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Osaka University
JØRGENSEN SUBGROUPS OF THE PICARD GROUP

FRANCISCO GONZÁLEZ-ACUÑA and ARTURO RAMÍREZ

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Abstract

Let $G$ be a subgroup of rank two of the Möbius group $PSL(2, \mathbb{C})$. The Jørgensen number $J(G)$ of $G$ is defined by

$$J(G) = \inf \{ |\text{tr}^2 A - 4| + |\text{tr}[A, B] - 2| : \langle A, B \rangle = G \}.$$

We describe all subgroups $G$ of the Picard group $PSL(2, \mathbb{Z} + i\mathbb{Z})$ with $J(G) = 1$.

1. Introduction

Let $G$ be a subgroup of rank two of the Möbius group $\text{Möb} = PSL(2, \mathbb{C})$. The Jørgensen number $J(G)$ of $G$ is dened by

$$J(G) = \inf \{ |\text{tr}^2 A - 4| + |\text{tr}[A, B] - 2| : \langle A, B \rangle = G \}.$$

A subgroup $G$ of $\text{Möb}$ is elementary if the cardinality of its limit set $\Lambda(G)$ is at most 2 see [8, p.266]. If $G = \langle A, B \rangle$ is a discrete group with $A$ parabolic, then $G$ is elementary iff $\text{tr}[A, B] = 2$ (that is, iff $J(A, B) = 0$).

Jørgensen has proved that if $G$ is a discrete nonelementary rank two subgroup of $\text{Möb}$ then $J(G) > 1$.

It has been conjectured [10, p.273] that if $G$ is nonelementary rank two subgroup of $\text{Möb}$ which does not contain elliptic elements of infinite order and $J(G) = 1$ then $G$ is discrete.

Groups $G$ with $J(G) = 1$ have been studied in the literature ([3], [4], [13], [10], [12]). Following [10] we call a discrete nonelementary rank two subgroup $G$ of $\text{Möb}$ with $J(G) = 1$ a Jørgensen group.

An important subgroup of $\text{Möb}$ is the Picard group $\text{Pic} = PSL(2, \mathbb{Z} + i\mathbb{Z})$. We are interested in the Jørgensen numbers of rank two subgroups of $\text{Pic}$.

Our motivation for the present paper is the article [12] by H. Sato in which he considers the Whitehead link group $W = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle \subset \text{Pic}$ (see [5], [9], [15]) and proves that $J(W) = 2$. Here we will give a brief proof of this result.

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We now describe a family of rank two subgroups of Pic. Let

\[ \text{Mod} = \text{Mod}^1 = PSL(2, \mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3, \]

\[ \text{Mod}^i = \left\{ \begin{pmatrix} a & -ib \\ ic & d \end{pmatrix} \in PSL(2, \mathbb{C}) : a, b, c, d \in \mathbb{Z} \right\} \cong \mathbb{Z}_2 * \mathbb{Z}_3, \]

\[ G_{\alpha, \beta}^k = \left( \text{Mod}^\alpha, \left( \begin{array}{cc} \alpha \beta & -k\alpha^2 \beta i \\ 0 & (\alpha \beta)^{-1} \end{array} \right) \right) \text{ where } \alpha, \beta \in \{1, i\} \text{ and } k \in \mathbb{Z}. \]

For example \( G_{0, 1}^1 = \text{Mod}, G_{0, i}^1 = \text{Mod}^i \) and one can show that \( G_{1, 1}^1 = G_{1, i}^1 = \text{Pic}. \)

The group Pic is generated by Mod and Mod\(^i\), and these two subgroups are conjugate in Möb by a 90° rotation \( R = \begin{pmatrix} (1 + i)/\sqrt{2} & 0 \\ 0 & (1 - i)/\sqrt{2} \end{pmatrix} \).

Denoting by \( \mathbb{D}_\infty \) the infinite dihedral group, we will see that (Theorem 11) for \( k \geq 2 \) we have \( G_{\alpha, \beta}^k \cong \left\{ \begin{array}{c} \text{Mod} * \mathbb{Z}^2 \text{ if } \alpha \beta = \pm 1, \\ \text{Mod} * \mathbb{D}_\infty \text{ if } \alpha \beta = i \end{array} \right\} \), where the element \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) of Mod is amalgamated to a primitive element of infinite order of \( \mathbb{Z}^2 \) or \( \mathbb{D}_\infty \).

The symbol \( \text{Pic} \) denotes conjugation in Pic, the symbol \( \sim \) denotes conjugation in Möb.

Our main result is the following:

**Theorem 1.** Let \( I = \{ \alpha, \beta, k \} : \alpha, \beta \in \{1, i\}, k \text{ a non negative integer} \}. \) Let \( G \) be a rank two subgroup of Pic, with \( J(G) = 1 \). Then:

1) \( G \) is conjugate, in Pic, to \( G_{\alpha, \beta}^k \) for some \((\alpha, \beta, k) \in I\).

2) \( G \) is isomorphic to exactly one of the groups \( G_{0, 1}^1, G_{1, 1}^1, G_{2, 1}^1, G_{1, i}^1 \) and \( G_{2, i}^1 \).

