# Coalescing stochastic flows on the real line

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## 1. Introduction

Stochastic differential equations have been considered to construct diffusion processes on $R^d$ and on manifolds. From 1970's to the early part of 80's, it was proved (see, e.g., [3],[5],[11],[16]) that the solutions of stochastic differential equations with smooth coefficients define the Brownian motions of diffeomorphisms group as in the case of the deterministic dynamical systems. After that, Brownian motions on the diffeomorphisms groups themselves have been considered and the theory of stochastic flows has been ~~developped~~ developed by many authors. See, for example, [2],[7],[17],[18] and so on. All of the works above are concerned with the flows of smooth maps.

Recently Harris [8], taking into account of a model of infinite particle systems called coalescing Brownian motions, considered stochastic flows of non-smooth maps on $R^1$. In this paper, following [8], we will study the stochastic flows of non-smooth maps on $R^1$ called coalescing flows, which we will describe below. We refer Arratia [1] for the coalescing Brownian motions.

Let us imagine that, from each $x \in R^1$, the Brownian motion $\{X_{0,t}(x)\}_{t \geq 0}$ starts and that, if there exists time $\tau$ satisfying $X_{0,\tau}(x)=X_{0,\tau}(y)$ for $x \neq y$, the paths of the two Brownian motions coalesce into a single one. Firstly let us consider the case where the infinitesimal correlation function a(x,y) defined by

$$a(x,y) = \lim_{t \downarrow 0} \frac{1}{t} E[(X_{0,t}(x) - x)(X_{0,t}(y) - y)],$$

which determines the stochastic flow uniquely under suitable assumptions and is called the local characteristic of X, is written as b(x-y) by some function b on $R^1$. Then whether such coalescence in finite time occurs or not is known by a integral condition for b. That is, sinse $\{X_{0,t}(x)-X_{0,t}(y)\}_{t\geq 0}$ is proved to be a diffusion process generated by

$$(1 - b(z))\frac{d^2}{dz^2},$$

the coalescence occurs if

$$\int_{0+} (1 - b(z))^{-1} z dz \tag{1.1}$$

is finite and does not occur if infinite by Feller's criterion. For details, see Section 3.

Now let us consider the map $X_{0,t}: x \to X_{0,t}(x)$ for each fixed t. Then, taking adequate modification, we have the stochastic flow $\{X_{0,t}\}_{t\geq 0}$ valued in $\Lambda$, the space of all right-continuous, non-decreasing functions on $R^1$.

At first let us consider the case (1.1) is finite. Then coalescence in finite time occurs and, moreover Harris [8] has showed that, if there exists $\varepsilon$ with $0<\varepsilon<2$, such that

(1.2) $$b(0) - b(z) \geq |z|^{2-\varepsilon}$$

holds in some neighborhood of $z=0$, we have

(1.3) $$E[{}^{\#}X_{0,t}(K)] < \infty,$$

and, in particular,

$$\begin{aligned} &P(X_{0,t}(K) \text{ is a finite set}) \\ &= P(X_{0,t}(R^1) \cap K \text{ is a finite set}) \\ &= 1 \end{aligned}$$

for every $t>0$ and every compact interval $K$. On the other hand, Harris [8] also has showed that, if $b''(0)$ exists, which implies that (1.1) is infinite, we have

(1.4) $$P(X_{0,t} \text{ is a homeomorphism of } R^1) = 1$$

for every $t>0$.

We will study in detail when (1.3) or (1.4) occurs and characterize it in terms of the function b. Then we will see that (1.3) holds if (1.1) is finite and b satisfies a monotonicity condition in some sense (see (3.3) below). On the other hand, if (1.1) is infinite, we will see that (1.4) holds, that is, we have a stochastic flow of homeomorphisms of $R^1$.

In the proof of the countable range case (1.3), we will consider the eigenfunction expansion for the transition probability density of one-dimensional diffusion process with respect to the speed measure. The purpose there is to give some estimates for the eigenfunction by a similar method to that in Hille [9]. Those estimates are related to the boundedness of the fundamental solution for some one-dimensional diffusion operators, which might be called the ultra-contractivity for the diffusion operator. For details, see Section 4 and for the ultra-contractivity, see,e.g., Davies-Simon [5] 4 and references therein.

Moreover in the final part of this paper, we will consider some spatially inhomogeneous stochastic flows whose local characteristic $a(x,y)$ is not written as $b(x-y)$. We will consider the case where $\{X_{0,t}(x)-X_{0,t}(y)\}_{t\geq 0}$ can be compared with the one-dimensional diffusion process with the same probability law as that of the diffusion process defined in the same way as that for the spatially homogeneous stochastic flows. Then we will have spatially inhomogeneous flows satisfying (1.3) or (1.4).

This paper is organized as follows. In Section 2, following Harris [8], we will mention the construction theorem for the stochastic flows. The main theorems for the spatially homogeneous flows will be stated in Section 3 and their proofs will be stated in Sections 5 and 6. In Section 4 we will prove the boundedness of transition densities of certain one-dimensional diffusion processes with respect to the speed measures. The result in Section 4 will play a key role in the proof of the countable range case. In the last section, Section 7, we will study the spatially inhomogeneous flows which has the same properties mentioned above.

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## 2. Construction of stochastic flows

In this section we will give the definition of the stochastic flows treated in this paper and state when there exist unique (in law) stochastic flows with non-smooth local characteristics. The result is due to Harris [8]. The formulation here is a little bit different from his, but the proof is the same. Therefore we state only the result without proof. For details, see [8].

