $F : S^4 \setminus S(K) \rightarrow S^1$ . Define a function  $F_0$  on  $S^4 \setminus (S(K) \cup \Sigma)$  by

$$F_0(y, \theta) = \phi(y).$$

In other words,  $F_0 = (\phi \circ p_1)(x)$ , where  $p_1 : S^4 \setminus \Sigma \rightarrow B_+^3$  is the projection of the first factor of the Cartesian product. It is clear that  $F_0$  is  $C^\infty$  on  $S^4 \setminus (S(K) \cup \Sigma)$ . Using our assumptions on  $\phi$  (see Subsection 4.2. 1)) we deduce that  $F_0$  extends to a  $C^\infty$  function on the whole of  $S^4 \setminus S(K)$ . We have  $\text{Crit}F_0 \approx S^1 \times \text{Crit}\phi$ , therefore  $F_0$  is not a Morse function. We will now construct a small perturbation of  $F_0$  which will have only non-degenerate critical points. Let  $h : S^1 \rightarrow \mathbb{R}$  be a Morse function with two non-degenerate critical points (e.g.  $h(\theta) = \sin(\theta)$ ). Define a function  $\tilde{h} : S^4 \setminus \Sigma \rightarrow S^1$  by  $\tilde{h}(y, \theta) = h(\theta)$ .

Extend the functions  $\alpha, \beta : D_+^3 \rightarrow [0, 1]$  constructed in Subsection 4.2 1) to the functions on  $S^4$  invariant with respect to action of  $S^1$ . We will denote the resulting functions by the same symbols  $\alpha, \beta$ . Then  $\alpha$  is a  $C^\infty$  function, vanishing in a neighbourhood of  $\Sigma$  and equal to 1 outside  $N(\Sigma)$ . The function  $\beta$  vanishes in a neighbourhood of  $S(K)$  and equals 1 outside  $N(S(K))$ . Define a function  $G : S^4 \rightarrow S^1$  as follows:

$$\begin{aligned} G(x) &= \alpha(x)\beta(x)\tilde{h}(x) \text{ for } x \notin \Sigma; \\ G(x) &= 0 \text{ for } x \in \Sigma. \end{aligned}$$

Then  $G$  is a  $C^\infty$  function vanishing in a neighbourhood of  $\Sigma \cup S(K)$  and equal to 1 outside  $N(\Sigma) \cup N(S(K))$ .

**Proposition 4.1.** *For any  $\epsilon$  sufficiently small, the function  $F$  defined by*

$$F(x) = F_0(x) + \epsilon G(x),$$

*is a Morse function, and*

$$m_i(F) = m_i(\phi) + m_{i-1}(\phi) \text{ for every } i.$$

*Proof.* In the domain  $N(\Sigma) \cup (N(S(K)) \setminus S(K))$  the norm of  $F'_0$  is bounded from below by a strictly positive constant, therefore the gradient of  $F$  is non-zero everywhere in this domain if  $\epsilon$  is sufficiently small. In the domain

$$S^4 \setminus (N(\Sigma) \cup N(S(K))) \approx (B_+^3 \setminus K_+) \times S^1$$

the function  $F$  is diffeomorphic to the function  $\tilde{F} : (B_+^3 \setminus K_+) \times S^1 \rightarrow S^1$  defined by

$$\tilde{F}(y, \theta) = \phi(y) + \epsilon h(\theta).$$

The proposition follows. □

**4.4. The Novikov complex of  $F$ .** We will now construct a suitable gradient for  $F$ . Let  $v$  be a  $\phi$ -gradient on  $B_+^3 \setminus K$  satisfying the assumptions from Subsection 4.2. Define a vector field  $\tilde{v}$  on the product  $(B_+^3 \setminus K) \times S^1$  by  $\tilde{v}(y, \theta) = v(y)$ , where  $\theta \in S^1, y \in B_+^3$ . Carry over the vector field  $\tilde{v}$  to  $S^4 \setminus S(K)$  (we will keep the same notation for the resulting field). Let  $u$  be any gradient for  $h : S^1 \rightarrow \mathbb{R}$ . Define a vector field  $\tilde{u}$  on the product  $B_+^3 \times S^1$  by the formula  $\tilde{u}(y, \theta) = u(\theta)$  (where  $y \in B_+^3, \theta \in S^1$ ). Carry over this vector field to  $S^4 \setminus \Sigma$  and define a vector field  $\bar{u}$  on  $S^4 \setminus \Sigma$  setting  $\bar{u}(x) = \alpha(x)\beta(x)\tilde{u}(x)$ . The vector field  $\bar{u}$  extends to a  $C^\infty$  vector



field on  $S^4$  (which will be denoted by the same symbol  $\bar{u}$ ), vanishing in a neighbourhood of  $\Sigma \cup S(K)$ . It is clear that for  $\epsilon$  small enough, the vector field

$$w = \bar{v} + \epsilon \bar{u}$$

is an  $F$ -gradient.