3) If \( G_{\alpha, \beta}^k \cong G_{\alpha', \beta'}^{k'} \) where \((\alpha, \beta, k) \) and \((\alpha', \beta', k') \) are different elements of \( I \) then \( k = k' = 1, \alpha = \beta \) and \( \alpha' = \beta' \).

4) \( G_{\alpha, \beta}^k \cong G_{\alpha', \beta'}^{k'} \) (with \((\alpha, \beta, k), (\alpha', \beta', k') \in I \}) iff \( k = k' \) and \( \alpha \beta = \pm \alpha' \beta' \).

Notice that no Jørgensen subgroup \( G \) of Pic is the group of a link in \( S^3 \), because \( G \supset \mathbb{Z}_2 * \mathbb{Z}_3 \).

In Section 1 we give another proof of Sato’s theorem.

In Section 2 we give a different description of \( G_{\alpha, \beta}^k \) which shows its rank is two. With this description we extend our family to a family of rank two subgroups \( G_{\alpha, \beta}^k \) with \( \alpha, \beta \) and \( k \in \mathbb{C} - \{0\} \) and compare it with a family dened by Sato ([10], [12]). At the end of the section we prove Theorem 1 1).

In Section 3 we prove Theorem 1 4).

In Section 4, using the structure of Pic as an amalgamated product, we prove Theorem 1 3) and 2).

In Section 5 we exhibit a table that gives algebraic information of the groups \( G_{\alpha, \beta}^k \), as their abelianizations, their images under the abelianization map of Pic, and the num-
ber of conjugacy classes of elements of order two. These facts are proved in Sections 3 and 4 and are used in the proof of Theorem 1.3) 2).

Some of the results of this paper were presented at the Tôhoku Seminar in Akita, in February 2005, while the first author was visiting Osaka City University. He would like to thank this institution for its kind hospitality.

We would like to thank the referee. His suggestions improved the presentation of the paper.

2. Section 1

Proposition 2. If \( G \) is a rank two subgroup of \( \text{Pic} \) then \( J(G) \in [0, 1, 2] \cup [3, \infty) \).

Proof. Let \( A \) and \( B \in \text{GL}(2, \mathbb{C}) \) and \( C = AB \). Then

\[
\text{tr}[A, B] - 2 = \text{tr}^2 A + \text{tr}^2 B + \text{tr}^2 C - \text{tr} A \text{ tr} B \text{ tr} C - 4
\]

in particular if \( \text{tr} A = 2 \) one has then \( \text{tr}[A, B] - 2 = (\text{tr} C \text{ tr} B)^2 \) and therefore \( J(A, B) = |\text{tr} C - \text{tr} B|^2 \). Hence if \( A \) is parabolic and \( A, B \in \text{Pic} \), we have that \( J(A, B) \) is the modulus of the square of an element of \( \mathbb{Z} + i\mathbb{Z} \), that is, an integer that is the sum of two squares. If \( A \in \text{Pic} \) and \( A \) is not parabolic then \( J(A, B) \geq |\text{tr} A - 2| |\text{tr} A + 2| \geq 3 \). \( \square \)

Proposition 3. Let \( \tilde{\phi}: \text{PSL}(2, \mathbb{Z} + i\mathbb{Z}) \to \text{PSL}(2, \mathbb{Z}_2) \) be the homomorphism induced by the ring homomorphism \( \phi: \mathbb{Z} + i\mathbb{Z} \to \mathbb{Z}_2 \). If \( G \) is a nonelementary \( e \) rank two subgroup of \( \text{Pic} \) and \( |\tilde{\phi}(G)| < 6 \) then \( J(G) \geq 2 \).

Proof. Suppose \( G = \langle A, B \rangle \). Notice that \( \tilde{\phi}(G) \) is Abelian since it is a proper subgroup of \( \text{PSL}(2, \mathbb{Z}_2) \approx S_3 \). Hence \( \tilde{\phi}(\text{tr}[A, B]) = \text{tr}[\tilde{\phi}(A), \tilde{\phi}(B)] = \text{tr} I = 2 = 0 \) and so \( \text{tr}[A, B] \in \ker \phi = \langle 1 + i \rangle \). Therefore \( |\text{tr}[A, B] - 2| \neq 1 \) and also \( |\text{tr}[A, B] - 2| \neq 0 \) since \( G \) is nonelementary. Hence \( J(G) \geq |\text{tr}[A, B] - 2| > 1 \) and, by Proposition 2, \( J(G) \geq 2 \). \( \square \)

Corollary 4 (H. Sato). If \( \mathcal{W} \) is the Whitehead link group then \( J(\mathcal{W}) = 2 \).

