Kojima, H.
Osaka J. Math.
48 (2011), 1063–1084

OPEN ALGEBRAIC SURFACES WITH $\bar{\kappa} = \bar{p}_g = 0$ AND $\bar{P}_2 > 0$

HIDEO KOJIMA

(Received February 1, 2010, revised May 19, 2010)

Abstract

Let S be a smooth open rational surface with $\bar{\kappa}(S) = \bar{p}_g(S) = 0$ and $\bar{P}_2(S) > 0$. We construct a certain minimal model of S, which is called a strongly minimal model of S in [15], and determine the strongly minimal model in the case where S has non-contractible boundary at infinity. As an application, we classify the log affine surfaces with $\bar{\kappa} = \bar{p}_g = 0$ and $\bar{P}_2 > 0$ under the minimality condition.

0. Introduction

Throughout the present article, we work over the complex number field $\mathbb{C}$.

In the theory of logarithmic Kodaira dimension due to Iitaka, the class of (not necessarily complete) algebraic varieties with logarithmic Kodaira dimension zero is very important because such varieties can appear as general fibers of Iitaka fiber spaces. It is clear that a smooth open (non-complete) algebraic curve with logarithmic Kodaira dimension zero is $\mathbb{A}^1_* := \mathbb{A}^1 - \{0\}$. Open algebraic surfaces with logarithmic Kodaira dimension zero have been studied by several authors. Irrational open algebraic surfaces with $\bar{\kappa} = 0$ were studied in Iitaka [7], Sakai [22, Section 2], Miyanishi [15, Theorem 6.4.1 (p. 184)], etc. Tsunoda [23] proved that, for a smooth open algebraic surface S with $\bar{\kappa}(S) = 0$, $\bar{P}_n(S) = 1$ for some n, $1 \leq n \leq 66$. Iitaka [7] and Zhang [24] considered open rational surfaces with $\bar{\kappa} = 0$ and $\bar{p}_g > 0$ and Zhang [24] classified the Iitaka surfaces which are almost minimal open rational surfaces with $\bar{\kappa} = 0$ and $\bar{p}_g > 0$. Log Enriques surfaces (normal projective rational surfaces with only quotient singular points and with numerically trivial canonical divisors), whose smooth parts are interesting examples of open algebraic surfaces with $\bar{\kappa} = \bar{p}_g = 0$, were studied by Blache, Kudryavtsev, Oguiso and Zhang. For more details, see Blache [3], Kudryavtsev [12] and [13], Oguiso–Zhang [19], [20] and [21], Zhang [25], [26], [27], [28] and [29]. In [9] and [10], the author established a classification theory of smooth open rational surfaces with $\bar{\kappa} = 0$ and with connected boundaries at infinity in any characteristic and gave a classification of the strongly minimal smooth affine surfaces with $\bar{\kappa} = 0$, which gives a generalization of Fujita's result concerning the smooth affine surfaces with $\bar{\kappa} = 0$ and with finite Picard groups (see [4, Section 8]).

2000 Mathematics Subject Classification. Primary 14J26; Secondary 14R05.

The purpose of the present article is to study smooth open rational surfaces with $\bar{\kappa} = \bar{p}_g = 0$ and $\bar{P}_2 > 0$. Let S be a smooth open rational surface with $\bar{\kappa}(S) = \bar{p}_g(S) = 0$ and $\bar{P}_2(S) > 0$ and let (X, B) be a pair of a smooth projective rational surface X and a simple normal crossing divisor B on X such that $S = X - B$ (we call such a pair (X, B) an SNC-completion of S). In Sections 1 and 2, following [15, Chapter 2] (see also [16]), we construct an almost minimal model (W, C) and a strongly minimal model (V, D) of the pair (X, B). Here the pairs (W, C) and (V, D) are SNC-pairs and there exist birational morphisms $f\colon X \to W$ and $g\colon W \to V$ such that $f_*(B) = C$ and $g_*(C) = D$. Further, in Section 1, we give a rough classification of possible connected components of $\operatorname{Supp} C$. In Section 3, we determine the pair (V, D) when $\lfloor D^\# \rfloor \neq 0$. The main result of the present article is Theorem 3.6 which gives a classification of the strongly minimal models. In Section 4, by using the result in Section 3, we classify the strongly minimal log affine surfaces with $\bar{\kappa} = \bar{p}_g = 0$ and $\bar{P}_2 > 0$ (cf. Theorem 4.4).

In a forthcoming paper, we study smooth open rational surfaces with $\bar{\kappa} = \bar{P}_2 = 0$.

1. Preliminaries

The terminology is the same as the one in [15]. By a $(-n)$-curve, we mean a smooth complete rational curve (on a smooth algebraic surface) with self-intersection number $-n$. A reduced effective divisor D is called an NC-divisor (resp. an SNC-divisor) if D has only normal crossings (resp. simple normal crossings). Let V be a smooth projective surface, let D, D_1 and D_2 be divisors on V and let S be a smooth open algebraic surface. We then employ the following notations. For the definitions of $\bar{\kappa}$, $\bar{p}_g$ and $\bar{P}_m$, see [15, Chapter 2].