Let a be a real valued function on $R^1 \times R^1$. Throughout this paper we call $X=\{X_{s,t}; 0 \leq s \leq t < \infty\}$ a stochastic flow on $R^1$ with a as its local characteristic if and only if X satisfies the following:

(i) for each s and t, $X_{s,t}$ is a non-decreasing and right-continuous mapping from $R^1$ into $R^1$ a.s.,

(ii) for s,t and u with $s \leq t \leq u$, $X_{t,u} \circ X_{s,t} = X_{s,u}$ a.s.,

(iii) for each $x_0 \in R^1$, $\{X_{s,t}(x_0)\}_{t \geq s}$ is a diffusion process on $R^1$ starting from $x_0$ at time s which is generated by

$$\frac{1}{2}\, a(x,x) \frac{d^2}{dx^2} .$$

(iv) for each $x_1, x_2 \in R^1$ and $s \geq 0$, $\{(X_{s,t}(x_1), X_{s,t}(x_2))\}_{t \geq s}$ is a diffusion process starting from $(x_1,x_2)$ at s which is generated by

$$\frac{1}{2} a(x,x) \frac{\partial^2}{\partial x^2} + a(x,y) \frac{\partial^2}{\partial x \partial y} + \frac{1}{2} a(y,y) \frac{\partial^2}{\partial y^2}$$

up to the coalescing time $\tau = \inf\{t \geq s; X_{s,t}(x_1) = X_{s,t}(x_2)\}$ and, for $t \geq \tau$, $X_{s,t}(x_1) = X_{s,t}(x_2)$.

In order to state the result, we give some assumptions on the local characteristic a. We assume throughout that a satisfies

(2.1) $a:R^1\times R^1\to R^1$ is (i) symmetric, bounded, continuous, (ii) locally Lipschitz continuous outside each neighborhood of the diagonal set in $R^1\times R^1$ and (iii) strictly positive definite in the sense that, for each $x_1,\dots,x_k$ with $x_1<\dots<x_k$ and $\xi=(\xi_1,\dots,\xi_k)\in R^k\setminus\{0\}$, it holds that

$$\sum_{i,j=1}^{k} a(x_i,x_j)\xi^i\xi^j > 0.$$

Then, by Harris [8], we have the following fundamental result: under the assumption (2.1), there exists a unique (in law) stochastic flow $X=\{X_{s,t};0\le s\le t<\infty\}$ on $R^1$ with a as its local characteristic.

## 3. Spatially homogeneous stochastic flows

In this section and Sections 5 and 6, we will consider stochastic flows whose local characteristic is given in the form $b(x-y)$. We assume that $b(x-y)$ satisfies the assumption (2.1) as a function of $(x,y)\in R^2$ and call also $b$ the local characteristic of the flow. Moreover we assume $b(0)=1$ for simplicity only. In particular, the assumption (iii) reads that $b$ is a positive definite function on $R^1$ and, therefore, that $b$ is written as

$$(3.1) \qquad b(z) = \int_{-\infty}^{\infty} e^{iz\xi}\, dF(\xi) = \int_{-\infty}^{\infty} \cos(z\xi)\, dF(\xi), \quad z \in R^1,$$

by some probability measure $F$ on $R^1$. By the strictly positive definiteness of $b$, we see that the support of $F$ is not a finite set.

Denote by $X=\{X_{s,t};0\leq s\leq t<\infty\}$ the stochastic flow with $b$ as its local characteristic defined on some probability space $(\Omega,\mathcal{F},P)$. Such stochastic flows were studied by Harris [8] for the first time. A typical property of such spatially homogeneous stochastic flows is the following. In order to mention it, denote by $\{Q_z\}_{z\geq 0}$ the probability law of the one-dimensional diffusion processes on $(0,\infty)$ with its absorbing boundary 0 and generator

$$(1 - b(z))\frac{d^2}{dx^2} f(z), \qquad f \in C_b^{\infty}((0,\infty)).$$

Then Harris [8] has showed:

<u>Lemma 3.1.([8])</u> Let $x,y\in R^1$ with $x>y$ and define $\eta=\{\eta_t\}_{t\geq 0}$ by $\eta_t=X_{0,t}(x)-X_{0,t}(y)$. Then the probability law of $\eta$ coincides with $Q_{x-y}$.

From Lemma 3.1 we see that the coalescence of any two distinct trajectories in finite time occurs with probability 1 if and only if b satisfies

$$(3.2) \qquad \int_{0+} \frac{zdz}{1 - b(z)} < \infty$$

by Feller's criterion for accesibility of one-dimensional diffusion processes. See, for example, Ito [12]. Note that, under the integral condition (3.2), the boundary 0 is regular or exit in the sense of Feller (see, McKean [19] or [12]). On the other hand, when the integral in (3.2) is infinite, the boundary 0 is natural in the sense of Feller and, therefore, the coalescence in finite time does not occur with probability 1.

Firstly we consider the case where (3.2) holds. In this case we will consider the problem under the following assumption:

<u>Assumption.</u> There exists a continuous function $\beta:(0,1]\to(0,\infty)$ such that

(i) $1 - b(z) \geq \beta(z)$

(ii) $\int_{0+} z(\beta(z))^{-1}dz < \infty$

(iii) $z^{-2}\beta(z)$ is monotone decreasing on $(0,\alpha)$ for some $\alpha>0$.

Before stating our main result, we introduce some notations. We define $\mathbf{B}$ by

$$(1-b(z))\frac{d^2}{dz^2}$$

as an operator on $[0,1]$ whose boundaries 0 and 1 are absorbing and reflecting, respectively, and denote by $\{e_n\}_{n=1}^{\infty}$ the eigenfunctions of $\mathbf{B}$ corresponding to the eigenvalues $\{-\lambda_n\}_{n=1}^{\infty}$. Sinse $e_n(0)=0$, we assume that $e_n(x)$ is positive for sufficiently small $x$. Then we see easily that $x^{-1}e_n(x)$ is bounded for each $n$, so we set

$$M_n = \sup_{0\leq x\leq 1} \frac{1}{x}\,|e_n(x)|.$$

Now we can state our main result.

<u>Theorem 3.2.</u> If $b$ satisfies the assumption above, then we have

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \log M_n = 0$$

and

$$E[{}^{\#}X_{0,t}(K)] \leq 1 + |K| \sum_{n=1}^{\infty} e^{-\lambda_n t} M_n^2 \tag{3.3}$$

for any bounded interval $K$ with length $|K|$ and for any $t>0$, where $E$ denotes the expectation with respect to $P$.

