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One-to-one Continuous Mappings on Locally Compact Spaces

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It is a classical theorem of set-theoretical topology that a one-to-one continuous mapping φ of a bicompact Hausdorff space X onto a Hausdorff space Y is a homeomorphism[1]. But, in general, it cannot be said that such a mapping φ is homeomorphic if the spaces X and Y are both locally compact. In this paper we consider the problem for locally compact spaces with some special conditions.

Throughout this paper, we shall use the word "space" for "Hausdorff space". Let X and Y be two spaces, and let Y be a one-to-one continuous image of X under a mapping φ. If the inverse mapping φ^{-1} is not continuous, there exists a point x in X and a neighborhood U of x such that $\varphi(U)$ is not a neighborhood of $\varphi(x)$ in Y. Let D_φ denote the set of all such points x in X, i.e., D_φ is the set of all points x in X such that the inverse mapping $\varphi^{-1}(y)$ are not continuous at the points $\varphi(x)$. If D_φ is empty, the mapping φ is a homeomorphism from X to Y. When X is locally compact, a subset of X will be said to be bounded if the subset is contained in a bicompact subset of X. A° denotes always the interior of A for any subset A in X.

We shall prove the following theorem.

Theorem. *Let X be a locally compact Hausdorff space, and let X be represented as a union $\sum_{i=1}^{\infty} X_i$, where for each i X_i is bicompact, $X_i \subset X_{i+1}^\circ$ and $X-X_i$ is connected. Let Y be a locally compact but not bicompact and Hausdorff space, and let it be a one-to-one continuous image of X under φ. If the set D_φ is bounded in X, then the mapping φ is a homeomorphism.*

We shall prove first the following lemmas.

Lemma 1. *Let X be a locally compact space, and let φ be a one-to-one continuous mapping from X onto a space Y. Then the image $\varphi(D_\varphi)$ of D_φ under φ is closed in Y, and therefore D_φ is closed in X.*

Proof. Suppose that $\varphi(D_\varphi)$ is not closed in Y. Then there exists

1) See, for example, [1] p. 95, Satz III. Numbers in brackets refer to the references cited at the end of the paper.

a point p_0 in X such that $\overline{\varphi(D_\varphi)}-\varphi(D_\varphi)\ni\varphi(p_0)$. Since $D_\varphi \not\ni p_0$, there is a bicompact neighborhood U of p_0 such that $\varphi(U)$ is a neighborhood of $\varphi(p_0)$. It follows from $\overline{\varphi(D_\varphi)}\ni\varphi(p_0)$ that there exists a point p in D_φ such that $\varphi(U)$ is a neighborhood of $\varphi(p)$. But since $p\in D_\varphi$, there is a phalaux $\{p_\alpha\}$ in X such that $\{p_\alpha\}$ does not converge to p in X and $\{\varphi(p_\alpha)\}$ converges to $\varphi(p)$ in Y. We may here assume that the phalaux $\{p_\alpha\}$ has not p as a cluster point. (cf. [5]). Since $\varphi(U)$ is a neighborhood of $\varphi(p)$ and $\varphi(p_\alpha)$ converges to $\varphi(p)$, the subset U contains the point p_α if α is greater than a certain α_0. This is a contradiction by the compactness of U.

Lemma 2. *Let X be a locally compact space, and let it be represented as a union $\sum_{i=1}^{\infty} X_i$, where for each i X_i is bicompact, $X_i \subset X_{i+1}^{\circ}$ and $X-X_i$ is connected. If Y is a bicompact space and if φ is a one-to-one continuous mapping from X onto Y, then the image $\varphi(D_\varphi)$ of D_φ under φ is connected. Therefore D_φ is connected if it is bounded.*

Proof. If $\varphi(D_\varphi)$ is not connected, it follows from Lemma 1 that $\varphi(D_\varphi)$ is a sum of two non void mutually disjoint subsets B_1 and B_2 which are both closed in Y. If $\varphi(p)$ and $\varphi(q)$ are fixed points of B_1 and B_2 respectively, then there exist two phalaux $\{p_\alpha\}$ and $\{q_\beta\}$ in X, where $\{p_\alpha\}$ (or $\{q_\beta\}$) does not converge to p (or q) and $\{\varphi(p_\alpha)\}$ (or $\{\varphi(q_\beta)\}$) converges to $\varphi(p)$ (or $\varphi(q)$). Without any loss of generality, we may assume that $\{p_\alpha\}$ (or $\{q_\beta\}$) has not p (or q) as a cluster point and that the subset $A_1=B_1\cup\{\varphi(p_\alpha)\}$ and the subset $A_2=B_2\cup\{\varphi(q_\beta)\}$ are mutually disjoint since $\varphi(p)\neq\varphi(q)$. Since A_1 and A_2 are both closed in Y and Y is normal, there exists a real valued continuous function f on Y such that