Proof. If \( A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) and \( B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \), then \( \mathcal{W} = \langle A, B \rangle \) and \( J(A, B) = 2 \). Since \( \tilde{\phi}(B) = I \) then \( |\tilde{\phi}(\mathcal{W})| = 2 \) and therefore, by Proposition 3, \( J(\mathcal{W}) = 2 \). \( \square \)

3. Section 2

Proposition 5. Let \( \alpha, \beta \in [1, i] \) and \( k \in \mathbb{Z} \). Then:

i) \( G^\alpha_\beta \cong \mathbb{Z}/(\alpha \beta) \)

ii) \( G^\alpha_\beta \cong G^\alpha_{-k} \) and \( G^\alpha_\beta \cong G^\alpha_{\text{Pic}} \).

iii) The rank of \( G^\alpha_\beta \) is two and \( J(G^\alpha_\beta) = 1 \).
Proof. Write \( A = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \) and \( B = \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & k\alpha \beta i \end{pmatrix} \).

i) Since \( BAB^{-1} = \begin{pmatrix} 1 & 0 \\ -\beta^2 \alpha & 1 \end{pmatrix} \) and this matrix together with \( A \) generates \( \text{Mod}^\alpha \), it follows that \( \langle A, B \rangle = \langle \text{Mod}^\alpha, B \rangle \). As

\[
\begin{pmatrix} 0 & \alpha \beta^2 \beta \\ -\alpha^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & k\alpha \beta i \end{pmatrix} = \begin{pmatrix} \alpha \beta & ik\alpha^2 \beta \\ 0 & (\alpha \beta)^{-1} \end{pmatrix}
\]

and \( \begin{pmatrix} 0 & \alpha \\ -\alpha^{-1} & 0 \end{pmatrix} \in \text{Mod}^\alpha \), i) follows.

ii) If \( \alpha = \beta \), then \( \begin{pmatrix} \alpha \beta & -k\alpha^2 \beta i \\ 0 & (\alpha \beta)^{-1} \end{pmatrix} \) is the inverse of \( \begin{pmatrix} \alpha \beta & k\alpha^2 \beta i \\ 0 & (\alpha \beta)^{-1} \end{pmatrix} \) and so \( G^\alpha_{\mu} = G^\alpha_{-k} \). Else if \( \alpha \neq \beta \) conjugating \( \text{Mod}^\alpha \) and \( B^{-1} \) with \( \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & 0 \end{pmatrix} \) we obtain \( \text{Mod}^\alpha \) and

\[
\begin{pmatrix} 0 & -\beta^{-1} \\ \beta & -k\alpha \beta i \end{pmatrix} ; \text{ hence } G^\alpha_{k,\beta} \sim G^\alpha_{-k}.
\]

iii) As \( G^\alpha_{k,\beta} \) is a discrete group \( J(G^\alpha_{k,\beta}) \geq 1 \). Since \( A \) is parabolic we have \( J(A, B) = \|\text{tr} AB - \text{tr} B\|^2 = |\alpha \beta|^2 = 1 \) (see the proof of Proposition 2). Hence \( J(G^\alpha_{k,\beta}) = 1 \).

We now compare our groups \( G^\alpha_{k,\beta} \) with groups considered by Sato. Suppose a pair of elements of Möb generates a nonelementary subgroup and the first element is parabolic. Then his pair is conjugate to a pair \( (A, B_{\sigma,\mu}) \) where \( A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) and \( B = \begin{pmatrix} \mu \sigma & (\mu^2 \sigma^2 - 1)/\sigma \\ \mu \sigma & 1/\sigma \end{pmatrix} \) and \( \sigma \neq 0 \) (see [10], [12]). Define \( G_{\sigma,\mu} = \langle A, B_{\sigma,\mu} \rangle \).

Notice that \( G_{\sigma,\mu} = G_{-\sigma,\mu} = G_{\sigma,-\mu} = G_{\sigma,\mu+1} \) and \( G_{\sigma,\mu} \) is conjugate in Möb to \( G_{\sigma,\mu+1/2} \). This follows from \( \langle A, B \rangle = \langle A, -B \rangle = \langle A, -B^{-1} \rangle = \langle A, AB \rangle \) and \( \langle A, B \rangle \sim \langle A, (A^{1/2})^{-1} BA^{1/2} \rangle \) where \( B = B_{\sigma,\mu} \) and \( A^{1/2} = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix} \).

For example the Whitehead link group \( \mathcal{W} \) is

\[
\mathcal{W} = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}
\]

\[= G_{1-i,1+i/2} \sim G_{1-i,1/2} = G_{1-i,-i/2} \]

(cf. [12, Theorem 2]).

We now extend our definition of \( G^\alpha_{k,\beta} \). If \( \alpha, \beta, k \in \mathbb{C} - \{0\} \) define

\[
G^\alpha_{k,\beta} = \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & k\alpha \beta i \end{pmatrix} \right\}.
\]

Because of the last proposition this definition coincides with the one given in the introduction if \( \alpha, \beta \in [1, i] \) and \( k \in \mathbb{Z} \). Conjugating with \( \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \), where \( \lambda^2 = \alpha \), we see that \( G^\alpha_{k,\beta} \sim G^{1+\alpha}_{k,\beta} \), and conjugating with \( \begin{pmatrix} 1 & -ki/2 \\ 0 & 1 \end{pmatrix} \), we get \( G^{1+\lambda}_{k,\beta} \sim G_{\sigma,ki/2} \).
We have following equalities $G^\alpha_\beta = G^{\alpha,-\beta}_k = G^{-\alpha}_k = G_\beta^{\alpha}$ (the last equality follows from $(A, B) = (A, B\alpha)$) and, conjugating with $\begin{pmatrix} 1 & -k\alpha \beta \\ 0 & 1 \end{pmatrix}$, we get $G^\alpha_\beta \sim G^{-\alpha}_k$.