**DEFINITION 4.2.** For a chain complex  $A_*$  over a ring  $R$  we denote by  $\sigma A_*$  the suspension of  $A_*$ , that is,  $(\sigma A_*)_k = A_{k-1}$ .

In the next proposition we compute the Novikov complex of  $F$  in terms of the Novikov complex of  $\phi$ . Observe that the base ring of both complexes is the same, it is the Novikov completion of  $\mathbb{Z}G$ , where  $G = \pi_1(S^3 \setminus K) \approx \pi_1(S^4 \setminus S(K))$ .

**Proposition 4.3.** *We have*

$$\mathcal{N}_*(F, w) \approx \mathcal{N}_*(\phi, v) \oplus \sigma \mathcal{N}_*(\phi, v).$$

*Proof.* It is easy to check (using our assumptions on  $v$ ) that any flow line of  $w$  joining critical points of  $F$  does not intersect the subset  $N(K) \cup N(\Sigma)$ , and therefore remains in the domain, where the vector field  $w$  is the direct product of vector fields  $\epsilon \bar{u}$  and  $\bar{w}$ . Therefore

$$\mathcal{N}_*(F, w) \approx \mathcal{N}_*(\phi, v) \otimes \mathcal{M}_*(h, u)$$

(where  $\mathcal{M}_*(h, u)$  is the Morse complex of  $h$ ) and the Proposition follows. □

**4.5. Superspinning.** The classical Artin construction has several generalizations, in particular *the superspinning*. This construction, due to E.C. Zeeman [23] and D.B.A. Epstein [3], associates to any  $n$ -knot  $K^n \subset S^{n+2}$  and a natural number  $p \geq 1$  an  $(n + p)$ -knot in  $S^{n+p+2}$  (see [4], p. 196). We will denote the resulting knot by  $S_p(K)$ . The results of the previous subsection generalize directly to the superspinning case. Let  $\phi : S^{n+2} \setminus K^n \rightarrow S^1$  be a regular Morse map, and  $v$  a regular  $\phi$ -gradient.

**Theorem 4.4.** *There is a regular Morse function  $F : S^{n+p+2} \setminus S_p(K) \rightarrow S^1$  and a regular  $F$ -gradient  $w$  such that*

- 1)  $m_i(F) = m_i(\phi) + m_{i-1}(\phi)$ ,
- 2)  $\mathcal{N}_*(F, w) = \mathcal{N}_*(\phi, v) \oplus \sigma^p \mathcal{N}_*(\phi, v)$ .

*Proof.* The argument repeats the proof of Propositions 4.1 and 4.3 with minor modifications. □

**5. Morse-Novikov numbers for spun knots**

In this section we gathered some consequences of the constructions developed in Section 4. The next Corollary is immediate from Proposition 4.1.

**Corollary 5.1.**  $\mathcal{MN}(S_p(K)) \leq 2\mathcal{MN}(K)$ . □

In particular, if  $K$  is fibred, then  $S_p(K)$  is fibred. The case of fibred knots was observed in [1], together with the inverse implication: if  $K$  is a classical knot and  $S(K)$  is fibred, then  $K$  is fibred. This last property is not valid for 3-knots, as an example of C. Kearton [11]

shows.

**Theorem 5.2.** *Let  $K \subset S^3$  be a classical knot with  $\mathcal{MN}(K) = 2$ . Then  $\mathcal{MN}(S(K)) = 4$  and there is a strongly minimal Morse function on  $S^4 \setminus S(K)$ .*

Proof. The inequality  $\mathcal{MN}(S(K)) \leq 4$  follows from Corollary 5.1. Proceeding to the proof of the inverse inequality, let  $\phi : S^3 \setminus K \rightarrow S^1$  be a regular Morse function without local maxima or minima, with  $m_1(\phi) = m_2(\phi) = 1$ . By Proposition 4.1 there is a regular Morse function  $F : S^4 \setminus S(K) \rightarrow S^1$  with  $m_1(F) = m_3(F) = 1$ ,  $m_2(F) = 2$ ,  $m_0(F) = m_4(F) = 0$ . Therefore  $\mathcal{MN}(S(K)) \leq 4$ . We will show that  $F$  is actually a strongly minimal Morse map.

