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JØRGENSEN SUBGROUPS OF THE PICARD GROUP

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Abstract

Let G be a subgroup of rank two of the Möbius group $PSL(2,\mathbb{C})$. The Jørgensen number $J(G)$ of G is defined by

$$J(G) = \inf\{|\mathrm{tr}^2 A - 4| + |\mathrm{tr}[A, B] - 2| : \langle A, B\rangle = G\}.$$

We describ e all subgroups G of the Picard group $PSL(2, \mathbb{Z} + i\mathbb{Z})$ with $J(G) = 1$.

1. Introduction

Let G be a subgroup of rank two of the Möbius group Möb $= PSL(2,\mathbb{C})$. The Jørgensen number $J(G)$ of G is dened by

$$J(G) = \inf\{|\mathrm{tr}^2 A - 4| + |\mathrm{tr}[A, B] - 2| : \langle A, B\rangle = G\}.$$

A subgroup G of Möb is elementary if the cardinality of its limit set $\Lambda(G)$ is at most 2 see [8, p.266]. If $G = \langle A, B\rangle$ is a discrete group with A parabolic, then G is elementary iff $\mathrm{tr}[A, B] = 2$ (that is, iff $J(A, B) = 0$).

Jørgensen has proved that if G is a discrete nonelementary rank two subgroup of Möb then $J(G) > 1$.

It has been conjectured [10, p.273] that if G is nonelementary rank two subgroup of Möb which does not contain elliptic elements of infinite order and $J(G) = 1$ then G is discrete.

Groups G with $J(G) = 1$ have been studied in the literature ([3], [4], [13], [10], [12]). Following [10] we call a discrete nonelementary rank two subgroup G of Möb with $J(G) = 1$ a *Jørgensen group*.

An important subgroup of Möb is the Picard group Pic $= PSL(2, \mathbb{Z} + i\mathbb{Z})$. We are interested in the Jørgensen numbers of rank two subgroups of Pic.

Our motivation for the present paper is the article [12] by H. Sato in which he considers the Whitehead link group $\mathcal{W} = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1-i & 1 \end{pmatrix} \right\rangle \subset$ Pic (see [5], [9], [15]) and proves that $J(W) = 2$. Here we will give a brief proof of this result.

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We now describe a family of rank two subgroups of Pic. Let

$$\mathrm{Mod} = \mathrm{Mod}^1 = PSL(2,\mathbb{Z}) \simeq \mathbb{Z}_2 * \mathbb{Z}_3,$$

$$\mathrm{Mod}^i = \left\{ \begin{pmatrix} a & -ib \\ ic & d \end{pmatrix} \in PSL(2,\mathbb{C})\colon a,b,c,d \in \mathbb{Z} \right\} \simeq \mathbb{Z}_2 * \mathbb{Z}_3,$$

$$\mathcal{G}_k^{\alpha,\beta} = \left\langle \mathrm{Mod}^\alpha, \begin{pmatrix} \alpha\beta & -k\alpha^2\beta i \\ 0 & (\alpha\beta)^{-1} \end{pmatrix} \right\rangle \quad \text{where} \quad \alpha,\beta \in \{1,i\} \quad \text{and} \quad k \in \mathbb{Z}.$$

For example $\mathcal{G}_0^{1,1} = \mathrm{Mod}$, $\mathcal{G}_0^{i,i} = \mathrm{Mod}^i$ and one can show that $\mathcal{G}_1^{1,1} = \mathcal{G}_1^{i,i} = \mathrm{Pic}$.

The group Pic is generated by Mod and Mod^i, and these two subgroups are conjugate in Möb by a 90° rotation $R = \begin{pmatrix} (1+i)/\sqrt{2} & 0 \\ 0 & (1-i)/\sqrt{2} \end{pmatrix}$.

Denoting by $\mathbb{D}_\infty$ the infinite dihedral group, we will see that (Theorem 11) for $k \geq 2$ we have $\mathcal{G}_k^{\alpha,\beta} \simeq \begin{cases} \mathrm{Mod} *_{\mathbb{Z}} \mathbb{Z}^2 & \text{if } \alpha\beta = \pm 1 \\ \mathrm{Mod} *_{\mathbb{Z}} \mathbb{D}_\infty & \text{if } \alpha\beta = i \end{cases}$, where the element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of Mod is amalgamated to a primitive element of infinite order of $\mathbb{Z}^2$ or $\mathbb{D}_\infty$.

The symbol $\overset{\mathrm{Pic}}{\sim}$ denotes conjugation in Pic, the symbol $\sim$ denotes conjugation in Möb.

Our main result is the following:

Theorem 1. *Let* $\mathcal{I} = \{(\alpha,\beta,k)\colon \alpha,\beta \in \{1,i\},\ k \text{ a non negative integer}\}$. *Let G be a rank two subgroup of* Pic, *with* $J(G) = 1$. *Then*:

1) *G is conjugate, in* Pic, *to* $\mathcal{G}_k^{\alpha,\beta}$ *for some* $(\alpha,\beta,k) \in \mathcal{I}$.

2) *G is isomorphic to exactly one of the groups* $\mathcal{G}_0^{1,1}$, $\mathcal{G}_1^{1,1}$, $\mathcal{G}_2^{1,1}$, $\mathcal{G}_1^{1,i}$, $\mathcal{G}_1^{1,i}$ *and* $\mathcal{G}_2^{1,i}$.

3) *If* $\mathcal{G}_k^{\alpha,\beta} \overset{\mathrm{Pic}}{\sim} \mathcal{G}_{k'}^{\alpha',\beta'}$ *where* (α,β,k) *and* (α',β',k') *are different elements of* $\mathcal{I}$ *then* $k = k' = 1$, $\alpha = \beta$ *and* $\alpha' = \beta'$.