Note that the finiteness of the right hand side of (3.3) is seen by virtue of the first assertion and the fact that the Green operator for $\underline{B}_{\bar{x}\eta}$ is of trace class under the assumption. See, for example, [13] or [19].

As an immediate consequence of (3.3), we get:

<u>Corollary 3.3.</u> For any $t_0>0$ and compact interval K, it holds that

$$P(X_{0,t}(K) \text{ is a finite set for all } t \geq t_0) = 1.$$

Moreover, combining the proof of Theorem 3.2 with the result mentioned in Section 10 of Harris [8], we will prove:

<u>Theorem 3.4.</u> Under the same assumption as in Theorem 3.2,

$$P(X_{0,t}(R^1) \cap I \text{ is a finite set for all } t \geq t_0) = 1$$

holds for any compact interval I and $t_0>0$.

The proofs will be given in Section 5. Note that (1.2) implie the assumption and that, if it holds that

$$1 - b(z) = O(z^2(\log z^{-1})\ldots(\log_{n-1} z^{-1})(\log_n z^{-1})^\alpha) \quad (1)$$

as $z\downarrow 0$ for some $\alpha>1$ and for some positive integer n, the assumption is also satisfied. For this we will mention again in Section 6.

Secondly we consider the non-coalescing case, that is, the case the integral in (3.2) is infinite. Then we will prove:

Theorem 3.5. If the integral in (3.2) is infinite, then $X_{0,t}$ is a homeomorphism of $R^1$ with probability 1 for any $t \geq 0$.

The proof of Theorem 3.5 will be given in Section 6. We note here that, if b is smooth, we can apply the general theory of stochastic flows and have that X is a stochastic flow of diffeomorphisms of $R^1$. Since $b(z)$ attains its maximum 1 only at $z=1$ by the strictly positive definiteness of b, we have $b'(0)=0$ and, therefore, that the integral in (3.2) is infinite, which shows that Theorem 3.5 is consistent with the general theory.

## 4. Some asymptotics of eigenfunctions of one-dimensional diffusion operators

In this section, apart from the stochastic flows, we will consider the one-dimensional generalized diffusion operator $\underset{\text{スク}}{\underline{G}}$ defined by

$$\underset{\text{スク}}{\underline{G}} = \frac{d}{dM(x)} \frac{d}{dx}$$

on $[1,\infty)$, where M is a real valued nontrivial right-continuous non-decreasing function on $[1,\infty)$ with $M(1)=0$ and dM is the naturally induced measure by M. The purpose of this section is to study the increasing order of $\sup_x |v_n(x)|$ for the eigenfunction $v_n(x)$ as n tends to infinity, which will play the most important role in the proof of Theorem 3.2 in the next section and is of independent interest because of its relation to the ultra-contractivity of this operator.

We assume the following for M: it holds that

$$\text{(4.1)} \qquad \int_1^\infty x dM(x) < \infty$$

and

$$\text{(4.2)} \qquad \int_1^\infty \frac{\rho(x)}{x} dx < \infty,$$

where

$$\rho(x) = \sup_{y \geq x} g(y), \quad g(y) = y\int_y^\infty dM(z).$$

A sufficient condition for (4.2) in the case $dM$ has its density $m$ is that there exists a continuous function $\tilde{m}$ such that

$$
\text{(4.3)} \qquad
\begin{aligned}
&\text{(i) } m(x) \leq \tilde{m}(x) \text{ and } \tilde{m} \text{ satisfies (4.1),} \\
&\text{(ii) } x^2\tilde{m}(x) \text{ is monotone decreasing on } [\alpha,\infty) \text{ for some } \alpha \geq 1.
\end{aligned}
$$

(4.1) means that the boundary $\infty$ is of entrance type and we give the boundary condition at $x=1$ by

$$pu(1) + (1 - p)u^+(1) = 0$$

for $0 \leq p \leq 1$, where $u^+$ denotes the right derivative of $u$ and

$$u(1) = \lim_{x \downarrow 1} u(x), \quad u^+(1) = \lim_{x \downarrow 1} u^+(x).$$

Now denote by $\{v_n\}_{n=1}^{\infty}$ the eigenfunctions for $\underset{\sim}{G}$ corresponding to the eigenvalues $\{-\mu_n\}_{n=1}^{\infty}$ under the boundary conditions mentioned above. Since it is easily seen that $\lim_{x\uparrow\infty} v_n(x)$ exists and is not equal to zero for each $n$, we assume that $v_n(x)$ is positive for large $x$.

The main result of this section is the following theorem. This immediately implies that the fundamental solution $p(t,x,y)$ for $\underset{\sim}{G}$, for which we have the following eigenfunction expansion

$$p(t,x,y) = \sum_{n=1}^{\infty} e^{-\mu_n t} v_n(x)v_n(y),$$

is bounded in $(x,y)$ for each fixed $t>0$.

Theorem 4.1. Under the assumptions (4.1) and (4.2), it holds that

$$(4.4) \qquad \lim_{n\uparrow\infty} \frac{1}{\mu_n} \log\Big( \sup_{x\in[1,\infty)} |v_n(x)| \Big) = 0.$$

Remark 4.2. Without the assumption (4.2), (4.4) does not hold in general and some counterexamples are known ([13],[14]). In those cases $p(t,\infty,\infty)=\infty$ holds for all $t>0$.

Proof. Hille [9] has mentioned the same result for classical ordinary differential operators like

$$a(x)\frac{d^2}{dx^2} .$$

His idea is applicable also to the generalized operators. We will give the sketch of it for the completeness. Let $X_n$ be the maximum root of $4\mu_n\rho(x)=1$. It is easy to see that $X_n$ exists and tends to $\infty$ as n tends to $\infty$, because $\rho(x)$ decreases monotonically to 0 .