K_V: the canonical divisor on V.
$\rho(V)$: the Picard number of V.
$\bar{\kappa}(S)$: the logarithmic Kodaira dimension of S.
$\bar{p}_g(S)$ (or $\bar{P}_1(S)$): the logarithmic geometric genus of S.
$\bar{P}_m(S)$ ($m \geq 2$): the logarithmic m-genus of S.
$\mathbb{F}_n$ ($n \geq 0$): a Hirzebruch surface of degree n.
M_n ($n \geq 0$): a minimal section of $\mathbb{F}_n$.
$\bar{M}_n$ ($n \geq 0$): a section of the fixed ruling on $\mathbb{F}_n$ with $\bar{M}_n \cdot M_n = 0$.
$\#(D)$: the number of all irreducible components in $\operatorname{Supp} D$.
$f^*(D)$: the total transform of D.
$f_*(D)$: the direct image of D.
$f'(D)$: the proper transform of D.
$D_1 \sim D_2$: D_1 and D_2 are linearly equivalent.
$D_1 \equiv D_2$: D_1 and D_2 are numerically equivalent.
$\lfloor D^\# \rfloor$: the integral part of a $\mathbb{Q}$-divisor $D^\#$.

Now we recall some basic notions in the theory of peeling. For more details, see [15, Chapter 2] and [16, Chapter 1].

Let (X, B) be a pair of a smooth projective surface X and an SNC-divisor B. We call such a pair (X, B) an *SNC-pair*. A connected curve T consisting of irreducible components of B (a connected curve in B, for short) is a *twig* if each irreducible component of T is rational, the dual graph of T is a linear chain and T meets $B - T$ in a single point at one of the end components of T, the other end of T is called the *tip* of T. A connected curve R (resp. F) in B is a *rational rod* (resp. a *rational fork*) if R (resp. F) is a connected component of B and consists only of rational curves and if the dual graph of R (resp. F) is a linear chain (resp. the dual graph of the exceptional curves of the minimal resolution of a non-cyclic quotient singular point). A connected curve E in B is *admissible* if there are no (-1)-curves in $\operatorname{Supp} E$ and the intersection matrix of E is negative definite. An admissible rational twig T in B is *maximal* if T is not extended to an admissible rational twig with more irreducible components of B. By a (-2)-rod (resp. a (-2)-fork), we mean a rod (resp. a fork) consisting only of (-2)-curves.

Let $\{T_\lambda\}$ (resp. $\{R_\mu\}$, $\{F_\nu\}$) be the set of all maximal admissible rational twigs (resp. all admissible rational rods, all admissible rational forks), where no irreducible components of T_λ's belong to R_μ's or F_ν's. Then there exists a unique decomposition of B as a sum of effective $\mathbb{Q}$-divisors $B = B^\# + \operatorname{Bk}(B)$ such that the following two conditions (i) and (ii) are satisfied:

(i) $\operatorname{Supp}(\operatorname{Bk}(B)) = \left(\bigcup_\lambda T_\lambda\right) \cup \left(\bigcup_\mu R_\mu\right) \cup \left(\bigcup_\nu F_\nu\right)$.

(ii) $(B^\# + K_X) \cdot Z = 0$ for every irreducible component Z of $\operatorname{Supp}(\operatorname{Bk}(B))$.

We call the divisor $\operatorname{Bk}(B)$ the *bark* of B and say that $B^\# + K_X$ is produced by the *peeling* of B. Let $\pi\colon X \to \bar{X}$ be the contraction of $\operatorname{Supp}(\operatorname{Bk}(B))$ to quotient singular points and put $\bar{B} := \pi_*(B)$. Then, by the condition (ii) as above, we have $\pi^*(\bar{B} + K_{\bar{X}}) = B^\# + K_X$.

Lemma 1.1. *Each connected component of $B - (B^\#)_{\mathrm{red}}$ is either a (-2)-rod or a (-2)-fork.*

Proof. See [15, p. 94]. □

DEFINITION 1.2. An SNC-pair (X, B) is *almost minimal* if, for every irreducible curve C on X, either $(B^\# + K_X) \cdot C \geq 0$ or $(B^\# + K_X) \cdot C < 0$ and the intersection matrix of $C + \operatorname{Bk}(B)$ is not negative definite.