4) $\mathcal{G}_k^{\alpha,\beta} \sim \mathcal{G}_{k'}^{\alpha',\beta'}$ *(with* $(\alpha,\beta,k),(\alpha',\beta',k') \in \mathcal{I}$*) iff* $k = k'$ *and* $\alpha\beta = \pm\alpha'\beta'$.

Notice that no Jørgensen subgroup G of Pic is the group of a link in S^3, because $G \supset \mathbb{Z}_2 * \mathbb{Z}_3$.

In Section 1 we give another proof of Sato's theorem.

In Section 2 we give a different description of $\mathcal{G}_k^{\alpha,\beta}$ which shows its rank is two. With this description we extend our family to a family of rank two subgroups $\mathcal{G}_k^{\alpha,\beta}$ with α,β and $k \in \mathbb{C} - \{0\}$ and compare it with a family dened by Sato ([10], [12]). At the end of the section we prove Theorem 1 1).

In Section 3 we prove Theorem 1 4).

In Section 4, using the structure of Pic as an amalgamated product, we prove Theorem 1 3) and 2).

In Section 5 we exhibit a table that gives algebraic information of the groups $\mathcal{G}_k^{\alpha,\beta}$, as their abelianizations, their images under the abelianization map of Pic, and the number of conjugacy classes of elements of order two. These facts are proved in Sections 3 and 4 and are used in the proof of Theorem 1 3) 2).

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2. Section 1

Proposition 2. *If G is a rank two subgroup of* Pic *then* $J(G) \in \{0,1,2\} \cup [3,\infty)$.

Proof. Let A and $B \in GL(2,\mathbb{C})$ and $C = AB$. Then

$$\operatorname{tr}[A, B] - 2 = \operatorname{tr}^2 A + \operatorname{tr}^2 B + \operatorname{tr}^2 C - \operatorname{tr} A \operatorname{tr} B \operatorname{tr} C - 4$$

in particular if $\operatorname{tr} A = 2$ one has then $\operatorname{tr}[A, B] - 2 = (\operatorname{tr} C \operatorname{tr} B)^2$ and therefore $J(A, B) = |\operatorname{tr} C - \operatorname{tr} B|^2$. Hence if A is parabolic and $A, B \in$ Pic, we have that $J(A, B)$ is the modulus of the square of an element of $\mathbb{Z} + i\mathbb{Z}$, that is, an integer that is the sum of two squares. If $A \in$ Pic and A is not parabolic then $J(A,B) \geq |\operatorname{tr} A - 2|\,|\operatorname{tr} A + 2| \geq 3$. □

Proposition 3. *Let $\tilde{\phi}\colon PSL(2,\mathbb{Z}+i\mathbb{Z}) \to PSL(2,\mathbb{Z}_2)$ be the homomorphism induced by the ring homomorphism $\phi\colon \mathbb{Z} + i\mathbb{Z} \to \mathbb{Z}_2$. If G is a nonelementary e rank two subgroup of* Pic *and $|\tilde{\phi}(G)| < 6$ then $J(G) \geq 2$.*

Proof. Suppose $\mathcal{G} = \langle A, B\rangle$. Notice that $\tilde{\phi}(G)$ is Abelian since it is a proper subgroup of $PSL(2,\mathbb{Z}_2) \approx S_3$. Hence $\tilde{\phi}(\operatorname{tr}[A, B]) = \operatorname{tr}[\tilde{\phi}(A), \tilde{\phi}(B)] = \operatorname{tr} I = 2 = 0$ and so $\operatorname{tr}[A, B] \in \ker\phi = \langle 1 + i\rangle$. Therefore $|\operatorname{tr}[A, B] - 2| \neq 1$ and also $|\operatorname{tr}[A, B] - 2| \neq 0$ since $\mathcal{G}$ is nonelementary. Hence $J(G) \geq |\operatorname{tr}[A, B] - 2| > 1$ and, by Proposition 2, $J(G) \geq 2$. □

Corollary 4 (H. Sato). *If $\mathcal{W}$ is the Whitehead link group then $J(\mathcal{W}) = 2$.*

Proof. If $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1-i & 1 \end{pmatrix}$, then $\mathcal{W} = \langle A, B\rangle$ and $J(A, B) = 2$. Since $\tilde{\phi}(B) = I$ then $|\tilde{\phi}(\mathcal{W})| = 2$ and therefore, by Proposition 3, $J(\mathcal{W}) = 2$. □

3. Section 2

Proposition 5. *Let $\alpha, \beta \in \{1, i\}$ and $k \in \mathbb{Z}$. Then*:

i) $\mathcal{G}_k^{\alpha,\beta} = \left\langle \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & k\alpha\beta i \end{pmatrix} \right\rangle$.

ii) $\mathcal{G}_k^{\alpha,\alpha} = \mathcal{G}_{-k}^{\alpha,\alpha}$ *and* $\mathcal{G}_k^{\alpha,\beta} \overset{\text{Pic}}{\sim} \mathcal{G}_{-k}^{\alpha,\beta}$.

iii) *The rank of $\mathcal{G}_k^{\alpha,\beta}$ is two and $J\left(\mathcal{G}_k^{\alpha,\beta}\right) = 1$.*

Proof. Write $A = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & k\alpha\beta i \end{pmatrix}$.