We now describe which of the groups $G^\alpha_\beta$ are subgroups of Pic. First, if $G^\alpha_\beta \subset Pic$ we must have $\alpha, \beta, k \in \mathbb{Z} + i\mathbb{Z}$ and $|\beta| = 1$. Since $G^\alpha_\beta = G^{\alpha,-\beta}_k = G^{\alpha}_k$ we may assume $k \in \mathbb{Z}$ and $\beta \in [1, i]$.

The following theorem describes all the Jørgensen subgroups of Pic, up to conjugation in Pic.

**Theorem 6.** If $G$ is a rank two subgroup of Pic with $J(G) = 1$ then $G$ is conjugate in Pic to $G^\alpha_\beta$ where $\alpha, \beta \in [1, i]$ and $k$ is a nonnegative integer.

**Proof.** Let $A$ and $B$ be generators of $G$ such that

$$J(A, B) = |\text{tr}^2 A - 4| + |\text{tr}[A, B] - 2| = 1.$$

If $\text{tr} A \neq \pm 2$ then $|\text{tr}^2 A - 4| \geq 3$ hence $|\text{tr}^2 A - 4| = 0$ and $|\text{tr}[A, B] - 2| = 1$. $A$ is then parabolic with fixed point $a/c$ where $a$ and $c$ are relatively prime Gaussian integers. Let $b$ and $d$ be Gaussian integers such that $ad - bc = 1$. Conjugating $A$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Pic$ we obtain a parabolic element which fixes $\infty$. Hence we can assume that $A = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$, with $\alpha$ a nonzero Gaussian integer.

Write $B = \begin{pmatrix} x & y \\ \beta & z \end{pmatrix} \in Pic$. Then, as in the proof of Proposition 2,

$$1 = |\text{tr}[A, B] - 2| = |\text{tr}(AB) - \text{tr} B|^2 = |\alpha \beta|^2.$$

Hence $|\alpha| = |\beta| = 1$. Conjugating with $\begin{pmatrix} 1 & x \beta^{-1} \\ 0 & 1 \end{pmatrix}$ we see that the pair $(A, B)$ is conjugate in Pic to the pair $(A, \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & k\alpha \beta \end{pmatrix})$ where $k\alpha \beta i = x + z$ and $k$ is Gaussian integer.

Than $G \sim G^\alpha_\beta$ and since $G^\alpha_\beta = G^{\alpha,-\beta}_k = G^{-\alpha}_k = G^{\alpha}_k$ and $G^\alpha_\beta \sim G^\alpha_\beta$ we may assume that $\alpha, \beta \in [1, i]$ and $k$ is a nonnegative integer. \qed

4. **Section 3**

In this section and the next one we will use free products with amalgamation (see [6]).

Also we will use the $90^\circ$ rotation $R \in \text{Möb}$. Let $R = \begin{pmatrix} (1+i)/\sqrt{2} & 0 \\ 0 & (1-i)/\sqrt{2} \end{pmatrix} \in \text{Möb}$ (multiplication by $i$); this element does not belong to Pic. Then $R^{-1}\text{Pic} R = \text{Pic}$, $R^{-1}\text{Mod}^\alpha R = \text{Mod}^{\alpha'}$, $R^{-1}G^\alpha_\beta R = G^{\alpha'}_\beta$ where $\{\alpha, \alpha'\} = \{\beta, \beta'\} = [1, i]$; thus $G^\alpha_\beta \sim G^{\alpha'}_\beta$. This proves the if part of Theorem 1 4).
A presentation of Pic can be given as follows (see [1], [14]):

\[ \text{Pic} = \langle x, y, u, v : x^3 = y^3 = u^2 = v^2 = (uy)^2 = (yx)^2 = (uv)^2 = 1 \rangle \]

where \( u = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \), \( x = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \), \( v = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \) and \( y = \begin{pmatrix} 0 & i \\ i & 1 \end{pmatrix} \).

From this one can show \( \text{Pic} = V \cdot \text{Mod} \) where \( V = \langle \text{Mod}, v \rangle \) and \( Y = \langle \text{Mod}, y \rangle \).

We have also \( \text{Mod} = \langle u, x \rangle \) and \( \text{Mod}^d = \langle v, y \rangle \) and of course \( \text{Pic} = \langle \text{Mod}, \text{Mod}^d \rangle \).