For fixed $y\geq X_n$ let us consider the integral equation

$$v(x) = x\int_x^\infty v(z)^2 z^{-2}\, dz + \mu_n x\int_x^\infty dM(z) \qquad x \geq y \geq X_n.$$

Then, by using the method of succecive approximation, we see that it has a unique solution $v(x)$ and that $0<v(x)\leq\sigma$, where $\sigma$ is the smaller root of the quadratic equation $\cancel{s^2-s+g(y)=0}$ $s^2-s+\mu_n\rho(y)=0$. Moreover if we set $w(x)=v(x)/x$, $w$ satisfies the following integral equation of Riccati type;

$$\int_x^\infty w^+(z)\,dz + \int_x^\infty w(z)^2\,dz = -\mu_n\int_x^\infty dM(z).$$

Therefore $w(x)$ coincides with $(\log v_n(x))^+$, and since

$$\sigma = \frac{1}{2} - \left(\frac{1}{4} - \mu_n\rho(y)\right)^{1/2} \leq 2\mu_n\rho(y),$$

we get

$$\text{(4.5)}\qquad \log v_n(x) - \log v_n(y) \leq 2\mu_n\rho(y)\{\log x - \log y\}.$$

Deviding $[y,x]$ into small intervals, applying (4.5) to each interval and using the limiting argument, we have

$$\log v_n(x) - \log v_n(y) \leq 2\mu_n\int_y^x \frac{1}{z}\rho(z)\,dz .$$

We have then proved that for every $x\geq X_n$

$$v_n(x) \leq v_n(X_n)\exp\{2\mu_n\int_{X_n}^\infty \frac{1}{z}\rho(z)\,dz\}.$$

On the other hand, by the eigenfunction expansion for the Green function, we have

$$\frac{v_n(x)^2}{\mu_n + 1} \leq u_1(x)u_2(x),$$

where $u_1$ [$u_2$] is the positive and non-decreasing [non-increasing, respectively] solution for $u=\underline{G}u$ whose Wronskian

$$u_1^+(x)u_2(x) - u_1(x)u_2^+(x)$$

is equal to 1. Moreover it is known ([13],[19]) that, under the boundary conditions, $u_1^+$ and $u_2$ are bounded functions. Therefore there exists a constant C such that

$$|v_n(x)| \leq C(1 + \mu_n)^{1/2} X_n^{1/2}$$

holds for $1 \leq x \leq X_n$.

Now we have, for all $x \in [1,\infty)$,

$$|v_n(x)| \leq C(1 + \mu_n)^{1/2} X_n^{1/2} \exp\{2\mu_n \int_{X_n}^{\infty} \frac{1}{z} \rho(z)dz\}.$$

Furthermore, since $\rho(x)$ is monotone decreasing, we have

$$\log X_n = 4\mu_n \rho(X_n) \log X_n$$

$$\leq 8\mu_n \int_{\sqrt{X_n}}^{X_n} \frac{1}{z} \rho(z)\, dz.$$

Now the fact that $\mu_n$ and $X_n$ tend to $\infty$ as n tends to $\infty$ implies

$$\limsup_{n\to\infty} \frac{1}{\mu_n} \log\left( \sup_{x\in[1,\infty)} |v_n(x)|\right) \leq 0.$$

Finally we prove

(4.6) $$\liminf_{n\to\infty} \frac{1}{\mu_n} \log\left( \sup_{x\in[1,\infty)} |v_n(x)|\right) \geq 0.$$

Let us assume the contrary. Then there exist $\varepsilon>0$ and a subsequence $\{n(i)\}_{i=1}^{\infty}$ tending to $\infty$ such that

$$\sup_x |v_{n(i)}(x)| \leq \exp(-\varepsilon\mu_{n(i)}).$$

For such $n(i)$ we have

$$1 = \|v_{n(i)}\|^2_{L^2(dM)} \leq \exp(-2\varepsilon\mu_{n(i)})\, dM([1,\infty)),$$

which gives a contradiction for large $n(i)$. Therefore we have proved (4.6) and the assertion of Theorem 4.1.

$$\log X_n = 4\mu_n \rho(X_n) \log X_n$$

$$\leq 8\mu_n \int_{\sqrt{X_n}}^{X_n} \frac{1}{z}\, \rho(z)\, dz.$$

Now the fact that $\mu_n$ and $X_n$ tend to $\infty$ as n tends to $\infty$ implies (4.4).

Remark 4.4 The above proof shows that the assumption (4.3)(ii) can be replaced by

$$\text{(ii)}' \quad \lim_{x\uparrow\infty} (\log x) \int_x^\infty dy \int_y^\infty dM(z) = 0.$$

We close this section by giving some examples, where we assume that dM has its density m(x) and that (4.3) holds.

Example 1.(cf.[9]) Assume that $m(x) \leq x^{-2-\varepsilon}$ for large x and for $0<\varepsilon<2$. Then $X_n$ is defined by

$$4\mu_n X_n \int_{X_n}^\infty x^{-2-\varepsilon}\, dx = 1$$

and we see $X_n = O(\mu_n^{1/\varepsilon})$. Therefore in this case we have

$$|v_n(x)| \leq \text{const} \times (1 + \mu_n)^{(1+\varepsilon)/2\varepsilon}.$$

Example 2.(cf. [15]) Assume that there exist a positive integer k and $\alpha>1$ such that

$$m(x) \le x^{-2}(\log x)^{-1}\ldots(\log_{k-1} x)^{-1}(\log_k x)^{-\alpha} \quad ( = \tilde{m}(x),\ \text{say})$$

holds for every sufficiently large x. Then, using

$$2\mu_n X_n \int_{X_n}^{\infty} \tilde{m}(x)\, dx = 1,$$

direct calculations show that

$$\log X_n = O(\mu_n(\log_{k-1} \mu_n)^{1-\alpha})$$

and

$$|v_n(x)| \le \exp\{\text{const} \times \mu_n(\log_{k-1}\mu_n)^{1-\alpha}\}.$$

## 5. Proof of Theorems 3.2 and 3.4

In this section we will prove Theorems 3.2 and 3.4 by using the result in the previous section. Before giving the complete proof, we give a formula for the expectation of the number of the ranges of finite sets under the stochastic flows, which was found by Harris [8]. This formula not only shows the connection between (1.3) and one-dimensional diffusion processes but also it will be used in the proof of Theorem 3.5 (Section 6) and of the same results for spatially inhomogeneous flows (Section 7).