i) Since $BAB^{-1} = \begin{pmatrix} 1 & 0 \\ -\beta^2\alpha & 1 \end{pmatrix}$ and this matrix together with A generates Mod^α it follows that $\langle A, B\rangle = \langle \mathrm{Mod}^\alpha, B\rangle$. As

$$\begin{pmatrix} 0 & \alpha \\ -\alpha^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & k\alpha\beta i \end{pmatrix} = \begin{pmatrix} \alpha\beta & ik\alpha^2\beta \\ 0 & (\alpha\beta)^{-1} \end{pmatrix}$$

and $\begin{pmatrix} 0 & \alpha \\ -\alpha^{-1} & 0 \end{pmatrix} \in \mathrm{Mod}^\alpha$, i) follows.

ii) If $\alpha = \beta$, then $\begin{pmatrix} \alpha\beta & -k\alpha^2\beta i \\ 0 & (\alpha\beta)^{-1} \end{pmatrix}$ is the inverse of $\begin{pmatrix} \alpha\beta & k\alpha^2\beta i \\ 0 & (\alpha\beta)^{-1} \end{pmatrix}$ and so $\mathcal{G}_k^{\alpha,\alpha} = \mathcal{G}_{-k}^{\alpha,\alpha}$. Else if $\alpha \neq \beta$ conjugating Mod^α and B^{-1} with $\begin{pmatrix} 0 & -\beta^{-1} \\ \beta & 0 \end{pmatrix}$ we obtain Mod^α and $\begin{pmatrix} 0 & -\beta^{-1} \\ \beta & -k\alpha\beta i \end{pmatrix}$; hence $\mathcal{G}_k^{\alpha,\beta} \overset{\mathrm{Pic}}{\sim} \mathcal{G}_{-k}^{\alpha,\beta}$.

iii) As $\mathcal{G}_k^{\alpha,\beta}$ is a discrete group $J\big(\mathcal{G}_k^{\alpha,\beta}\big) \geq 1$. Since A is parabolic we have $J(A,B) = |\mathrm{tr}\, AB - \mathrm{tr}\, B|^2 = |\alpha\beta|^2 = 1$ (see the proof of Proposition 2). Hence $J\big(\mathcal{G}_k^{\alpha,\beta}\big) = 1$. □

We now compare our groups $\mathcal{G}_k^{\alpha,\beta}$ with groups considered by Sato. Suppose a pair of elements of Möb generates a nonelementary subgroup and the first element is parabolic. Then his pair is conjugate to a pair $(A, B_{\sigma,\mu})$ where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} \mu\sigma & (\mu^2\sigma^2 - 1)/\sigma \\ \sigma & \mu\sigma \end{pmatrix}$ and $\sigma \neq 0$ (see [10], [12]). Define $\mathcal{G}_{\sigma,\mu} = \langle A, B_{\sigma,\mu}\rangle$.

Notice that $\mathcal{G}_{\sigma,\mu} = \mathcal{G}_{-\sigma,\mu} = \mathcal{G}_{\sigma,-\mu} = \mathcal{G}_{\sigma,\mu+1}$ and $\mathcal{G}_{\sigma,\mu}$ is conjugate in Möb to $\mathcal{G}_{\sigma,\mu+1/2}$. This follows from $\langle A, B\rangle = \langle A, -B\rangle = \langle A, -B^{-1}\rangle = \langle A, ABA\rangle$ and $\langle A, B\rangle \sim \langle A, (A^{1/2})^{-1} B A^{1/2}\rangle$ where $B = B_{\sigma,\mu}$ and $A^{1/2} = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$.

For example the Whitehead link group $\mathcal{W}$ is

$$\mathcal{W} = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1-i & 1 \end{pmatrix} \right\rangle$$
$$= \mathcal{G}_{1-i,(1+i)/2} \sim \mathcal{G}_{1-i,i/2} = \mathcal{G}_{1-i,-i/2}$$

(cf. [12, Theorem 2]).

We now extend our definition of $\mathcal{G}_k^{\alpha,\beta}$. If $\alpha, \beta, k \in \mathbb{C} - \{0\}$ define

$$\mathcal{G}_k^{\alpha,\beta} = \left\langle \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & k\alpha\beta i \end{pmatrix} \right\rangle.$$

Because of the last proposition this definition coincides with the one given in the introduction if $\alpha, \beta \in \{1, i\}$ and $k \in \mathbb{Z}$. Conjugating with $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, where $\lambda^2 = \alpha$, we see that $\mathcal{G}_k^{\alpha,\beta} \sim \mathcal{G}_k^{1,\alpha\beta}$, and conjugating with $\begin{pmatrix} 1 & -ki/2 \\ 0 & 1 \end{pmatrix}$, we get $\mathcal{G}_k^{1,\sigma} \sim \mathcal{G}_{\sigma,ki/2}$.

We have following equalities $\mathcal{G}_k^{\alpha,\beta} = \mathcal{G}_k^{\alpha,-\beta} = \mathcal{G}_{-k}^{-\alpha,\beta} = \mathcal{G}_{k+1}^{\alpha,\beta}$ (the last equality follows from $\langle A, B\rangle = \langle A, BA\rangle$) and, conjugating with $\begin{pmatrix} 1 & -k\alpha i \\ 0 & 1 \end{pmatrix}$, we get $\mathcal{G}_k^{\alpha,\beta} \sim \mathcal{G}_{-k}^{\alpha,\beta}$.

We now describe which of the groups $\mathcal{G}_k^{\alpha,\beta}$ are subgroups of Pic. First, if $\mathcal{G}_k^{\alpha,\beta} \subset$ Pic we must have $\alpha, \beta, k \in \mathbb{Z} + i\mathbb{Z}$ and $|\beta| = 1$. Since $\mathcal{G}_k^{\alpha,\beta} = \mathcal{G}_k^{\alpha,-\beta} = \mathcal{G}_{k+1}^{\alpha,\beta}$ we may assume $k \in \mathbb{Z}$ and $\beta \in \{1, i\}$.