Let  $H : S^4 \setminus S(K) \rightarrow S^1$  be any Morse function. Denote the fundamental group of  $S^3 \setminus K$  by  $G$ . It is known that  $\pi_1(S^4 \setminus S(K)) \approx G$ . If  $m_1(H) = 0$ , then a standard Morse-theoretic argument applied to the infinite cyclic cover  $\overline{S^4 \setminus S(K)}$  of  $S^4 \setminus S(K)$  implies that  $\pi_1(\overline{S^4 \setminus S(K)})$  is finitely generated, which is impossible, since  $K$  is not fibred. Therefore  $m_1(H) \geq 1$ . A similar argument shows that  $m_3(H) \geq 1$ . Assume now that  $H$  is a minimal Morse function. Then

$$m_0(H) = m_4(H) = 0, m_1(H) \geq 1, m_3(H) \geq 1,$$

and this implies  $m_2(H) \geq 2$  (since  $\chi(S^4 \setminus S(K)) = 0$ ). □

In the work [6] based on his earlier paper [5] H. Goda proved that  $\mathcal{MN}(K) = 2$  for every non-fibred prime knot with  $\leq 10$  crossings. Therefore  $\mathcal{MN}(S(K)) = 4$  for these knots.

In the work [17] the second author proved that for any classical knot  $K$  we have

$$(3) \quad \mathcal{MN}(K) \leq 2t(K),$$

where  $t(K)$  is the tunnel number of  $K$ .

**Corollary 5.3.** *Let  $K \subset S^3$  be a non-fibred classical knot with tunnel number 1. Then  $\mathcal{MN}(S(K)) = 4$ , and there is a strongly minimal Morse function on  $S^4 \setminus S(K)$ .* □

Proof. By (3) we have

$$\mathcal{MN}(K) \leq 2;$$

on the other hand  $S(K)$  is not fibred, since  $K$  is not fibred. Hence  $\mathcal{MN}(S(K)) = 4$  by Theorem 5.2. □

These results allow to compute the numbers  $\mathcal{MN}(S(K))$  for many classical knots  $K$ . In the paper [15] K. Morimoto, M. Sakuma and Y. Yokota explicited many examples of tunnel number one knots, in particular, an infinite series of Montesinos knots. They prove that the Montesinos knot  $K = M(3, (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))$  has the tunnel number 1, if  $r = 3$ , and  $\beta_2/\alpha_2 \cong \beta_3/\alpha_3 \pmod{\mathbb{Z}}$  and  $b - \sum_{i=1}^3 \beta_i/\alpha_i = 1/(\alpha_1\alpha_2)$ .

**Corollary 5.4.** *Let  $K$  be a non-fibred Montesinos knot satisfying the conditions above. Then  $\mathcal{MN}(S(K)) = 4$ .* □

REMARK 5.5. In the work [9] of M. Hirasawa and K. Murasugi it is shown that the fibredness of most Montesinos knots with tunnel number 1 is detected by the monicness of the Alexander polynomial.

REMARK 5.6. Proposition 4.3 and Theorem 4.4 allow also to compute the Novikov homology for spun knots. Let us consider the example of the classical knot  $K = 5_2$ . We have  $t(K) = 1$ , thus there is a Morse function  $\phi$  on the complement to  $S(K)$  with  $m_1(\phi) = m_3(\phi) = 1$ ,  $m_2(\phi) = 2$ . The module  $\widehat{H}_1(C_K)$  is cyclic, therefore isomorphic to  $\widehat{L}/\Delta\widehat{L}$ , where  $\Delta = -2 + 3t - 2t^2$  is the Alexander polynomial of  $K$ . Thus we have  $\widehat{q}_1(S(K)) = 1 = \widehat{q}_2(S(K))$ .