Lemma 1.3. *Let (X, B) be an SNC-pair. Then there exists a birational morphism $\mu\colon X \to W$ onto a smooth projective surface W such that the following four conditions (1)–(4) are satisfied*:

(1) *$C := \mu_*(B)$ is an SNC-divisor.*

(2) *$\mu_*(\operatorname{Bk}(B)) \leq \operatorname{Bk}(C)$ and $\mu_*(B^\# + K_X) \geq C^\# + K_W$.*

(3) *$\bar{P}_n(X - B) = \bar{P}_n(W - C)$ for every integer $n \geq 1$. In particular, $\bar{\kappa}(X - B) = \bar{\kappa}(W - C)$.*

(4) *The pair* (W, C) *is almost minimal.*

Proof. See [15, Theorem 3.11.1 (p. 107)]. □

We call the SNC-pair (W, C) as in Lemma 1.3 an *almost minimal model* of (X, B).

Lemma 1.4. *Let* (W, C) *be an almost minimal SNC-pair with* $\bar{\kappa}(W - C) = 0$. *Then* $n(C^{\#} + K_W) \sim 0$ *for some integer* $n > 0$. *In particular,* $C^{\#} + K_W \equiv 0$.

Proof. See [15, Chapter 2, Section 6]. (See also [8].) □

Hereafter in the present section, let (W, C) be an almost minimal SNC-pair with $\bar{\kappa}(W - C) = 0$. Then $\kappa(W) \leq 0$, where $\kappa(W)$ denotes the Kodaira dimension of W. We prove the following two lemmas, which are well-known for experts.

Lemma 1.5. *Assume that* $\kappa(W) = 0$. *Then the following assertions hold.*
(1) W *is minimal.*
(2) *If* $C \neq 0$, *then each connected component of* C *is either a* (-2)*-rod or a* (-2)*-fork.*

Proof. Let H be an ample divisor on W. Since $\kappa(W) = 0$ and $C^{\#} + K_W \equiv 0$ by Lemma 1.4, we have $H \cdot K_W = 0$. So the assertion (1) follows. Moreover, since $C^{\#} \cdot H = 0$, we have $C^{\#} = 0$. So the assertion (2) follows from Lemma 1.1. □

Lemma 1.6 (cf. [15, Theorem 6.4.1 (2) (p. 184)]). *Assume that* W *is an irrational ruled surface. Let* $p\colon W \to B$ *be a* $\mathbb{P}^1$*-fibration onto a smooth projective curve* B *of genus* $q(W)$ (≥ 1) *and let* $C_1, \ldots, C_s$ $(s \geq 0)$ *be all the irrational components of* C. *Then the following assertions hold true.*
(1) $s = 1$ *or* 2.
(2) *For a fiber* F *of* p, *we have* $\left(\sum_{i=1}^{s} C_i\right) \cdot F = 2$.
(3) *Each* C_i $(1 \leq i \leq s)$ *is an elliptic curve and becomes a connected component of* C, *i.e.,* $C_i \cdot (C - C_i) = 0$.
(4) $q(W) = 1$, *i.e.,* W *is an elliptic ruled surface.*
(5) *If* $C - \sum_{i=1}^{s} C_i \neq 0$, *then each connected component of* $C - \sum_{i=1}^{s} C_i$ *is either a* (-2)*-rod or a* (-2)*-fork.*
(6) *If* $s = 1$ *(resp.* $s = 2$*), then* $C^{\#} + K_W \not\sim 0$ *and* $2(C^{\#} + K_W) \sim 0$ *(resp.* $C^{\#} + K_W \sim 0$*).*

Proof. Let F be a fiber of p.

(1) If $s = 0$, then every irreducible component of C is contained in a fiber of p. Then $\bar{\kappa}(W - C) = -\infty$, a contradiction. So, $s \geq 1$. Since $C^{\#} + K_W \equiv 0$ by Lemma 1.4, we have

$$F \cdot C^{\#} = -F \cdot K_W = 2.$$

Note that the coefficient of C_i $(1 \le i \le s)$ in $C^\#$ is equal to one because C_i is an irrational curve. Hence $s = 1$ or 2.

(2) Since each irreducible component of $C - \sum_{i=1}^{s} C_i$ is contained in a fiber of p, we have $F \cdot C^\# = F \cdot \left(\sum_{i=1}^{s} C_i\right)$. So, $F \cdot \left(\sum_{i=1}^{s} C_i\right) = 2$ since $C^\# + K_V \equiv 0$.

(3) Since $C^\# + K_W \equiv 0$ and the coefficient of C_i $(1 \le i \le s)$ in $C^\#$ is equal to one, we have

$$\begin{aligned} 0 = C_i \cdot (C^\# + K_W) &= C_i \cdot (C^\# - C_i) + C_i \cdot (C_i + K_W) \\ &\ge C_i \cdot (C_i + K_W) \\ &\ge 0 \end{aligned}$$

for $1 \le i \le s$. So, C_i is a smooth elliptic curve and $C_i \cdot (C^\# - C_i) = C_i \cdot (C - C_i) = 0$ for $1 \le i \le s$.

(4) The assertion easily follows from the assertion (3).