The following theorem describes all the Jørgensen subgroups of Pic, up to conjugation in Pic.

Theorem 6. *If G is a rank two subgroup of* Pic *with $J(G) = 1$ then G is conjugate in* Pic *to $\mathcal{G}_k^{\alpha,\beta}$ where $\alpha, \beta \in \{1, i\}$ and k is a nonnegative integer.*

Proof. Let A and B be generators of G such that

$$J(A, B) = |\text{tr}^2 A - 4| + |\text{tr}[A, B] - 2| = 1.$$

If $\text{tr}\, A \neq \pm 2$ then $|\text{tr}^2 A - 4| \geq 3$ hence $|\text{tr}^2 A - 4| = 0$ and $|\text{tr}[A, B] - 2| = 1$. A is then parabolic with fixed point a/c where a and c are relatively prime Gaussian integers. Let b and d be Gaussian integers such that $ad - bc = 1$. Conjugating A with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in$ Pic we obtain a parabolic element which fixes ∞. Hence we can assume that $A = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$, with α a nonzero Gaussian integer.

Write $B = \begin{pmatrix} x & y \\ \beta & z \end{pmatrix} \in$ Pic. Then, as in the proof of Proposition 2,

$$1 = |\text{tr}[A, B] - 2| = |\text{tr}(AB) - \text{tr}\, B|^2 = |\alpha\beta|^2.$$

Hence $|\alpha| = |\beta| = 1$. Conjugating with $\begin{pmatrix} 1 & x\beta^{-1} \\ 0 & 1 \end{pmatrix}$ we see that the pair (A, B) is conjugate in Pic to the pair $\left(A, \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & k\alpha\beta i \end{pmatrix}\right)$ where $k\alpha\beta i = x + z$ and k is Gaussian integer. Than $\mathcal{G} \overset{\text{Pic}}{\sim} \mathcal{G}_k^{\alpha,\beta}$ and since $\mathcal{G}_k^{\alpha,\beta} = \mathcal{G}_k^{\alpha,-\beta} = \mathcal{G}_{-k}^{-\alpha,\beta} = \mathcal{G}_{k+i}^{\alpha,\beta}$ and $\mathcal{G}_k^{\alpha,\beta} \overset{\text{Pic}}{\sim} \mathcal{G}_{-k}^{\alpha,\beta}$ we may assume that $\alpha, \beta \in \{1, i\}$ and k is a nonnegative integer. □

4. Section 3

In this section and the next one we will use free products with amalgamation (see [6]).

Also we will use the 90° rotation $R \in$ Möb. Let $R = \begin{pmatrix} (1+i)/\sqrt{2} & 0 \\ 0 & (1-i)/\sqrt{2} \end{pmatrix} \in$ Möb (multiplication by i); this element does not belong to Pic. Then $R^{-1}\text{Pic}R = \text{Pic}$, $R^{-1}\text{Mod}^{\alpha} R = \text{Mod}^{\alpha'}$, $R^{-1}\mathcal{G}_k^{\alpha,\beta} R = \mathcal{G}_k^{\alpha',\beta'}$ where $\{\alpha, \alpha'\} = \{\beta, \beta'\} = \{1, i\}$; thus $\mathcal{G}_k^{\alpha,\beta} \sim \mathcal{G}_k^{\alpha',\beta'}$. This proves the if part of Theorem 1 4).

A presentation of Pic can be given as follows (see [1], [14]):

$$\mathrm{Pic} = \langle x, y, u, v : x^3 = y^3 = u^2 = v^2 = (uy)^2 = (yx)^2 = (xv)^2 = (vu)^2 = 1\rangle$$

where $u = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $x = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$, $v = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & i \\ i & 1 \end{pmatrix}$.

From this one can show $\mathrm{Pic} = V *_{\mathrm{Mod}} Y$ where $V = \langle \mathrm{Mod}, v\rangle$ and $Y = \langle \mathrm{Mod}, y\rangle$. We have also $\mathrm{Mod} = \langle u, x\rangle$ and $\mathrm{Mod}^i = \langle v, y\rangle$ and of course $\mathrm{Pic} = \langle \mathrm{Mod}, \mathrm{Mod}^i\rangle$.

We have the following presentations:

$$\begin{aligned} V &= \langle x, u, v : x^3 = u^2 = v^2 = (xv)^2 = (vu)^2 = 1\rangle \\ &= \langle u, v\rangle *_{\langle v\rangle} \langle v, x\rangle = \mathbb{Z}_2^2 *_{\mathbb{Z}_2} \mathbb{D}_3, \\ Y &= \langle x, y, u : x^3 = y^2 = u^2 = (uy)^2 = (yx)^2 = 1\rangle \\ &= \langle u, y\rangle *_{\langle y\rangle} \langle y, x\rangle = \mathbb{D}_3 *_{\mathbb{Z}_3} A_4, \\ \mathrm{Mod} &= \langle x, u : x^3 = u^2 = 1\rangle = \mathbb{Z}_2 * \mathbb{Z}_3 \end{aligned}$$

where $\mathbb{D}_3$ is the dihedral group of order six and A_4 is the alternating group in four elements.

We have that $\mathcal{G}_1^{i,i} = \mathrm{Pic}$ because

$$\begin{aligned} \mathcal{G}_1^{i,i} &= \left\langle \mathrm{Mod}^i, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right\rangle \\ &= \left\langle \mathrm{Mod}^i, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right\rangle \\ &= \langle \mathrm{Mod}^i, \mathrm{Mod}\rangle = \mathrm{Pic} \end{aligned}$$

since $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = v\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}v$. This implies $\mathcal{G}_1^{1,1} = R^{-1}\mathcal{G}_1^{i,i}R = \mathrm{Pic}$.