Lemma 5.1. For any $x_1,x_2,\ldots,x_n$ with $x_1<x_2<\ldots<x_n$, we have

$$(5.1)\qquad E[\#X_t(\{x_1,x_2,\ldots,x_n\})] = 1 + \sum_{k=1}^{n-1} P(X_t(x_{k+1}) > X_t(x_k)).$$

In particular, if we set $D_n=\{k2^{-n};k=0,1,\ldots,2^n\}$, we have

$$(5.2)\qquad E[\#X_{0,t}(D_n)] = 1 + 2^n Q_{2^{-n}}(\eta_t>0),$$

where E denotes the expectation with respect to P and Q is the probability law of the diffusion process $\eta$ mentioned in Lemma 3.1.

Now we are in the position to give the proof of Theorem 3.2.

Proof of Theorem 3.2. By the spatial homogeneity it is sufficient to show in the case K=[0,1] and to show that $Q_x(\eta_t>0)\le Cx$ for some C.

At first we note that, denoting by $\xi=\{\xi_t\}_{t\geq 0}$ the diffusion process on $(0,1]$ generated by

$$B = (1-b(z))\frac{d^2}{dx^2}$$

whose boundaries 0 and 1 are absorbing and reflecting, respectively. Moreover the assumption (iii) in (2.1), the strictly positive definiteness of b, implies that $b(z)=1$ if and only if $z=0$ and, therefore ,that the boundary 1 is regular in the sense of Feller. Then we have, denoting by R the probability law for $\xi$,

$$\begin{aligned} Q_x(\eta_t > 0) &= Q_x(\eta_t > 0 \text{ and } 0 < \eta_s < 1 \text{ for all } s \text{ with } 0 < s \leq t) \\ &\quad + Q_x(\eta_t > 0 \text{ and } \eta_s = 1 \text{ for some } s \text{ with } 0 < s \leq t) \\ &\leq R_x(\xi_t > 0) + x. \end{aligned}$$

Therefore, combining this with Lemma 5.1, if we can prove that there exists a constant $C>0$ such that

$$R_x(\xi_t > 0) \leq Cx$$

holds for any x, the proof is completed.

Now let us denote by $p(t,x,y)$ the transition probability density of $\xi$ with respect to its speed measure $(1-b(z))^{-1}dz$. Then we have its eigenfunction expansion which we will denote by

$$p(t,x,y) = \sum_{n=1}^{\infty} e^{-\lambda_n t} e_n(x)e_n(y),$$

so $e_n$ satisfies $Be_n=-\lambda_n e_n$, $\lim_{x\downarrow 0} e_n(x)=\lim_{x\uparrow 1} e_n^+(x)=0$. Since $e_n^+(0)=\lim_{x\downarrow 0} e_n(x)$ exists for each n, we assume $e_n^+(0)>0$.

Moreover we have

$$R_x(\eta_t > 0) = \int_0^1 p(t,x,y)(1 - b(y))^{-1}dy.$$

Now we prove the first assertion of Theorem 3.2, which says

$$\text{(5.3)} \qquad \lim_{n\uparrow\infty} \frac{1}{\lambda_n} \log M_n = 0.$$

It is easy to see that (5.3) implies (5.1) and assertion of the second assertion of the theorem.

To see (5.3), define $u_n(x)=e_n(x)/x$. Then it is easy to see that $u_n$ is an eigenfunction corresponding to the eigenvalue $-\lambda_n$ for

$$A = \frac{1 - b(x)}{x^2} \frac{d}{dx} (x^2 \frac{d}{dx})$$

under the boundary conditions

$$\lim_{x\downarrow 0} x^2u_n^+(x) = \lim_{x\uparrow 1} (x^2u_n^+(x) + u_n(x)) = 0.$$

Moreover, to change the scale into the Lebesgue measure, set $\tilde{u}_n(x)=u_n(x^{-1})$ for $x\geq 1$. We have that $\tilde{u}_n$ is an eigenfunction with eigenvalue $-\lambda_n$ for

$$A_1 = x^4(1 - b(x^{-1}))\frac{d^2}{dx^2}$$

under the boundary conditions

$$\lim_{x\uparrow\infty} \tilde{u}_n^+(x) = \lim_{x\downarrow 1} (\tilde{u}_n(x) + \tilde{u}_n^+(x)) = 0.$$

Since the assumption is translated into (4.3) under the above transformation ~~translation~~, we can use the result in the previous section and have proved (5.3). The proof is completed.

Secondly we will give the proof of Theorem 3.4.

Proof of Theorem 3.4. By virtue of Theorem 3.2 and the right-continuity of $X_t(\cdot)$, we see that $X_t$ is a step function on $R^1$ with finite steps on each compact interval. Now denote by $\{\alpha_i\}$ the end point of the steps of $X_t(\cdot)$ and set $\beta_i = X_t(\alpha_i)$ according to the notation in [8]. Then Harris [8] has proved that, for any fixed $t>0$, the probability law of the random sets $\{\alpha_i\}$ and $\{\beta_i\}$ are the same. In particular, we have that

$$E[{}^{\#}X_{t_0+s}(R^1)\cap I] = E[{}^{\#}X_{t_0+s}(I)] \leq c_{t_0+s}|I|,$$

where $c_t$ is the constant which appeared in Theorem 3.2. From this the assertion of Theorem 3.4 follows.

Before closing this section we give several examples related to those mentioned in the last part of the previous section. Now we assume that b can be written as a Fourier transform of a probability measure with density, that is,

$$b(z) = \int_{-\infty}^{\infty} e^{iz\xi} f(\xi)d\xi = 2\int_{0}^{\infty} \cos(z\xi)f(\xi)d\xi.$$

<u>Example 1.(cf. Harris [8])</u> Let us assume that $f(\xi)=O(\xi^{-3+\varepsilon})$ with $0<\varepsilon<2$ as $\xi$ tends to $\infty$. Then we see easily that

$$1 - b(z) \geq c|z|^{2-\varepsilon}$$

in some neighborhood of $z=0$ for some $c>0$ and b satisfies the assumption in Section 3.