(5) The assertion (2) and [15, Theorem 2.5.1 (p. 76)] imply that $\bar{\kappa}\left(W - \sum_{i=1}^{s} C_i\right) \ge 0$. In particular, $\bar{\kappa}\left(W - \sum_{i=1}^{s} C_i\right) = 0$. Since $C^\# + K_W \equiv 0$ and some multiple of $\sum_{i=1}^{s} C_i + K_W$ is linearly equivalent to an effective divisor, we deduce that $C^\# - \sum_{i=1}^{s} C_i = 0$. Hence the assertion follows from Lemma 1.1.

(6) See [15, Lemma 6.4.3 (p. 186)]. Here we note that if $s = 1$ then $H^0(W, C^\# + K_W) = H^0(W, C_1 + K_W) = 0$ by [7, Proposition 20] (see also [22, (2.7) Theorem]). Hence $C^\# + K_W \not\sim 0$ if $s = 1$. □

In Lemmas 1.7 and 1.8, we consider the case where W is a rational surface.

Lemma 1.7. *Assume that W is a rational surface. Let I be the smallest positive integer such that $IC^\#$ is an integral divisor. Then*

$$\bar{P}_n(W - C) = \begin{cases} 1, & \text{if} \quad I \mid n, \\ 0, & \text{if otherwise}. \end{cases}$$

Proof. Since $\bar{\kappa}(W - C) = 0$, $\bar{P}_n(W - C) \le 1$ for any positive integer n. By [15, Lemma 3.10.1 (p. 106)], we have

$$\bar{P}_n(W - C) = h^0(W, n(C + K_W)) = h^0(W, \lfloor n(C^\# + K_W) \rfloor).$$

Since W is a rational surface and $C^\# + K_W \equiv 0$, we know that

$$\bar{P}_n(W - C) > 0 \Longleftrightarrow n(C^\# + K_W) \sim 0 \Longleftrightarrow I \mid n. \qquad \square$$

Lemma 1.8. *With the same notations and assumptions as in* Lemma 1.7, *assume further that $\bar{p}_g(W - C) = 0$ and $\bar{P}_2(W - C) > 0$ $(\Longrightarrow \bar{P}_2(W - C) = 1)$. Let $\bar{C}$ be a connected component of C. Assume that $\bar{C}$ is neither a* (-2)*-rod nor a* (-2)*-fork. Then we have*:

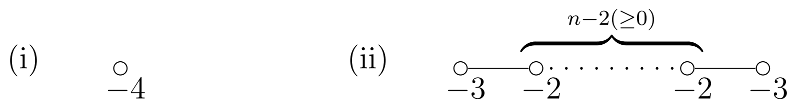

Fig. 1.

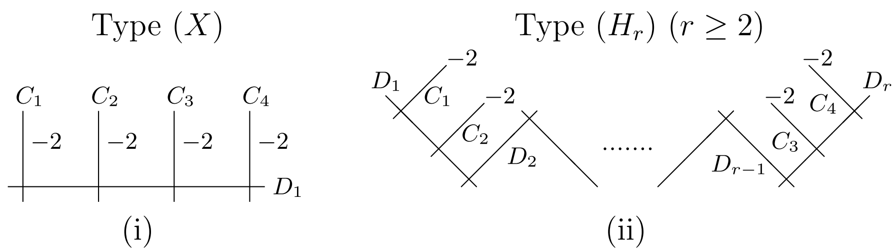

Fig. 2.

(1) *If* $\lfloor \bar{C}^{\#} \rfloor = 0$, *then* $\bar{C}$ *is either a single* (−4)*-curve or an admissible rational rod with* (−3)*-curves as tip components and* m ($m \geq 0$) (−2)*-curves as middle components. In particular, the weighted dual graph of* $\bar{C}$ *is given as one of* (i) *and* (ii) *in* Fig. 1.
(2) *If* $\lfloor \bar{C}^{\#} \rfloor \neq 0$, *then each component of* C *is a rational curve and the configuration of* C *is given as one of* (i) *and* (ii) *in* Fig. 2.

Proof. Note that $C^{\#} = (C - \bar{C})^{\#} + \bar{C}^{\#}$ because $\bar{C}$ is a connected component of C. Since W is a rational surface and $\bar{p}_g(W - C) = h^0(W, C + K_W) = 0$, each irreducible component of C is a (smooth) rational curve and the dual graph of each connected component of C is a tree by [14, Lemma I.2.1.3]. It follows from Lemma 1.7 and the assumption $\bar{P}_2(W - C) > 0$ that $2C^{\#}$ is an integral divisor. Hence, the coefficient of each irreducible component of $\mathrm{Supp}(C^{\#})$ in $C^{\#}$ is equal to 1/2 or 1.