Notice also that $\mathcal{G}_0^{1,1} = \mathrm{Mod} = \langle x, u\rangle$, $\mathcal{G}_0^{i,i} = \mathrm{Mod}^i = \langle y, v\rangle$, $\mathcal{G}_0^{1,i} = V$ and $\mathcal{G}_1^{i,1} \overset{\mathrm{Pic}}{\sim} \mathcal{G}_{-1}^{i,1} = Y$. One can see that $\mathcal{G}_k^{1,1} = \langle u, x, (vy)^k\rangle$, $\mathcal{G}_k^{i,i} = \langle v, y, (xu)^k\rangle$ and, using Proposition 5, $\mathcal{G}_k^{i,1} = \langle v, y, u(xu)^k\rangle$ and $\mathcal{G}_k^{1,i} = \langle u, x, (vy)^k v\rangle$.

The abelianizations of Pic, V, Y and Mod are $\mathbb{Z}_2^2, \mathbb{Z}_2^2, \mathbb{Z}_2, \mathbb{Z}_6$ respectively.

Denote by $\overline{\mathrm{Pic}}$ the abelianization of Pic and by $\mathrm{ab}\colon \mathrm{Pic} \to \overline{\mathrm{Pic}}$ the abelianizaction map. We will write $\overline{w} = \mathrm{ab}(w)$. We have that $\overline{\mathrm{Pic}} = \mathrm{Pic}/\langle x, y\rangle = \langle \overline{u}, \overline{v}\rangle \simeq \mathbb{Z}_2^2$.

Proposition 7. *If $\alpha = \beta$ and k is odd or if $\alpha \neq \beta$ and k is even then* $\mathrm{ab}\big(\mathcal{G}_k^{\alpha,\beta}\big) = \overline{\mathrm{Pic}}$. *Otherwise*

$$\mathrm{ab}\big(\mathcal{G}_k^{\alpha,\beta}\big) = \begin{cases} \langle \overline{u}\rangle & \text{if } \alpha = 1 \\ \langle \overline{v}\rangle & \text{if } \alpha = i \end{cases}.$$

Proof. We have $\mathrm{ab}\big(\mathcal{G}_k^{1,1}\big) = \langle \overline{u}, \overline{v}^k \rangle$, $\mathrm{ab}\big(\mathcal{G}_k^{1,i}\big) = \langle \overline{u}, \overline{v}^{k+1} \rangle$, $\mathrm{ab}\big(\mathcal{G}_k^{i,1}\big) = \langle \overline{v}, \overline{u}^{k+1} \rangle$ and $\mathrm{ab}\big(\mathcal{G}_k^{i,i}\big) = \langle \overline{v}, \overline{u}^k \rangle$. From these equalities the proposition follows. □

Corollary 8. $\mathcal{G}_k^{1,1} \overset{\mathrm{Pic}}{\not\approx} \mathcal{G}_k^{i,i}$ *(resp.* $\mathcal{G}_k^{1,i} \overset{\mathrm{Pic}}{\not\approx} \mathcal{G}_k^{i,1}$*) if k is even (resp. k is odd).*

The following lemma will be used in the classification of the groups $\mathcal{G}_k^{\alpha,\beta}$ in Möb.

Lemma 9. i) *The trace of any element of $\mathcal{G}_k^{1,\beta}$ is of the form $a+kbi$ or $ka+bi$ where $a, b \in \mathbb{Z}$.*
ii) *The trace of any element of $\mathcal{G}_k^{1,1}$ is of the form $a+kbi$ where $a, b \in \mathbb{Z}$.*
iii) *$\pm(1+ki)$ is the trace of an element of $\mathcal{G}_k^{1,1}$ and $\pm(i+k)$ is the trace of an element of $\mathcal{G}_k^{1,i}$.*

Proof. The natural ring homomorphism from $\mathbb{Z}+i\mathbb{Z} \approx \mathbb{Z}[X]/(X^2+1)$ onto $\mathbb{Z}_k + i\mathbb{Z}_k \approx \mathbb{Z}_k[X]/(X^2+1)$ induces a group homomorphism $PSL(2,\mathbb{Z}+i\mathbb{Z}) \xrightarrow{\psi} PSL(2,\mathbb{Z}_k+i\mathbb{Z}_k)$. As $\mathcal{G}_k^{1,\beta} \supset \mathrm{Mod}$ we have, by Proposition 5,

$$\begin{aligned}\mathcal{G}_k^{1,\beta} &= \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & k\beta i \end{pmatrix} \right\rangle \\ &= \left\langle \mathrm{Mod}, \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & k\beta i \end{pmatrix} \right\rangle.\end{aligned}$$

Then $\psi\big(\mathcal{G}_k^{1,\beta}\big) = \left\langle \psi(\mathrm{Mod}), \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & 0 \end{pmatrix} \right\rangle$ which is contained in

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}_k + i\mathbb{Z}_k) \colon a, b, c, d \in \mathbb{Z}_k \text{ or } a, b, c, d \in i\mathbb{Z}_k \right\}$$

so the trace of any element of $\psi\big(\mathcal{G}_k^{1,\beta}\big)$ lies in $\mathbb{Z}_k \cup i\mathbb{Z}_k$. From this i) follows.

ii) is proved similarly.

To prove iii) observe that the trace of $\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\beta^{-1} \\ \beta & k\beta i \end{pmatrix}$ is $\beta^{-1} + k\beta i$. □

The following theorem gives the classification of the groups $\mathcal{G}_k^{\alpha,\beta}$, up to conjugation in Möb.