We have the following presentations:

\[ V = \langle x, u, v : x^3 = u^2 = v^2 = (xy)^2 = (uv)^2 = 1 \rangle \]
\[ = \langle u, v \rangle \ast_{\langle v \rangle} \langle v, x \rangle = \mathbb{Z}_2^2 \ast_{\mathbb{Z}_2} D_3, \]
\[ Y = \langle x, y, u : x^3 = y^2 = u^2 = (uy)^2 = (yx)^2 = 1 \rangle \]
\[ = \langle u, y \rangle \ast_{\langle y \rangle} \langle y, x \rangle = D_3 \ast_{\mathbb{Z}_3} A_4, \]
\[ \text{Mod} = \langle x, u : x^3 = u^2 = 1 \rangle = \mathbb{Z}_2 \ast \mathbb{Z}_3 \]

where \( D_3 \) is the dihedral group of order six and \( A_4 \) is the alternating group in four elements.

We have that \( G_1^{i,j} = \text{Pic} \) because

\[ G_1^{i,j} = \left[ \text{Mod}^d, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right] \]
\[ \left[ \text{Mod}^d, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right] \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]
\[ = \langle \text{Mod}^d, \text{Mod} \rangle = \text{Pic} \]

since \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = u \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \). This implies \( G_1^{1,1} = R^{-1} G_1^{i,j} R = \text{Pic} \).

Notice also that \( G_0^{1,1} = \text{Mod} = \langle x, u \rangle, G_0^{0,i} = \text{Mod}^d = \langle y, v \rangle, G_0^{1,i} \text{Pic} = G_0^{1,1} = Y \). One can see that \( G_0^{1,1} = \langle u, x, (xy)^k \rangle, G_0^{i,j} = \langle v, y, (xy)^k \rangle \) and, using Proposition 5, \( G_0^{i,j} = \langle v, y, u(xu)^k \rangle \) and \( G_0^{1,j} = \langle u, x, (xy)^k v \rangle \).

The abelianizations of Pic, \( V, Y \) and \( \text{Mod} \) are \( \mathbb{Z}_2^2, \mathbb{Z}_2^2, \mathbb{Z}_2, \mathbb{Z}_0 \) respectively.

Denote by \( \overline{\text{Pic}} \) the abelianization of Pic and by ab: Pic \( \rightarrow \) Pic the abelianization map. We will write \( w = \text{ab}(w) \). We have that \( \overline{\text{Pic}} = \text{Pic} / \langle x, y \rangle = \langle \overline{u}, \overline{v} \rangle \simeq \mathbb{Z}_2^2 \).

**Proposition 7.** If \( \alpha = \beta \) and \( k \) is odd or if \( \alpha \neq \beta \) and \( k \) is even then \( \text{ab}(G_k^{\alpha,\beta}) = \overline{\text{Pic}} \). Otherwise

\[ \text{ab}(G_k^{\alpha,\beta}) = \begin{cases} \langle \overline{u} \rangle & \text{if } \alpha = 1 \\ \langle \overline{v} \rangle & \text{if } \alpha = i \end{cases} \]
Proof. We have $ab(G^{1,1}_{k}) = \langle \bar{u}, \bar{v}^k \rangle$, $ab(G^{1,i}_{k}) = \langle \bar{u}, \bar{v}^{k+1} \rangle$, $ab(G^{1,i}_{k}) = \langle \bar{u}, \bar{v}^k \rangle$. From these equalities the proposition follows.

Corollary 8. $G^{1,1}_{k} \cong G^{1,i}_{k}$ (resp. $G^{1,i}_{k} \cong G^{1,1}_{k}$) if $k$ is even (resp. $k$ is odd).

The following lemma will be used in the classification of the groups $G_{k}^{\alpha,\beta}$ in Möb.

Lemma 9. i) The trace of any element of $G^{1,\beta}_{k}$ is of the form $a + kbi$ or $ka + bi$ where $a, b \in \mathbb{Z}$.

ii) The trace of any element of $G^{1,1}_{k}$ is of the form $a + kbi$ where $a, b \in \mathbb{Z}$.

iii) $\pm(1 + ki)$ is the trace of an element of $G^{1,1}_{k}$ and $\pm(i + k)$ is the trace of an element of $G^{1,i}_{k}$.

Proof. The natural ring homomorphism from $\mathbb{Z} + i\mathbb{Z} \cong \mathbb{Z}[X]/(X^2 + 1)$ onto $\mathbb{Z}_k + i\mathbb{Z}_k \cong \mathbb{Z}_k[X]/(X^2 + 1)$ induces a group homomorphism $PSL(2, \mathbb{Z} + i\mathbb{Z}) \rightarrow PSL(2, \mathbb{Z}_k + i\mathbb{Z}_k)$. As $G^{1,\beta}_{k} \supset \text{Mod}$ we have, by Proposition 5,

$$G^{1,\beta}_{k} = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & kbi \end{pmatrix} \right\} = \left\{ \text{Mod}, \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & kbi \end{pmatrix} \right\}.$$

Then $\psi(G^{1,\beta}_{k}) = \left\{ \psi(\text{Mod}), \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & kbi \end{pmatrix} \right\}$ which is contained in

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}_k + i\mathbb{Z}_k) : a, b, c, d \in \mathbb{Z}_k \right\}$$

so the trace of any element of $\psi(G^{1,\beta}_{k})$ lies in $\mathbb{Z}_k + i\mathbb{Z}_k$. From this i) follows.

ii) is proved similarly.