**6. On fibering of high-dimensional spun knots**

Let  $K^n \subset S^{n+2}$  be a knot with  $n \geq 4$ . Let  $C_K$  denote the complement of  $K^n$  in  $S^{n+2}$ , and  $\widetilde{C}_K$  be the universal covering of  $C_K$ . Let  $\xi \in H^1(C_K, \mathbb{Z}) \approx \text{Hom}(\pi_1(C_K), \mathbb{Z})$  be the canonical generator of the cohomology of  $C_K$ . The knot  $K^n$  is fibred if and only if the following two conditions hold (see [16], [20], [12]):

- F0) The subgroup  $\text{Ker } \xi$  of  $\pi_1(C_K)$  is finitely presented.
- F1) The Novikov homology  $\widehat{H}_*(\widetilde{C}_K)$  vanishes,
- F2) The Whitehead torsion  $\tau(K) \in \text{Wh}(\widehat{\Lambda}_\xi)$  of the completed chain complex

$$C_*(\widetilde{C}_K) \otimes_{\widehat{\Lambda}_\xi}$$

is equal to 0.

The results obtained in the previous sections allow to compare the conditions F0) – F2) for a knot  $K$  and its spun knots. For  $m \geq 1$ ,  $p \geq 1$  we denote by  $S_p^m(K)$  the result of  $m$  iterations of the  $p$ -spinning construction of  $K$ . Observe that the fundamental group of the complement to the knot is isomorphic to that of the complement to its spun knot, therefore the condition F0) holds for  $K$  if and only if it holds for  $S_p^m(K)$  (with any  $p, m$ ). Let  $p \geq 1$ . Theorem 4.4 implies that

$$\widehat{H}_*(C_{S_p(K)}) \approx \widehat{H}_*(C_K) \oplus \sigma^p \widehat{H}_*(C_K)$$

therefore the Novikov homology of  $C_K$  vanishes if and only if the Novikov homology of  $C_{S_p(K)}$  vanishes. The situation with the torsion is different, since Theorem 4.4 implies

$$\tau(S_p(K)) = (1 + (-1)^p)\tau(K).$$

In particular,  $\tau(S_p(K)) = 0$  for any  $K$  if  $p$  is odd, and  $\tau(S_p(K)) = 2\tau(K)$  if  $p$  is even.

The next two propositions are now immediate.

**Proposition 6.1.** *If  $p \geq 1$  and  $q \geq 1$  have the same parity, then  $S_p(K)$  is fibred if and only if  $S_q(K)$  is fibred.* □

Let  $S_p^m(K)$  denote the result of  $m$  iterations of the  $p$ -spinning construction of  $K$ .

**Proposition 6.2.** *If  $p$  is odd and  $l, m \geq 1$  then  $S_p^m(K)$  is fibred if and only if  $S_p^l(K)$  is fibred.* □

REMARK 6.3. If for some knot  $K^n \subset S^{n+2}$  the conditions F0), F1) above hold, but F2) does not hold, then  $S(K)$  is fibred, and  $K$  is not fibred. However we do not know if such knots exist.

## 7. Open questions

1. Is it true that for any 2-knot  $K$  there exists a strongly minimal Morse function  $C_K \rightarrow S^1$ ? (This is true for spun knots  $S(K)$  with  $\mathcal{MN}(K) = 2$ , see Theorem 5.2.)

2. Is it true that for any classical knot  $K$  we have  $\mathcal{MN}(S(K)) = 2\mathcal{MN}(K)$ ? (This is true for any classical knot  $K$  with  $\mathcal{MN}(K) = 2$ , again by Theorem 5.2.)

3. Is it true that for a knot  $K$  of dimension  $\geq 4$  we have  $\mathcal{MN}(S(K)) = 2\mathcal{MN}(K)$ ? In particular is it true that  $K$  is fibred if and only if  $S(K)$  is fibred?

4. It is not difficult to prove that for knots  $K_1, K_2$  of any dimension we have  $\mathcal{MN}(K_1 \# K_2) \leq \mathcal{MN}(K_1) + \mathcal{MN}(K_2)$  (the argument repeats the proof for the classical knots, see [19]). Is it true that

$$\mathcal{MN}(K_1 \# K_2) = \mathcal{MN}(K_1) + \mathcal{MN}(K_2)$$

for 2-knots?

5. What is the relation between the saddle number and the unknotting number of a 2-knot? Is it true that  $sd(K) \leq u(K)$ ?

ACKNOWLEDGEMENTS. We began working on this project in 2014 during the visit of the first author to the Nantes University. The work was accomplished when the second author was visiting the Tokyo Institute of Technology in 2015. The first author thanks the Nantes University and the GeanPyl program for the support and warm hospitality. The second author thanks the Tokyo Institute of Technology for support and warm hospitality. The first author was partially supported by JSPS KAKENHI Grant Number 25400082.