<u>Example 2.</u> Let us set, for a positive integer k and $\alpha>1$,

$$L(\xi) = \xi^{-3}(\log_2\xi)\ldots(\log_{k-1}\xi)(\log_k\xi)^{\alpha}$$

and assume that

$$f(\xi) = O(L(\xi))$$

as $\xi$ tends to $\infty$. Then we can show that

$$(5.4) \qquad 1 - b(z) \geq z^2(\log z^{-1})\ldots(\log_{k-1} z^{-1})(\log_k z^{-1})^{\alpha}$$

$$= z^{-1}(\log z^{-1})L(z^{-1})$$

holds in some neighborhood of $z=0$. Therefore, also in this case, the assumption is satisfied and we have (1.3). (5.4) can be seen as follows. Fix $\delta>0$ satisfying $z\delta<1$, then direct calculations show

$$\int_0^{\delta} (1 - \cos(z\xi))f(\xi)d\xi \leq C_1 z^2\delta^2$$

and

$$\int_{1/z}^{\infty} (1 - \cos(z\xi))f(\xi)d\xi$$

$$\leq C_2 \int_{1/z}^{\infty} L(\xi)d\xi$$

$$= o(z^2(\log z^{-1})\ldots(\log_{k-1} z^{-1})(\log_k z^{-1})^{\alpha}).$$

Moreover the integration by parts shows

$$\int_{\delta}^{1/z} (1 - \cos(z\xi))f(\xi)d\xi$$

$$\geq C_3 z^2 \int_{\delta}^{1/z} \frac{1 - \cos(z\xi)}{z^2\xi^2} \xi^2 L(\xi)d\xi$$

$$\geq C_4 z^2 \int_{\delta}^{1/z} \xi^{-1}(\log_2\xi)\ldots(\log_{k-1}\xi)(\log_k\xi)^{\alpha}d\xi$$

$$= C_4 z^{-1}(\log z^{-1})L(z^{-1}) + \text{the negligible terms.}$$

Here $C_i'$s are positive constants. Therefore, since the first two integrals are also negligible and the last is the main term, we have (5.4).

## 6. Proof of Theorem 3.5

Before giving the complete proof of Theorem 3.5, we state a key lemma for it. The statement is stronger than is needed for the proof of Theorem 3.5 but this stronger result will be needed for the proof of the similar result on the spatially inhomogeneous stochastic flows studied in Section 7.

Lemma 6.1. Let $\eta$ be the one-dimensional diffusion process mentioned in Lemma 3.1. Then for any $t>0$ and $\varepsilon>0$, it holds that

and assume that the integral in (3.1) is infinite

$$\lim_{x \downarrow 0} \frac{1}{x} Q_x(\sigma_\varepsilon \geq t) = 0,$$

$\leqq$

where $\sigma_\varepsilon$ is the first hitting time to $\varepsilon$ of $\eta$. In particular, we have

$$\lim_{x \downarrow 0} \frac{1}{x} Q_x(\eta_t > 0) = 0.$$

$\geqq \varepsilon$

Proof. First of all we have, by Tchebycheff's inequality,

$$Q_x(\sigma_\varepsilon \leq t) \leq e^{\lambda t} E_x[\exp(-\lambda\sigma_\varepsilon)]$$

for all $\lambda>0$, where $E_x$ denotes the expectation with respect to $Q_x$. Then it is known ([13],p.129) that $u(x)=E_x[\exp(-\lambda\sigma_\varepsilon)]$ , $0<x<\varepsilon$, is an increasing solution of

$$(1-b(x)) \frac{d^2u}{dx^2} = \lambda u.$$

Such solutions are unique up to the multiplicative constant and, moreover, it is known (see, e.g., [13] or [19]) that, since the boundary 0 is natural in the sense of Feller,

$$\lim_{x \downarrow 0} u(x) = \lim_{x \downarrow 0} \frac{1}{x} u(x) = 0,$$

which immediately implies that $u(x)=o(x)$ as x tends to 0 and the assertion of the lemma.

Now we are in the position to give the complete proof of Theorem 3.5.

Proof of Theorem 3.5. Firstly we prove the one-to-one property of $X_t=X_{0,t}$.

We note that, under the integral condition,

$$\int_{0+} m(z)zdz = \infty, \qquad \int_{0+} \frac{z\,dz}{1-b(z)} = \infty,$$

the boundary 0 of $\eta=\{\eta_t\}_{t\geq 0}$ is natural and, therefore, that $Q_x(\eta_t>0)=1$ for any $x>0$. Then we have, by (5.2),

$$E[{}^{\#}X_t(D_n)] = 1 + 2^n$$

for any n. But, since ${}^{\#}X_t(D_n) \leq 1+2^n$, it means that ${}^{\#}X_t(D_n)=1+2^n$ with probability 1 for any n, which, by virtue of the monotonicity of $X_t(\cdot)$, implies the one-to-one property of $X_t$.

Secondly we prove the continuity of $X_t(x)$ in x on (0,1). We have, by using the monotonicity of $X_t$, that, for any $\varepsilon>0$ and sufficiently small $\delta>0$,

$$P(\sup\{X_t(y) - X_t(x); 0 \le x \le y \le 1, |y-x| < \delta\} \ge \varepsilon)$$

$$= P(\sup\{X_t(x+\delta) - X_t(x); 0 \le x \le 1 - \delta\} \ge \varepsilon)$$

$$\le P(\sup\{X_t((k+2)\delta) - X_t(k\delta); k=0,1,\dots,[\tfrac{1}{\delta}]\} \ge \varepsilon)$$

$$\le \sum_{k=0}^{[1/\delta]} P(X_t((k+\underset{2}{1})\delta) - X_t(k\delta) \ge \varepsilon)$$

$$\le (1 + \frac{1}{\delta}) \underset{2\delta}{Q_{\delta}}(\eta_t \underset{\ge \varepsilon}{>0}).$$

Now, applying Lemma 6.1, these can be arbitrary small if we choose $\delta$ sufficiently small. Therefore $X_t(x)$ is continuous in x on (0,1) with probability 1. By using the spatial homogeneity of $X_t(x)$, we have proved the continuity of $X_t(x)$ in x on $R^1$.