$$\text{if} \quad y\in A_1, \quad f(y)=0$$

and

$$\text{if} \quad y\in A_2, \quad f(y)=1\,.$$

Since X_i is bicompact for each i, there are a point $p_{\alpha(i)}$ in the phalaux $\{p_\alpha\}$ and a point $q_{\beta(i)}$ in the sequence $\{q_\beta\}$ such that $X-X_i\ni p_{\alpha(i)}$ and $X-X_i\ni q_{\beta(i)}$. Since $Y-\varphi(X_i)$ is connected, there exists a point r_i such that $r_i\in X-X_i$ and $f(\varphi(r_i))=\frac{1}{2}$ for each i. The sequence $\{\varphi(r_i)\}$ is contained in the bicompact space Y, therefore the sequence $\{\varphi(r_i)\}$ has a cluster point $\varphi(r)$ and we see that $f(\varphi(r))=\frac{1}{2}$. If a point y in Y is contained in $\varphi(D_\varphi)$, the valve of $f(y)$ is equal to zero or to one. Hence $r\notin D_\varphi$. On the other hand, we see easily that the sequence $\{r_i\}$ has not r as a cluster point on account of the conditions of the family $\{X_i\}$. Therefore $r \in D_\varphi$. This contradiction shows that $\varphi(D_\varphi)$ is connected.

E. Hewitt[2] proved the following lemma.

Lemma 3. *If X is a locally compact space, there exist a bicompact space Y and a one-to-one continuous mapping from X onto Y such that D_φ contains only one point.*

We shall next prove our theorem.

Proof of Theorem. Let X be a locally compact Hausdorff space, and let it be represented as a union $\sum_{i=1}^{\infty} X_i$, where for each i X_i is bicompact, $X_i \subset X_{i+1}^{\circ}$ and $X - X_i$ is connected. Let the space Y be locally compact but not bicompact, and let it be a one-to-one continuous image of X under φ. Since D_φ is bounded, there exists a point y_0 such that $y_0 \in Y - \varphi(D_\varphi)$. It follows from Lemma 3 that there are a bicompact (Hausdorff) space Z and a one-to-one continuous mapping ψ from Y onto Z such that D_ψ contains only one point y_0. The mapping $\psi\varphi$ is one-to-one and continuous from X onto Z, therefore set $D_{\psi\varphi}$ in X can be defined and it is obvious that $D_{\psi\varphi} = D_\varphi \cup \varphi^{-1}(D_\psi) = D_\varphi \cup \varphi^{-1}(y_0)$. We see here that D_φ is closed in X, $D_\varphi \not\ni \varphi^{-1}(y_0)$ and $D_{\psi\varphi}$ is connected from Lemma 1 and Lemma 2. Therefore the subset D_φ must be empty. This fact shows that φ is a homeomorphism.

REMARK. (1) In the theorem, the boundedness of D_φ is necessary. If X is a 2-dimensional euclidean space with the ordinary topology, it satisfies the conditions of the theorem. We next introduce new neighborhoods $U_n(p)$ of a point p in X as follows:

If p is a point of the form $(0, y)$ in X, we put

$$U_n(p) = E\left\{(u, v) \mid |u| < \frac{1}{n}, \quad |v-y| < \frac{1}{n}\right\}$$
$$\cup E\left\{(u, v) \mid |u| > n, \quad |v-y| < \frac{1}{n}\right\},$$

otherwise,

$$U_n(p) = E\left\{(u, v) \mid |u-x| < \frac{1}{n}, \quad |v-y| < \frac{1}{n}\right\},$$

and let Y be the space with this topology. Let φ be the identical mapping from X to Y. Then the space Y is locally compact but not bicompact and φ is continuous. Here D_φ is not bounded and the mapping φ fails to be homeomorphic.

2) See Hewitt [2]. R. Sikorski shows also that if X is a locally compact separable space, there exist a compact metrizable space Y and a one-to-one continuous mapping from X onto Y such that D_φ contains only one point. (cf. [3] or [4]).

(2) If $n \geqslant 2$, an n-dimensional euclidean space $X = R^n$ satisfies the conditions of the theorem, but the theorem is false in the case of a 1-dimensional euclidean space R^1. We next construct new neighborhoods $U_n(p)$ of a point in X $(= R^1)$ as follows:

If $p = 0$, we put

$$U_n(p) = E\left\{y \mid y \in X \text{ and } |y| < \frac{1}{n}\right\} \cup E\{y \mid y \in X \text{ and } y > n\},$$

otherwise,

$$U_n(p) = E\left\{y \mid y \in X \text{ and } |y - p| < \frac{1}{n}\right\},$$

and let Y be the space with this topology. Then the identical mapping φ from X to Y is continuous. We see here that $D_\varphi = \{0\}$ is bounded in X but the mapping φ fails to be homeomorphic.

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References

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