Theorem 10. *If $\mathcal{G}_k^{1,\beta} \sim \mathcal{G}_{k'}^{1,\beta'}$ where $\beta, \beta' \in \{1, i\}$, $k \geq 0$ and $k' \geq 0$ then $\beta = \beta'$ and $k = k'$.*

Proof. As $\mathcal{G}_1^{1,1}$ (= Pic) and $\mathcal{G}_1^{1,i}$ (= Y) have nonisomorphic abelianizations the case $k = k' = 1$ follows. If $(k, k') \neq (1, 1)$ and $(k, \beta) \neq (k', \beta')$ then, using the lemma, one sees that

$$\left\{\text{traces of elements of } \mathcal{G}_k^{1,\beta}\right\} \neq \left\{\text{traces of elements of } \mathcal{G}_{k'}^{1,\beta'}\right\}$$

and therefore $\mathcal{G}_k^{1,\beta} \overset{\text{Möb}}{\not\approx} \mathcal{G}_{k'}^{1,\beta'}$. □

This completes the proof of Theorem 1 4).

5. Section 4

In this section we will think of Pic as $V *_{\text{Mod}} Y$. Define an integer valued function on Pic as follows:

$$\lambda(w) = \begin{cases} 1 & \text{if } w \overset{\text{Pic}}{\sim} w' \in V \cup Y \\ 2n & \text{if } w \overset{\text{Pic}}{\sim} v_1 y_1 \cdots v_n y_n,\ n \geq 1,\ v_i \in V,\ y_i \in Y\ (i = 1, \ldots, n). \end{cases}$$

The function is well defined (see for example [7, Theorems 4.4 and 4.6] or [6, Chapter IV, Theorems 2.6 and 2.8]). Clearly if $w \overset{\text{Pic}}{\sim} w'$, $\lambda(w) = \lambda(w')$.

Recall that $\mathcal{G}_k^{1,\beta} = \left\langle \text{Mod}, \begin{pmatrix} \beta & -k\beta i \\ 0 & \beta^{-1} \end{pmatrix} \right\rangle$. Write and $t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $T = \langle t \rangle$ and

$$s = \begin{pmatrix} \beta & -k\beta i \\ 0 & \beta^{-1} \end{pmatrix} = \begin{cases} (vy)^k & \text{if } \beta = 1 \\ u(yv)^{k-1} y & \text{if } \beta = i \end{cases}.$$

Proposition 11. *Consider the groups $\mathcal{G}_k^{1,\beta}$ with $k \geq 2$. Then*:

i) *If $\beta = 1$ (resp. $\beta = i$) then $\langle s, t \rangle \simeq \mathbb{Z}^2$ (resp. $\langle s, t \rangle \simeq \mathbb{D}_\infty$).*

ii) *There is an isomorphism $\mathcal{G}_k^{1,\beta} \simeq \text{Mod} *_T \langle s, t \rangle$.*

iii) *$\lambda(s^{e_1} m_1 s^{e_2} m_2 \cdots s^{e_r} m_r) > 1$ where $e_j \neq 0$ (resp. $e_j = 1$) if $\beta = 1$ (resp. if $\beta = i$), $r \geq 1$, $m_j \in \text{Mod} - T$ $(j = 1, \ldots, r)$.*

Proof. Write $w = s^{e_1} m_1 s^{e_2} m_2 \cdots s^{e_r} m_r$. Let $\beta = 1$ so that $s = \begin{pmatrix} 1 & -ki \\ 0 & 1 \end{pmatrix}$ and $\langle s, t \rangle \simeq \mathbb{Z}^2$. Using the matrix expressions for the elements one can see that, if $m \in \text{Mod} - T$, then $ymv \notin \text{Mod}$, $ymy^{-1} \notin \text{Mod}$, $vmy^{-1} \in \text{Mod}$, $vmv \in \text{Mod}$ and $y^{-1}vmvy \notin \text{Mod}$. Using these facts we see that

$$\begin{aligned} \lambda(w) &= \lambda((vy)^{e_1} m_1 (vy) s^{e_2} m_2 \cdots (vy)^{e_r} m_r) \\ &= 2k \sum_{j=1}^{r} |e_j| - \#\{l : e_l e_{l+1} < 0\} \geq 2kr - r > 1. \end{aligned}$$

Let $\beta = i$ so that $s = \begin{pmatrix} i & k \\ 0 & -i \end{pmatrix}$, $\langle s, t\rangle \simeq \mathbb{D}_\infty$. Then

$$\begin{aligned} w &= sm_1sm_2\cdots sm_r \\ &= u(yv)(yv)^{k-2}ym_1u(yv)(yv)^{k-2}ym_2u(yv)(yv)^{k-2}ym_3\cdots u(yv)(yv)^{k-2}ym_r \\ &\overset{\text{Pic}}{\sim} v(yv)^{k-2}y_1v(yv)^{k-2}y_2v(yv)^{k-2}y_3\cdots v(yv)^{k-2}y_r \end{aligned}$$

where $y_j = ym_juy$. As $m_j \in \text{Mod} - T$, one can verify that $y_j \in Y - \text{Mod}$. Therefore $\lambda(w) = r(2k-2) > 1$.