Finally we prove the onto property of $X_t$. To show this we note, as in Section 4 of Harris [8], that

$$(5.3) \qquad \sum_{n=1}^{\infty} P(X_t(n) \le a) < \infty \text{ and } \sum_{n=1}^{\infty} P(X_t(-n) \ge -a) < \infty$$

for any $a>0$, which can be easily seen because $\{X_t(x)\}_{t\ge 0}$ is a standard Brownian motion on R starting from x. By Borel-Cantelli's lemma, we have

$$\lim_{x \to \pm\infty} X_t(x) = \pm\infty \quad \text{with probability 1,}$$

which, combined with the two properties proved above, implies the onto property of $X_t$. The proof is completed.

7. Spatially inhomogeneous stochastic flows

In this section we will consider spatially inhomogeneous stochastic flows whose local characteristics cannot be written as $b(x-y)$. We will see that there exist flows which have the same property mentioned in Theorems 3.2 and 3.5. For this we let a be a function on $R^1 \times R^1$ which satisfies (2.1) and denote by $X=\{X_{s,t}; 0 \le s \le t < \infty\}$ the stochastic flow with its local characteristic a.

To study the property of X, the process $\eta=\{X_{0,t}(x)-X_{0,t}(y)\}_{t \ge 0}$, which is a diffusion process for a spatially homogeneous flows, plays an important role as was seen in Theorems mentioned in Section 3 and their proofs. Here we consider the case that, roughly speaking, $\eta$ is bounded from above or below by some diffusion process as appeared in the spatially homogeneous flows. We will apply the comparison theorems studied by Ikeda-Watanabe ([10],[11]). For this purpose we define functions $b^*$ and $b_*$ on $(0,\infty)$ by

$$b^*(\zeta) = \sup_{\xi \in R} \alpha(\xi,\zeta) \quad \text{and} \quad b_*(\zeta) = \inf_{\xi \in R} \alpha(\xi,\zeta),$$

respectively, where

$$\alpha(\xi,\zeta) = \frac{1}{2} a(\xi,\xi) - a(\xi,\xi-\zeta) + \frac{1}{2} a(\xi-\zeta,\xi-\zeta).$$

Then the main result in this section is the following. Hereafter we assume that $b^*$ and $b_*$ are locally Lipschitz continuous outside each neighborhood of 0.

Theorem 7.1. (1)(1.3) holds for the stochastic flow X if the assumption mentioned in Section 3 holds by replacing $1-b(z)$ with $b_*(z)$. (2)(1.4) holds for X if

$$\int_{0+} b^*(\zeta)^{-1}\zeta d\zeta = \infty. \tag{7.1}$$

Proof.(1) For $x,y \in R^1$ with $x>y$, let us consider the 2-point process $X(x,y)=\{X_t(x),X_t(y)\}_{t\geq 0}$, where $X_t=X_{0,t}$. Then it is a diffusion process on $R^2$ generated by the operator

$$L = \frac{1}{2}\, a(x_1,x_1)\frac{\partial^2}{\partial x_1^2} + a(x_1,x_2)\frac{\partial^2}{\partial x_1 \partial x_2} + \frac{1}{2}\, a(x_2,x_2)\frac{\partial^2}{\partial x_2^2}$$

up to the coalecing time $\tau=\inf\{t\geq 0; X_t(x)=X_t(y)\}$.

At first we note that, by virtue of the assumption (2.1), there exists a 2×2-matrix valued function $\sigma=\sigma(x_1,x_2)$ on $R^1\times R^1$ which is bounded, locally Lipschitz continuous outside each neighborhood of the diagonal set and satisfies $\sigma\sigma^*=A$. Here $A=(a(x_i,x_j))_{1\leq i,j\leq 2}$. Let us consider the stochastic differential equation

$$dx_t = \sigma(x_t)dw_t\ , \quad x_0 = (x,y),$$

on $D=\{(x_1,x_2); x_1>x_2\}$ under the assumption that $\partial D$, the diagonal set, is the absorbing boundary. Here $\{w_t\}_{t\geq 0}$ is the standard 2-dimensional Brownian motion. Then a solution $\tilde{X}(x,y)= \{(\tilde{X}_t(x),\tilde{X}_t(y))\}_{t\geq 0}$ exists and the pathwise uniqueness holds by virtue of the conditions for $\sigma$ above. $\tilde{X}(x,y)$ is the minimal

L-diffusion on D and we have, denoting by $P_1$ the probability law of $\tilde{X}(x,y)$,

$$P(X_t(x) > X_t(y)) = P_1(\tilde{X}_t(x) > \tilde{X}_t(y))$$

Now let $\eta^- = \{\eta_t^-\}_{t \geq 0}$ be the diffusion process on $(0,\infty)$ generated by

$$\frac{1}{2}\, b_*(z) \frac{d^2}{dz^2}$$

with absorbing boundary 0 and apply the comparison theorem of Ikeda-Watanabe, Theorem 3.1 in [10] or Theorem 4.1 in Chapter 6 of [11], which holds for the minimal diffusion processes on a domain as is pointed out in [10]. Then we can construct the processes, for which we will use the same notations, with the same probability laws on some probability space $(\tilde{\Omega}, \tilde{F}, \tilde{P})$ such that

$$\min_{0 \leq s \leq t} \tilde{\eta}_s \leq \min_{0 \leq s \leq t} \eta_s^-$$

for $t < \tilde{\tau}$, where $\tilde{\tau}$ is the first hitting time of $\tilde{X}(x,y)$ to $\partial D$ and $\tilde{\eta}_s = \tilde{X}_s(x) - \tilde{X}_s(y)$. Therefore we get