Assume that $\lfloor \bar{C}^{\#} \rfloor = 0$. Then $\bar{C}$ can be contracted to a quotient singular point. Since $\bar{C}$ is neither a (−2)-rod nor a (−2)-fork, $\bar{C}^{\#} \neq 0$. So, $\bar{C}^{\#} = (1/2)\bar{C}$. It then follows from [25, Lemma 1.8] that $\bar{C}$ is either a single (−4)-curve or an admissible rational rod with (−3)-curves as tip components and m ($m \geq 0$) (−2)-curves as middle components.

Assume that $\lfloor \bar{C}^{\#} \rfloor \neq 0$. Then, since $(C + K_W) \cdot C = \mathrm{Bk}(C) \cdot C$, it follows from [4, Lemma (8.7) and Corollary (8.8)] that $\bar{C}$ is of type (O), type (H), type (Y) or type (X) (for more details, see [4, Corollary (8.8)]). Since the dual graph of $\bar{C}$ is a tree and $2\bar{C}^{\#}$ is an integral divisor, we know that $\bar{C}$ is of type (H) or (X). Hence, the configuration of $\bar{C}$ is given as one of (i) and (ii) in Fig. 2. □

DEFINITION 1.9. Let C be a connected SNC-divisor on a smooth surface such that each component of C is a rational curve. Then we say that C is of type (K_1) (resp. (K_n) ($n \geq 2$), (X), (H_r) ($r \geq 2$)) if C is a single (−4)-curve (resp. C is an admissible rational rod with (−3)-curves as tip components and $(n-2)$ (−2)-curves as middle components, $\#(C)=5$ and the configuration of C is given as in Fig. 2-(i), $\#(C)=r+4$ and the configuration of C is given as in Fig. 2-(ii)).

2. Construction of strongly minimal models

In this section, we construct strongly minimal models of smooth open rational surfaces with $\bar{\kappa}=\bar{p}_g=0$ and $\bar{P}_2>0$.

Let S be a smooth open rational surface with $\bar{\kappa}=\bar{p}_g=0$ and $\bar{P}_2>0$ and let (X,B) be an SNC-pair such that $X-B\cong S$. We call the pair (X,B) an SNC-completion of S. Let (W,C) be an almost minimal model of (X,B). Then Lemma 1.8 implies that each connected component of $\mathrm{Supp}(C^{\#})$ is of type (K_n) $(n\geq 1)$, (X) or (H_r) $(r\geq 2)$. Throughout the present section, we retain this situation.

Lemma 2.1. *Assume that $(W, C-\lfloor C^{\#}\rfloor)$ is not almost minimal. Then there exists a* (−1)*-curve E such that $E\cdot((C-\lfloor C^{\#}\rfloor)^{\#}+K_W)<0$ and the intersection matrix of $E+\mathrm{Bk}(C-\lfloor C^{\#}\rfloor)$ is negative definite. Moreover, the following assertions hold true.*
(1) $E\cdot C=1$ *or* 2.
(2) *Assume that $E\cdot C=1$ and $E\not\subset \mathrm{Supp}\,C$. Let C_i be the irreducible component of C meeting E. Then the coefficient of C_i in $C^{\#}$ is equal to one.*
(3) *If $E\cdot C=1$ and $E\subset \mathrm{Supp}\,C$, then the connected component C' of C containing E is of type (H_r), $r\geq 3$ and $E=D_i$ $(2\leq i\leq r-1)$ with the same notations as in* Fig. 2-(ii).
(4) *If $E\cdot C=2$, then $E\not\subset \mathrm{Supp}\,C$ and E meets two connected components C' and C'' of C such that C' is of type (X) or (H_r), C'' is an admissible rational rod and E meets one of the tip components of C''. Furthermore, we have*:
(4-i) *If $E\cdot\lfloor C'^{\#}\rfloor>0$ (then $E\cdot C'=E\cdot C'^{\#}=1$), then C'' is a* (−2)*-rod.*
(4-ii) *If $E\cdot\lfloor C'^{\#}\rfloor=0$, then C'' is of type (K_n) and E meets one of the four terminal components of C'.*

Proof. Since $(W, C-\lfloor C^{\#}\rfloor)$ is not almost minimal, there exists an irreducible curve E such that $E\cdot((C-\lfloor C^{\#}\rfloor)^{\#}+K_W)<0$ and the intersection matrix of $E+\mathrm{Bk}(C-\lfloor C^{\#}\rfloor)$ is negative definite. Then $E^2<0$. Here we note that every connected component of $C-\lfloor C^{\#}\rfloor$ is a (−2)-rod, a (−2)-fork or a divisor of type (K_n) (see Definition 1.9). Then $E\not\subset \mathrm{Supp}(C-\lfloor C^{\#}\rfloor)$ and so $E\cdot K_W<0$. Hence, E is a (−1)-curve. By [15, Lemma 3.6.3 (p. 96)], $E\cdot(C-\lfloor C^{\#}\rfloor)\leq 2$. We consider the following three cases separately.