To prove iii) observe that the trace of $\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & kbi \end{pmatrix}$ is $\beta^{-1} + kbi$.

The following theorem gives the classification of the groups $G^{\alpha,\beta}_{k}$, up to conjugation in Möb.

Theorem 10. If $G^{1,\beta}_{k} \sim G^{1,\beta'}_{k'}$ where $\beta, \beta' \in \{1, i\}$, $k \geq 0$ and $k' \geq 0$ then $\beta = \beta'$ and $k = k'$. 

Proof. As $G^1_1 (= \text{Pic})$ and $G^1_2 (= Y)$ have nonisomorphic abelianizations the case $k = k' = 1$ follows. If $(k, k') \neq (1, 1)$ and $(k, \beta) \neq (k', \beta')$ then, using the lemma, one sees that

$$\{\text{traces of elements of } G^1_{k, \beta}\} \neq \{\text{traces of elements of } G^1_{k', \beta'}\}$$

and therefore $G^1_{k, \beta} \cong \text{Mob} \cong G^1_{k', \beta'}$.

This completes the proof of Theorem 1.4.

5. Section 4

In this section we will think of Pic as $V \ast_{\text{Mod}} Y$. Define an integer valued function on Pic as follows:

$$\lambda(w) = \begin{cases} 
1 & \text{if } w \stackrel{\text{Pic}}{\sim} w' \in V \cup Y \\
2n & \text{if } w \sim v_1 y_1 \cdots v_n y_n, n \geq 1, v_i \in V, y_i \in Y (i = 1, \ldots, n).
\end{cases}$$

The function is well defined (see for example [7, Theorems 4.4 and 4.6] or [6, Chapter IV, Theorems 2.6 and 2.8]). Clearly if $w \stackrel{\text{Pic}}{=} w'$, $\lambda(w) = \lambda(w')$.

Recall that $G^1_{k, \beta} = \left(\text{Mod}, \left( \begin{array}{cc} \beta & -k\beta i \\ 0 & \beta^{-1} \end{array} \right) \right)$. Write and $t = \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right)$ and $T = \langle t \rangle$ and

$$s = \left( \begin{array}{cc} \beta & -k\beta i \\ 0 & \beta^{-1} \end{array} \right) = \left\{ \begin{array}{ll} (vy)^k & \text{if } \beta = 1 \\
u(yv)^{k-1}y & \text{if } \beta = i \end{array} \right.$$

Proposition 11. Consider the groups $G^1_{k, \beta}$ with $k \geq 2$. Then:

i) \hspace{1em} If $\beta = 1$ \hspace{.5em} (resp. $\beta = i$) \hspace{.5em} then $\langle s, t \rangle \simeq \mathbb{Z}^2$ \hspace{.5em} (resp. $\langle s, t \rangle \simeq \mathbb{D}_\infty$).

ii) \hspace{1em} There is an isomorphism $G^1_{k, \beta} \simeq \text{Mod} \ast_T \langle s, t \rangle$.

iii) \hspace{1em} $\lambda(s^{e_1}m_1 s^{e_2}m_2 \cdots s^{e_r}m_r) > 1$ where $e_j \neq 0$ (resp. $e_j = 1$) if $\beta = 1$ (resp. if $\beta = i$), $r \geq 1$, $m_j \in \text{Mod} - T$ ($j = 1, \ldots, r$).

Proof. Write $w = s^{e_1}m_1 s^{e_2}m_2 \cdots s^{e_r}m_r$. Let $\beta = 1$ so that $s = \left( \begin{array}{cc} 1 & -ki \\ 0 & 1 \end{array} \right)$ and $\langle s, t \rangle \simeq \mathbb{Z}^2$. Using the matrix expressions for the elements one can see that, if $m \in \text{Mod} - T$, then $ymv \not\in \text{Mod}$, $ymv^{-1} \not\in \text{Mod}$, $vmy^{-1} \in \text{Mod}$, $vmv \in \text{Mod}$ and $y^{-1}vmy \not\in \text{Mod}$. Using these facts we see that

$$\lambda(w) = \lambda((vy)^{e_1}m_1(vy)s^{e_2}m_2 \cdots (vy)^{e_r}m_r)$$

$$= 2k \sum_{j=1}^r |e_j| - \#\{l : e_le_{l+1} < 0\} \geq 2kr - r > 1.$$
Let $\beta = i$ so that $s = \begin{pmatrix} i & k \\ 0 & -i \end{pmatrix}$, $(s, t) \simeq \mathbb{D}_\infty$. Then

$$w = sm_1sm_2 \cdots sm_r = u(yv)(yv)^{k-2}ym_1u(yv)(yv)^{k-2}ym_2u(yv)(yv)^{k-2}ym_3 \cdots u(yv)(yv)^{k-2}ym_r \simeq (yv)^{k-2}y_1v(yv)^{k-2}y_2v(yv)^{k-2}y_3 \cdots v(yv)^{k-2}y_r$$

where $y_j = ym_juy$. As $m_j \in \text{Mod} - T$, one can verify that $y_j \in Y - \text{Mod}$. Therefore $\lambda(w) = r(2k - 2) > 1$.