$$P(X_t(x) > X_t(y)) = \tilde{P}(\tilde{X}_t(x) > \tilde{X}_t(y))$$

$$= \tilde{P}(\min_{0 \leq s \leq t} \tilde{\eta}_s > 0)$$

$$\le \tilde{P}( \min_{0\le s\le t} \eta_s^- > 0 )$$

$$= Q_{x-y}^-(\eta_t^- > 0),$$

where $Q^-$ is the probability law of $\eta^-$. Therefore, since (5.1) holds for the spatially inhomogeneous flows, we obtain

$$E[{}^{\#}X_t(D_n)] \le 1 + 2^n Q_{2^{-n}}^-(\eta_t^- > 0).$$

Since $b_*$ satisfies the assumption mentioned in Section 3, we have $Q_z^-(\eta_t^- > 0) \le cz$ for some $c>0$ and this implies the assertion.

(2) Let $x,y\in R^1$ with $x>y$ and denote by $\eta^+=\{\eta_t^+\}_{t\ge 0}$ the diffusion process starting from x-y generated by the operator

$$\frac{1}{2} b^*(z) \frac{d^2}{dz^2}.$$

Then the Ikeda-Watanabe's comparison theorem shows that

$$\min_{0\le s\le t} \eta_s^+ \le \min_{0\le s\le t} \tilde{\eta}_s$$

for $t<\tilde{\tau}$ if we construct the processes with the same probability law as those of the above two processes, for which we have used the same notation. But, under our integral condition (7.1), the boundary 0 of $\eta^+$ is natural in the sense of Feller and so, since $\eta_s^+>0$ for any s almost surely, we have

$$\tilde{P}(\min_{0\le s\le t} \tilde{\eta}_s > 0 \text{ for any } t > 0) = 1$$

for any x,y with x>y, which implies the one-to-one property of $X_{0,t}$ by the similar argument to that of the proof of the spatially homogeneous flows.

To prove the continuity of $X_t(x)$ in x, we first note that the integral condition implies that

$$\int_{0+} b_*(\zeta)^{-1}\zeta d\zeta = \infty.$$

Therefore, for any $\varepsilon>0$, we have, by the comparison theorem, that

$$P(X_t(z+\delta) - X_t(z) \ge \varepsilon)$$

$$= \tilde{P}(\tilde{X}_t(z+\delta) - \tilde{X}_t(z) \ge \varepsilon)$$

$$\le P(\max_{0\le s\le t} (\tilde{X}_s(z+\delta) - \tilde{X}_s(z) \ge \varepsilon)$$

$$\le Q_\delta^-(\max_{0\le s\le t} \eta_s^- \ge \varepsilon)$$

$$= Q_\delta^-(\sigma_\varepsilon \le t),$$

for sufficiently small $\delta>0$, where $\sigma_\varepsilon$ is the first hitting time of $\eta^-$ to $\varepsilon$. Moreover we have proved in Lemma 6.1 that

$$\lim_{\delta\downarrow 0} \frac{1}{\delta} Q_\delta^-(\sigma_\varepsilon \le t) = 0.$$

Now, tracting the argument in the proof of Theorem 3.5, we get

$$\lim_{\delta \downarrow 0} P(\sup(X_t(x) - X_t(y); 0 \le y \le x \le 1 \text{ and } |y - x| < \delta) \ge \varepsilon)$$

$$\le \lim_{\delta \downarrow 0} \frac{1}{\delta} Q^{-}_{2\delta}(\sigma_\varepsilon \le t) = 0,$$

which implies the continuity of $X_t(x)$ in $x \in (0,1)$. Therefore we have proved that $X_t(x)$ is continuous in $x \in R^1$ with probability 1.

The onto property can be shown by comparing the 1-point process $\{X_t(x)\}_{t \ge 0}$ with a Brownian motion by means of the boundedness of $a(x,x)$ and by the same argument as in the proof of Theorem 3.5. The proof is completed.

Finally we would like to remark on the connection between Theorem 7.1 (2) and the known result for the solutions of one-dimensional stochastic differential equations. For this purpose let us consider the following;

$$(7.2) \qquad \begin{aligned} dx_t &= \sigma(x_t)dB_t, \quad t \ge s \\ x_s &= x, \end{aligned}$$

where $\{B_t\}_{t \ge 0}$ is a one-dimensional Brownian motion. Following Yamada-Ogura [20], we assume that

$$|\sigma(x) - \sigma(y)| \le \rho(|x - y|)$$

holds for some bounded continuous function $\rho$ satisfying $\rho(0)=0$ and

$$(7.3) \qquad \int_{0+} \rho(\eta)^{-2}\, \eta d\eta = \infty.$$

Although the local characteristic a for the stochastic flow constructed from the solution (7.2) should be given by

$$a(x,y) = \sigma(x)\sigma(y)$$

and this is not strictly positive definite in general, we can construct the stochastic flow $X=\{X_{s,t}\}_{t\geq s}$ with a as its local characteristic by virtue of the simple form of a. The 1-point process $\{X_{s,t}(x)\}_{t\geq s}$ has the same probability law as that of the solution of (7.2).

Under the assumption above, Yamada-Ogura [20] has shown that, for any $x>y$, $X_{s,t}(x)>X_{s,t}(y)$ holds for any $t\geq s$ and $X_{s,t}$ is a one-to-one map with probability 1. Now note that (7.3) implies

$$\int_{0+} (b^*(\eta))^{-1}\eta d\eta = \infty.$$

Therefore, by means of Theorem 7.1 (2), $X_{s,t}$ is a homeomorphism of $R^1$ with probability 1.

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(1) $\log_2 x = \log(\log x)$, $\log_k x = \log(\log_{k-1} x)$, $k=3,4,\ldots$

~~(2) There is no meaning in that the right end point is 1. The proof below still works however small this interval is.~~