CASE 1: $E\cdot(C-\lfloor C^{\#}\rfloor)=0$. If $E\not\subset \mathrm{Supp}(\lfloor C^{\#}\rfloor)$, then $E\cdot C=E\cdot C^{\#}=-E\cdot K_V=1$ and so E meets only one irreducible component, say C_i, of C. Moreover, the coefficient of C_i in $C^{\#}$ is equal to one. If $E\subset \mathrm{Supp}(\lfloor C^{\#}\rfloor)$, then $E\cdot C=E\cdot C^{\#}=1$ and $E\cdot(C-E)=2$. We can easily see that the connected component $\bar{C}$ of C containing E is of type (H_r), $r\geq 3$ and $E=D_i$ $(2\leq i\leq r-1)$ with the same notations as in Fig. 2-(ii).

CASE 2: $E\cdot(C-\lfloor C^{\#}\rfloor)=1$. In this case, let $\tilde{C}$ be the connected component of $C-\lfloor C^{\#}\rfloor$ meeting E. Suppose that $\tilde{C}$ is a divisor of type (K_n). Then

$$1=-E\cdot K_W=E\cdot C^{\#}=E\cdot\lfloor C^{\#}\rfloor+\frac{1}{2}E\cdot\tilde{C}=E\cdot\lfloor C^{\#}\rfloor+\frac{1}{2},$$

which is a contradiction. Hence, $\tilde{C}$ is a (-2)-rod and E meets a terminal component of $\tilde{C}$, here we note that the intersection matrix of $E+\tilde{C}$ is negative definite. If $\tilde{C}\subset\operatorname{Supp}(C^{\#})$, then

$$(\lfloor C^{\#}\rfloor+K_W)\cdot E=-\frac{1}{2}E\cdot\tilde{C}=-\frac{1}{2},$$

which is a contradiction. Hence, $\tilde{C}$ is a connected component of C. Then $E\cdot\lfloor C^{\#}\rfloor=E\cdot C^{\#}=-E\cdot K_W=1$ and so $E\cdot C=2$.

CASE 3: $E\cdot(C-\lfloor C^{\#}\rfloor)=2$. Then E meets two connected components $\tilde{C}_1$ and $\tilde{C}_2$ of $C-\lfloor C^{\#}\rfloor$ (see [15, Lemma 3.7.1 (p. 97)]). Since the intersection matrix of $E+\mathrm{Bk}(C)$ is negative definite and $E\cdot((C-\lfloor C^{\#}\rfloor)^{\#}+K_V)<0$, we may assume that $\tilde{C}_1$ is a (-2)-rod and $\tilde{C}_2$ is a divisor of type (K_n). Moreover, E meets a terminal component of $\tilde{C}_1$. If $\tilde{C}_1$ is a connected component of C, then

$$1=-E\cdot K_W=E\cdot C^{\#}=E\cdot\lfloor C^{\#}\rfloor+\frac{1}{2}E\cdot\tilde{C}_2=E\cdot\lfloor C^{\#}\rfloor+\frac{1}{2},$$

which is a contradiction. So, $\tilde{C}_1\subset\operatorname{Supp}(C^{\#})$. In particular, $\tilde{C}_1$ is a (-2)-curve. Since the intersection matrix of $E+(C-\lfloor C^{\#}\rfloor)$ is negative definite and $E\cdot(C^{\#}+K_W)=0$, we know that $E\cdot C=E\cdot(C-\lfloor C^{\#}\rfloor)=2$ and E meets a terminal component of $\tilde{C}_2$.

As seen from the arguments as in Cases 1–3, we obtain the assertions (1)–(4). □

Now, let E be a (-1)-curve on W such that $E\cdot((C-\lfloor C^{\#}\rfloor)^{\#}+K_W)<0$ and the intersection matrix of $E+\mathrm{Bk}(C-\lfloor C^{\#}\rfloor)$ is negative definite. Let $g\colon W\to W_1$ be a successive contraction of (-1)-curves in $\operatorname{Supp}(E+(C-\lfloor C^{\#}\rfloor))$ starting with the contraction of E such that the image of $E+(C-\lfloor C^{\#}\rfloor)$ has no (-1)-curves. Put $C^{(1)}:=g_*(C)$. From Lemma 2.1, we know that $C^{(1)}$ is an SNC-divisor, $(C^{(1)})^{\#}=g_*(C^{\#})$ and $2((C^{(1)})^{\#}+K_{W_1})=g_*(2(C^{\#}+K_W))\sim 0$. In particular, the pair $(W,C^{(1)})$ is an almost minimal SNC-pair with $\bar{\kappa}(W-C^{(1)})=\bar{p}_g(W-C^{(1)})=0$ and $\bar{P}_2(W-C^{(1)})>0$. By repeating this process, we obtain the following lemma.