This proves i) and iii). Assertion ii) follows from iii).

\[ \cdot \]

**Corollary 12.** For $k > 2$, $G^{1,j}_k \simeq G^{i,j}_k \simeq G^{1,1}_2 \simeq \text{Mod} \ast \mathbb{Z}_2$ and $G^{i,j}_k \simeq G^{i,1}_k \simeq G^{i,j}_2 \simeq \text{Mod} \ast \mathbb{Z} \mathbb{D}_\infty$.

**Corollary 13.** If $k \geq 2$ and $w \in G^{1,\beta}_k - \text{Mod}$ then $\lambda(w) \geq 1$.

**Proof.** It follows from the proposition observing that $\lambda(w) = 1$ if $w \in \text{Mod}$, $\lambda((vy)^{mk}) = 2|m|k > 1$ and $\lambda(u(yv)^{k-1}y) = 2k - 1 > 1$.

**Corollary 14.** The abelianization of $G^{1,1}_2$ is $\mathbb{Z} \oplus \mathbb{Z}_6$ and the abelianization of $G^{i,j}_2$ is $\mathbb{Z}_6$. We will use the number of conjugacy classes of elements of order two in $G^{\alpha,\beta}_k$; we will denote it by $c_2(G^{\alpha,\beta}_k)$.

**Corollary 15.** We have $c_2(G^{1,1}_0) = 1$, $c_2(G^{1,j}_0) = 3$, $c_2(G^{1,1}_1) = 4$, $c_2(G^{i,j}_1) = 2$, $c_2(G^{i,j}_2) = 1$ and $c_2(G^{i,j}_2) = 2$.

**Proof.** Recall that $G^{1,1}_0 = \text{Mod} = \mathbb{Z}_2 \ast \mathbb{Z}_3$, $G^{1,j}_0 = V = \mathbb{Z}_2^2 \ast \mathbb{Z}_2 \mathbb{D}_3$, $G^{1,1}_1 = \text{Pic} = V \ast \text{Mod} Y$, $G^{i,j}_1 \simeq G^{1,j}_1 = Y = \mathbb{D}_3 \ast \mathbb{Z}_3$, $A_4$, $G^{1,1}_2 \simeq \text{Mod} \ast \mathbb{Z}^2$ and $G^{i,j}_2 = \text{Mod} \ast \mathbb{Z} \mathbb{D}_\infty$. Using the fact that an element of finite order in a free product with amalgamation is conjugate to an element in a factor and using ab the corollary follows.

The following theorem states that if $(\alpha, \beta, k) \neq (\alpha', \beta', k')$ then $G^{\alpha,\beta}_k \simeq G^{\alpha',\beta'}_k$ with one exception (namely $G^{1,j}_1 = G^{1,j}_1 = \text{Pic}$).

**Theorem 16.** Let $\alpha, \beta, \alpha', \beta' \in \{1, i\}$, $k \geq 0$ and $k' \geq 0$. Suppose $G^{\alpha,\beta}_k \simeq G^{\alpha',\beta'}_k$ with $(\alpha, \beta, k) \neq (\alpha', \beta', k')$. Then $k = k' = 1$, $\alpha = \beta$ and $\alpha' = \beta'$.
Proof. As \( G^1_k \beta \sim G^1_k \beta' \) we have, by Theorem 1 4), that \( k = k' \) and \( \alpha \beta = \pm \alpha' \beta' \) and so we may assume that \( \alpha = 1 \) and \( \alpha' = i \). Hence \( k \leq 1 \).

Suppose \( k \geq 2 \). No conjugate, in Pic, of \( v \) lies in Mod because \( ab(\text{Mod}) = \langle \bar{u} \rangle \). Therefore, by Corollary 13, no conjugate, in Pic, of \( v \) lies in \( G^1_k \beta \). As \( v \in G^1_k \beta' \), we have \( G^1_k \beta \text{Pic} \cong G^1_k \beta' \).

Suppose \( k = 1 \), and \( \beta = i \). Then \( \beta' = 1 \) and \( ab(G^1_1 \beta) = \langle \bar{u} \rangle \neq \langle \bar{v} \rangle = G^1_k \beta' \) so \( G^1_1 \beta \text{Pic} \cong G^1_k \beta' \).

Suppose \( k = 0 \), and \( \beta = 1 \). Then \( \beta' = 1 \) and \( ab(G^1_0 \beta) = \langle \bar{u} \rangle \neq \langle \bar{v} \rangle = G^1_k \beta' \) so \( G^1_0 \beta \text{Pic} \cong G^1_k \beta' \).