This proves i) and iii). Assertion ii) follows from iii). □

Corollary 12. *For $k > 2$, $\mathcal{G}_k^{1,1} \simeq \mathcal{G}_k^{i,i} \simeq \mathcal{G}_2^{1,1} \simeq \text{Mod} *_{\mathbb{Z}} \mathbb{Z}_2$ and $\mathcal{G}_k^{1,i} \simeq \mathcal{G}_k^{i,1} \simeq \mathcal{G}_2^{1,i} \simeq$ $\text{Mod} *_{\mathbb{Z}} \mathbb{D}_\infty$.*

Corollary 13. *If $k \geq 2$ and $w \in \mathcal{G}_k^{1,\beta} - \text{Mod}$ then $\lambda(w) > 1$.*

Proof. It follows from the proposition observing that $\lambda(w) = 1$ if $w \in \text{Mod}$, $\lambda((vy)^{mk}) = 2|m|k > 1$ and $\lambda(u(yv)^{k-1}y) = 2k-1 > 1$. □

Corollary 14. *The abelianization of $\mathcal{G}_2^{1,1}$ is $\mathbb{Z} \oplus \mathbb{Z}_6$ and the abelianization of $\mathcal{G}_2^{1,i}$ is $\mathbb{Z}_6$.*

We will use the number of conjugacy classes of elements of order two in $\mathcal{G}_k^{\alpha,\beta}$; we will denote it by $c_2(\mathcal{G}_k^{\alpha,\beta})$.

Corollary 15. *We have $c_2(\mathcal{G}_0^{1,1}) = 1$, $c_2(\mathcal{G}_0^{1,i}) = 3$, $c_2(\mathcal{G}_1^{1,1}) = 4$, $c_2(\mathcal{G}_1^{1,i}) = 2$, $c_2(\mathcal{G}_2^{1,1}) = 1$ and $c_2(\mathcal{G}_2^{1,i}) = 2$.*

Proof. Recall that $\mathcal{G}_0^{1,1} = \text{Mod} = \mathbb{Z}_2 * \mathbb{Z}_3$, $\mathcal{G}_0^{1,i} = V = \mathbb{Z}_2^2 *_{\mathbb{Z}_2} \mathbb{D}_3$, $\mathcal{G}_1^{1,1} = \text{Pic} = V *_{\text{Mod}} Y$, $\mathcal{G}_1^{1,i} \approx \mathcal{G}_{-1}^{1,i} = Y = \mathbb{D}_3 *_{\mathbb{Z}_3} A_4$, $\mathcal{G}_2^{1,1} \simeq \text{Mod} *_{\mathbb{Z}} \mathbb{Z}^2$ and $\mathcal{G}_2^{1,i} = \text{Mod} *_{\mathbb{Z}} \mathbb{D}_\infty$. Using the fact that an element of finite order in a free product with amalgamation is conjugate to an element in a factor and using ab the corollary follows. □

The following theorem states that if $(\alpha, \beta, k) \neq (\alpha', \beta', k')$ then $\mathcal{G}_k^{\alpha,\beta} \overset{\text{Pic}}{\not\sim} \mathcal{G}_{k'}^{\alpha',\beta'}$ with one exception (namely $\mathcal{G}_1^{1,1} = \mathcal{G}_1^{i,i} = \text{Pic}$).

Theorem 16. *Let $\alpha, \beta, \alpha', \beta' \in \{1, i\}$, $k \geq 0$ and $k' \geq 0$. Suppose $\mathcal{G}_k^{\alpha,\beta} \overset{\text{Pic}}{\sim} \mathcal{G}_{k'}^{\alpha',\beta'}$ with $(\alpha, \beta, k) \neq (\alpha', \beta', k')$. Then $k = k' = 1$, $\alpha = \beta$ and $\alpha' = \beta'$.*

Proof. As $\mathcal{G}_k^{\alpha,\beta} \sim \mathcal{G}_{k'}^{\alpha',\beta'}$ we have, by Theorem 1 4), that $k = k'$ and $\alpha\beta = \pm\alpha'\beta'$ and so we may assume that $\alpha = 1$ and $\alpha' = i$. Hence $k \leq 1$.

Suppose $k \geq 2$. No conjugate, in Pic, of v lies in Mod because ab(Mod) = $\langle\overline{u}\rangle$. Therefore, by Corollary 13, no conjugate, in Pic, of v lies in $\mathcal{G}_k^{1,\beta}$. As $v \in \mathcal{G}_k^{i,\beta'}$, we have $\mathcal{G}_k^{1,\beta} \overset{\text{Pic}}{\not\sim} \mathcal{G}_k^{i,\beta'}$.

Suppose $k = 1$, and $\beta = i$. Then $\beta' = 1$ and $\text{ab}(\mathcal{G}_k^{\alpha,\beta}) = \langle\overline{u}\rangle \neq \langle\overline{v}\rangle = \mathcal{G}_{k'}^{\alpha',\beta'}$ so $\mathcal{G}_k^{\alpha,\beta} \overset{\text{Pic}}{\not\sim} \mathcal{G}_{k'}^{\alpha',\beta'}$.

Suppose $k = 0$, and $\beta = 1$. Then $\beta' = 1$ and $\text{ab}\big(\mathcal{G}_k^{\alpha,\beta}\big) = \langle\overline{u}\rangle \neq \langle\overline{v}\rangle = \mathcal{G}_k^{\alpha',\beta'}$ so $\mathcal{G}_k^{\alpha,\beta} \overset{\text{Pic}}{\not\sim} \mathcal{G}_{k'}^{\alpha',\beta'}$.