---

## References

- [1] J.J. Andrews and D.W. Sumners: *On higher-dimensional fibred knots*, Trans. Amer. Math. Soc. **153** (1971), 415–426.
- [2] E. Artin: *Zur Isotopie zweidimensionalen Flächen im  $R_4$* , Abh. Math. Sem. Univ. Hamburg **4** (1925), 174–177.
- [3] D.B.A. Epstein: *Linking spheres*, Proc. Cambridge Philos. Soc. **56** (1960), 215–219.
- [4] G. Friedman: *Knot Spinning*, Handbook of Knot Theory, edited by William Menasco, Morwen Thistlethwaite, ch.4. Elsevier, 2005.
- [5] H. Goda: *On handle number of Seifert surfaces in  $S^3$* , Osaka J. Math. **30** (1993) 63–80.
- [6] H. Goda: *Some estimates of the Morse-Novikov numbers for knots and links*, in Intelligence of low dimensional topology 2006, (J S Carter, S Kamada, L H Kauffman, A Kawachi, T Kohno, eds.), Ser. Knots Everything, vol.40, World Sci. Publ., Singapore (2007) 35–42.
- [7] R.E. Gompf and A.I. Stipsicz: *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics **20**, American Mathematical Society, 1999.
- [8] C. McA Gordon: *Homology of groups of surfaces in the 4-sphere*, Math. Proc. Camb. Philos. Soc. **89**, 113–117.
- [9] M. Hirasawa and K. Murasugi: *Genera and fibredness of Montesinos knots*, Pacific J. Math. **225** (2006).
- [10] S. Kamada: *Braid and knot theory in dimension four*, Math. Surveys Monogr. **95**, Amer. Math. Soc., Providence, RI, 2002.
- [11] C. Kearton: *Some non-fibred 3-knots*, Bull. London Math. Soc. **15** (1983), 365–367.
- [12] F. Latour: *Existence de 1-formes fermées non singulières dans une classe de cohomologie de de Rham*, Publ. IHES **80** (1995), 135–194.

- [13] J. Milnor: Lectures on  $h$ -cobordism theorem, Notes by L. Siebenmann and J. Sondow, Princeton Univ. Press, Princeton, New Jersey, 1965.
- [14] J. Milnor: *Infinite cyclic coverings*, In: Conference on the topology of manifolds, (1968).
- [15] K. Morimoto, M. Sakuma and Y. Yokota: *Identifying tunnel number one knots*, J. Math. Soc. Japan **48** (1996), 667–688.
- [16] A.V. Pajitnov: *On the Novikov complex for rational Morse forms*, preprint: Institut for Matematik og datalogi, Odense Universitet Preprints 1991, No 12, Oct. 1991; Ann. Fac. Sci. Toulouse Math. (6) **4** (1995), 297–338.
- [17] A. Pajitnov: *On the tunnel number and the Morse-Novikov number of knots*, Algebr. Geom. Topol. **10** (2010) 627–635.
- [18] A. Pajitnov: *Circle-Valued Morse Theory*, de Gruyter Studies in Mathematics 32, 2006.
- [19] A.V. Pajitnov, C. Weber and L. Rudolph: *Morse-Novikov number for knots and links*, Algebra i Analiz, **13** (2001), (in Russian), English translation: St. Petersburg. Math. J. **13** (2002), 417–426.
- [20] A. Ranicki: *Finite domination and Novikov rings*, Topology **34** (1995), 619–632.
- [21] M. Scharlemann: *Smooth Spheres in  $R^4$  with four critical points are standard*, Invent. Math. **79** (1985), 125–141.
- [22] K. Yoshikawa: *An enumeration of surfaces in four-space*, Osaka J. Math. **31** (1994), 497–522.
- [23] E.C. Zeeman: *Linking spheres*, Abh. Math. Sem. Univ. Hamburg **24** (1960), 149–153.

Hisaaki Endo  
Department of Mathematics  
Tokyo Institute of Technology  
2-12-1 Ookayama, Meguro-ku  
Tokyo 152-8551  
Japan  
e-mail: endo@math.titech.ac.jp

Andrei Pajitnov  
Laboratoire Mathématiques Jean Leray UMR 6629  
Université de Nantes  
Faculté des Sciences  
2, rue de la Houssinière  
44072, Nantes  
Cedex  
e-mail: andrei.pajitnov@univ-nantes.fr