Lemma 2.2. *With the same notations as above, there exists a birational morphism $\nu\colon X\to V$ onto a smooth projective rational surface V such that the following conditions* (1)–(4) *are satisfied*:

(1) $D:=\nu_*(B)$ *is an SNC-divisor.*

(2) $\nu_*(\mathrm{Bk}(B))\le\mathrm{Bk}(D)$ *and* $\nu_*(B^{\#}+K_X)\ge D^{\#}+K_V$.

(3) $\bar{P}_n(V - D) = \bar{P}_n(X - B)$ *for any integer* $n \geq 1$. *In particular,* $\bar{\kappa}(V - D) = \bar{\kappa}(X - B) = 0$.
(4) *The pairs* (V, D) *and* $(V, D - \lfloor D^{\#} \rfloor)$ *are almost minimal.*

We call the pair (V, D) (resp. the surface $V - D$) as in Lemma 2.2 an *strongly minimal model* of (X, B) (resp. the surface $S = X - B$).

3. Classification

In this section, we classify the strongly minimal open rational surfaces of $\bar{\kappa} = \bar{p}_g = 0$ and $\bar{P}_2 > 0$ with non-contractible boundaries at infinity (cf. Theorem 3.6). First of all, we give some examples (Examples 3.1–3.5). In the following examples, let M_n be a minimal section of the fixed ruling on a Hirzebruch surface $\mathbb{F}_n$ of degree n $(n \geq 0)$ and let $\bar{M}_n$ be a section of the ruling on $\mathbb{F}_n$ with $\bar{M}_n \cdot M_n = 0$.

EXAMPLE 3.1. Let $V_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and let C_1 be an irreducible curve such that $C_1 \sim 2M_0 + l$, where l is a fiber of the fixed ruling π on V_0. Let P_1 and P_2 be the two ramification points of a double covering $\pi|_{C_1} \colon C_1 \to \mathbb{P}^1$ and let l_i $(i = 1, 2)$ be the fiber of π passing through P_i. Let l_j $(j = 3, 4)$ be a fiber of π meeting C_1 in distinct two points, say P_j and P_j'. Let $f \colon V \to V_0$ be a composite of blowing-ups over $P_1, \ldots, P_4$ such that the following conditions are satisfied:
(i) For $i = 1, 2$, $r_i := \#(f^*(l_i)_{\mathrm{red}}) \neq 2$. Moreover, if $r_i \geq 3$, then $\mathrm{Supp}(f^*(l_i))$ consists entirely of a (-1)-curve E_i and (-2)-curves $D_{i,2}, \ldots, D_{i,r_i}$ and $f^*(l_i) = 2(E_i + D_{i,2} + \cdots + D_{i,r_i-2}) + D_{i,r_i-1} + D_{i,r_i}$.
(ii) For $i = 3, 4$, $f^*(l_i) = D_i + 2E_i + D_i'$, where D_i and D_i' are (-2)-curves and E_i is a (-1)-curve.
Put $D_0 := f'(C_1)$. Then $(D_0)^2 = 4 - (r_1 + r_2)$. For $i = 1, 2$, we put

$$D^{(i)} := \begin{cases} \sum_{k=2}^{r_i} D_{i,k} & \text{if} \quad r_i \neq 1, \\ 0 & \text{if} \quad r_i = 1. \end{cases}$$

The divisor $D^{(i)}$ $(i = 1, 2)$ can be contracted to two rational double points of type A_1 (resp. one rational double point of type A_3, one rational double point of type D_{r_i-1}) if $r_i = 3$ (resp. $r_i = 4$, $r_i \geq 5$). Put

$$D := D_0 + D^{(1)} + D^{(2)} + D_3 + D_3' + D_4 + D_4'.$$

Then it is easy to see that $D^{\#} = D_0 + (1/2) \sum_{j=3}^{4} (D_j + D_j')$ and $D^{\#} + K_V \equiv 0$. So, $\bar{\kappa}(V - D) = \bar{p}_g(V - D) = 0$ and $\bar{P}_2(V - D) = 1$. We say that the pair (V, D) is of type $X[4 - (r_1 + r_2)] + F_1 + F_2$, where $F_i = 0$ (resp. $F_i = 2A_1$, $F_i = A_3$, $F_i = D_{r_i-1}$) if $r_i = 1$ (resp. $r_i = 3$, $r_i = 4$, $r_i \geq 5$) for $i = 1, 2$. Note that (V, D) is the pair as in [10, Example 2.1] if $r_1 = r_2 = 1$.