Finally suppose \( k = 0 \), and \( \beta = i \). Then \( \beta' = 1 \). We have \( G^1_0 \beta = V = \langle v, u, x \rangle \) and \( G^1_0 \beta' = \langle v, y, u \rangle \). There is an inner automorphism \( \phi \) of Pic such that \( \phi(G^1_0 \beta) = G^1_0 \beta' \) and we have a commutative diagram

\[
\begin{array}{ccc}
G^1_k \beta & \xrightarrow{\phi} & G^1_k \beta' \\
\downarrow{\theta'} & & \downarrow{\theta} \\
\langle \bar{u}, \bar{v} \rangle & & \langle \bar{u}, \bar{v} \rangle
\end{array}
\]

where \( \theta' \) and \( \theta \) are the restrictions of \( ab \). Then \( \theta'^{-1}(\langle \bar{u} \rangle) \simeq \theta^{-1}(\langle \bar{u} \rangle) \) which is impossible because, since \( [V, \text{Mod}] = 2 \), \( \theta^{-1}(\langle \bar{u} \rangle) = \text{Mod} \) and \( \theta'(\langle u \rangle) \supset \langle y, u \rangle \simeq \mathbb{D}_3 \) and \( \mathbb{D}_3 \) is not isomorphic to a subgroup of \( \text{Mod} \).

\[\square\]

\textbf{Theorem 17.} Let \( \alpha, \beta \in \{1, i\}, k \geq 0 \). Then \( G^k_1 \beta \) is isomorphic to one of the groups \( \text{Mod}, V, \text{Pic}, Y, \text{Mod} \ast \mathbb{Z} \mathbb{Z}^2 \) and \( \text{Mod} \ast \mathbb{Z} \mathbb{D}_\infty \). These six groups are pairwise nonisomorphic.

Proof. The first assertion is a consequence of \( G^i_k \beta \simeq G^1_i \beta' \), where \( \{\beta, \beta'\} = \{1, i\} \), and Corollary 12. Now \( V, \text{Pic} \) and \( \text{Mod} \ast \mathbb{Z} \mathbb{D}_\infty \) have abelianization \( \mathbb{Z}^2_3 \) while \( \text{Mod}, Y \) and \( \text{Mod} \ast \mathbb{Z} \mathbb{Z}^2 \) have pairwise non isomorphic abelianizations different from \( \mathbb{Z}^2_3 \). Since \( c_2(V) = 3 \), \( c_2(\text{Pic}) = 4 \) and \( c_2(\text{Mod} \ast \mathbb{Z} \mathbb{D}_\infty) = 2 \), the theorem follows.

\[\square\]

6. Section 5

In what follow ab: \( \text{Pic} \to \overline{\text{Pic}} = \langle \bar{u}, \bar{v} \rangle \) is the abelianization map, where \( u = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \), \( \mathbb{D}_3 \) is the dihedral group of order six and \( A_4 \) is the alternating group in four letters.

In what follows \( \mathbb{D}_3 \) is the dihedral group of order six, \( A_4 \) is the alternating group in for letters ab: \( \text{Pic} \to \overline{\text{Pic}} = \langle \bar{u}, \bar{v} \rangle \) is the abelianization map, where \( u = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \), \( v = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \).
The following table has the information

<table>
<thead>
<tr>
<th>#(conjugacy classes of elements of order two)</th>
<th>a group isomorphic to $G_k^{\alpha, \beta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>image under ab of $G_k^{\alpha, \beta}$</td>
<td>abelianization of $G_k^{\alpha, \beta}$</td>
</tr>
</tbody>
</table>

for the group $G_k^{\alpha, \beta}$.

<table>
<thead>
<tr>
<th>$G_k^{\alpha, \beta}$</th>
<th>$(\alpha, \beta) = (1, 1) \ (\text{resp.} \ (i, i))$</th>
<th>$(\alpha, \beta) = (1, i) \ (\text{resp.} \ (i, 1))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k = 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\langle \bar{u} \rangle \ (\text{resp.} \ \langle \bar{v} \rangle)$</td>
<td>$\mathbb{Z}_6$</td>
<td>$\langle \bar{u}, \bar{v} \rangle$</td>
</tr>
<tr>
<td>$k = 1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\langle \bar{u}, \bar{v} \rangle$</td>
<td>$\mathbb{Z}_2^2$</td>
<td>$\langle \bar{u} \rangle \ (\text{resp.} \ \langle \bar{v} \rangle)$</td>
</tr>
<tr>
<td>$K = 2, 4, \ldots$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\langle \bar{u} \rangle \ (\text{resp.} \ \langle \bar{v} \rangle)$</td>
<td>$\mathbb{Z}^2 \oplus \mathbb{Z}_6$</td>
<td>$\langle \bar{u} \rangle \ (\text{resp.} \ \langle \bar{v} \rangle)$</td>
</tr>
<tr>
<td>$k = 3, 5, \ldots$</td>
<td></td>
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</tr>
<tr>
<td>$\langle \bar{u}, \bar{v} \rangle$</td>
<td>$\mathbb{Z}^2 \oplus \mathbb{Z}_6$</td>
<td>$\langle \bar{u}, \bar{v} \rangle$</td>
</tr>
</tbody>
</table>

References


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