Finally suppose $k = 0$, and $\beta = i$. Then $\beta' = 1$. We have $\mathcal{G}_k^{\alpha,\beta} = V = \langle v, u, x\rangle$ and $\mathcal{G}_{k'}^{\alpha',\beta'} = \langle v, y, u\rangle$. There is an inner automorphism ϕ of Pic such that $\phi(\mathcal{G}_k^{\alpha,\beta}) = \mathcal{G}_{k'}^{\alpha',\beta'}$ and we have a commutative diagram

$$\begin{array}{ccc} \mathcal{G}_{k'}^{i,1} & \xrightarrow{\quad\phi\quad} & \mathcal{G}_k^{1,i} \\ & {}_{\theta'}\searrow \quad \swarrow_{\theta} & \\ & \langle\bar{u}, \bar{v}\rangle & \end{array}$$

where θ' and θ are the restrictions of ab. Then $\theta'^{-1}(\langle\overline{u}\rangle) \simeq \theta^{-1}(\langle\overline{u}\rangle)$ which is impossible because, since $[V, \text{Mod}] = 2$, $\theta^{-1}(\langle\overline{u}\rangle) = \text{Mod}$ and $\theta'(\langle u\rangle) \supset \langle y, u\rangle \simeq \mathbb{D}_3$ and $\mathbb{D}_3$ is not isomorphic to a subgroup of Mod. □

Theorem 17. *Let* $\alpha, \beta \in \{1, i\}$, $k \geq 0$. *Then* $\mathcal{G}_k^{\alpha,\beta}$ *is isomorphic to one of the groups* Mod, V, Pic, Y, $\text{Mod} *_{\mathbb{Z}} \mathbb{Z}^2$ *and* $\text{Mod} *_{\mathbb{Z}} \mathbb{D}_\infty$. *These six groups are pairwise nonisomorphic*.

Proof. The first assertion is a consequence of $\mathcal{G}_k^{i,\beta} \simeq \mathcal{G}_k^{1,\beta'}$, where $\{\beta, \beta'\} = \{1, i\}$, and Corollary 12. Now V, Pic and $\text{Mod} *_{\mathbb{Z}} \mathbb{D}_\infty$ have abelianization $\mathbb{Z}_2^2$ while Mod, Y and $\text{Mod} *_{\mathbb{Z}} \mathbb{Z}^2$ have pairwise non isomorphic abelianizations different from $\mathbb{Z}_2^2$. Since $c_2(V) = 3$, $c_2(\text{Pic}) = 4$ and $c_2(\text{Mod} *_{\mathbb{Z}} \mathbb{D}_\infty) = 2$, the theorem follows. □

6. Section 5

In what follow ab: Pic $\to \overline{\text{Pic}} = \langle\overline{u}, \overline{v}\rangle$ is the abelianization map, where $u = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $v = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $\mathbb{D}_3$ is the dihedral group of order six and A_4 is the alternating group in four letters.

In what follows $\mathbb{D}_3$ is the dihedral group of order six, A_4 is the alternating group in for letters ab: Pic $\to \overline{\text{Pic}} = \langle\overline{u}, \overline{v}\rangle$ is the abelianization map, where $u = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $v = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$.

The following table has the information

#{conjugacy classes of elements of order two}	a group isomorphic to $\mathcal{G}_k^{\alpha,\beta}$
image under ab of $\mathcal{G}_k^{\alpha,\beta}$	abelianization of $\mathcal{G}_k^{\alpha,\beta}$

for the group $\mathcal{G}_k^{\alpha,\beta}$.

<table>
<tr><th>$\mathcal{G}_k^{\alpha,\beta}$</th><th>$(\alpha, \beta) = (1, 1)$ (resp. (i, i))</th><th>$(\alpha, \beta) = (1, i)$ (resp. $(i, 1)$)</th></tr>
<tr><td>$k = 0$</td><td><table><tr><td>1</td><td>Mod</td></tr><tr><td>$\langle \bar{u} \rangle$ (resp. $\langle \bar{v} \rangle$)</td><td>$\mathbb{Z}_6$</td></tr></table></td><td><table><tr><td>3</td><td>$\mathbb{Z}_2^2 *_{\mathbb{Z}_2} \mathbb{D}_3$</td></tr><tr><td>$\langle \bar{u}, \bar{v} \rangle$</td><td>$\mathbb{Z}_2^2$</td></tr></table></td></tr>
<tr><td>$k = 1$</td><td><table><tr><td>4</td><td>Pic</td></tr><tr><td>$\langle \bar{u}, \bar{v} \rangle$</td><td>$\mathbb{Z}_2^2$</td></tr></table></td><td><table><tr><td>2</td><td>$\mathbb{D}_3 *_{\mathbb{Z}_3} A_4$</td></tr><tr><td>$\langle \bar{u} \rangle$ (resp. $\langle \bar{v} \rangle$)</td><td>$\mathbb{Z}_2$</td></tr></table></td></tr>
<tr><td>$K = 2, 4, \ldots$</td><td><table><tr><td>1</td><td>$\mathrm{Mod} *_{\mathbb{Z}} \mathbb{Z}^2$</td></tr><tr><td>$\langle \bar{u} \rangle$ (resp. $\langle \bar{v} \rangle$)</td><td>$\mathbb{Z} \oplus \mathbb{Z}_6$</td></tr></table></td><td><table><tr><td>2</td><td>$\mathrm{Mod} *_{\mathbb{Z}} \mathbb{D}_\infty$</td></tr><tr><td>$\langle \bar{u} \rangle$ (resp. $\langle \bar{v} \rangle$)</td><td>$\mathbb{Z}_2^2$</td></tr></table></td></tr>
<tr><td>$k = 3, 5, \ldots$</td><td><table><tr><td>1</td><td>$\mathrm{Mod} *_{\mathbb{Z}} \mathbb{Z}^2$</td></tr><tr><td>$\langle \bar{u}, \bar{v} \rangle$</td><td>$\mathbb{Z} \oplus \mathbb{Z}_6$</td></tr></table></td><td><table><tr><td>2</td><td>$\mathrm{Mod} *_{\mathbb{Z}} \mathbb{D}_\infty$</td></tr><tr><td>$\langle \bar{u}, \bar{v} \rangle$</td><td>$\mathbb{Z}_2^2$</td></tr></table></td></tr>
</table>

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