EXAMPLE 3.2 (cf. [10, Example 2.2]). Let $V_0 = \mathbb{F}_n$ $(n \geq 1)$ and let l_0, l_1 and l_2 be three distinct fibers of the ruling on V_0. Put $P_i := l_i \cap \bar{M}_n$ $(i = 1, 2)$. Let $\mu_0\colon V_1 \to V_0$ be the blowing-up with centers P_1 and P_2. Put $E_i := \mu_0^{-1}(P_i)$ $(i = 1, 2)$. Furthermore, let $\mu_1\colon V_2 \to V_1$ be the blowing-up with centers $E_1 \cap \mu_0'(l_1)$ and $E_2 \cap \mu_0'(l_2)$. Put $V := V_2$ and

$$D := \mu_1'(E_1 + E_2 + \mu_0'(l_0 + l_1 + l_2 + M_n + \bar{M}_n)).$$

Then $\bar{\kappa}(V - D) = \bar{p}_g(V - D) = 0$ and $\bar{P}_2(V - D) = 1$. Further, the configuration of D is given as in Fig. 2-(ii), where $r = 3$, $(D_1)^2 = -n$, $(D_2)^2 = 0$ and $(D_3)^2 = n - 2$. We note that if $n > 1$ then the elementary transformations with centers at $P_0 := l_0 \cap \bar{M}_n$ and its infinitely near points will reduce the case $n > 1$ to the case $n = 1$. We say that the pair (V, D) is of type $H[-1, 0, -1]$.

EXAMPLE 3.3 (cf. [10, Examples 2.3 and 2.4]). Let $V_0 = \mathbb{F}_n$ $(n \geq 0)$. Let $C_1 = M_n$ and let C_2 be a smooth irreducible curve such that $C_2 \sim M_n + (n + 1)l$, where l is a fiber of the fixed ruling on $\mathbb{F}_n$. Let l_1 and l_2 be fibers of the ruling with $P_i := l_i \cap C_2 \notin C_1 \cap C_2$ $(i = 1, 2)$. Let $\mu_0\colon V_1 \to V_0$ be the blowing-up with centers P_1 and P_2. Put $E_i := \mu_0^{-1}(P_i)$ $(i = 1,2)$, $l_i' := \mu_0'(l_i)$ $(i = 1,2)$ and $C_i' := \mu_0'(C_i)$ $(i = 1,2)$. Let $\mu_1\colon V_2 \to V_1$ be the blowing-up with centers $Q_i := E_i \cap l_i'$ $(i = 1,2)$. Put $V := V_2$ and

$$D := \mu_1'(E_1 + l_1' + C_1' + E_2 + l_2' + C_2').$$

Then $\bar{\kappa}(V - D) = \bar{p}_g(V - D) = 0$ and $\bar{P}_2(V - D) = 1$. Further, the configuration of D is given as in Fig. 2-(ii), where $r = 2$, $(D_1)^2 = -n$ and $(D_2)^2 = n$. We say that the pair (V, D) is of type $H[n, -n]$.

EXAMPLE 3.4. Let (W, C) be an SNC-pair of type $H[1, -1]$ constructed as in Example 3.3 such that the configuration of C is given as in Fig. 2-(ii), where $r = 2$, $(D_1)^2 = -1$ and $(D_2)^2 = 1$. Then $F := 2D_1 + C_1 + C_2$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F|}\colon W \to \mathbb{P}^1$ and D_2 becomes a 2-section of Φ. Let G be the fiber of Φ containing C_3. Since $\rho(W) = 6$, we can easily see that $G = C_3 + C_4 + 2E'$, where E' is a (-1)-curve and $E' \cdot C_3 = E' \cdot C_4 = 1$. Since $\Phi|_{D_2}\colon D_2 \to \mathbb{P}^1$ is a double covering and $P = \operatorname{Supp} F \cap D_2$ is a ramification point of $\Phi|_{D_2}$, there exists uniquely a fiber H of Φ such that $Q := \operatorname{Supp} H \cap D_2$ is the ramification point of $\Phi|_{D_2}$ other than P. It is clear that H is irreducible. Let $\mu\colon V \to W$ be a composite of blowing-ups over Q such that $\mu^*(H) = 2(E + H_1 + \cdots + H_{s-2}) + H_{s-1} + H_s$, where $s \geq 2$, E is a (-1)-curve and $H_1, \ldots, H_s$ are (-2)-curves. Put

$$D := \mu'(C) + \sum_{i=1}^{s} H_i.$$

Then, $D^\# = \mu'(C^\#) = \mu'(D_1 + D_2) + (1/2)\mu'\left(\sum_{i=1}^{4} C_i\right)$ and $D^\# + K_V \equiv 0$. So, $\bar{\kappa}(V - D) = \bar{p}_g(V - D) = 0$ and $\bar{P}_2(V - D) = 1$. We say that the pair (V, D) is of type $H[1-s,-1]+F$, where $F=2A_1$ (resp. $F=A_3$, $F=D_s$) if $s=2$ (resp. $s=3$, $s\geq 4$).

EXAMPLE 3.5. Let $V_0=\mathbb{F}_n$ ($n\geq 0$) and let $l_1,\ldots,l_4$ be distinct four fibers of the fixed ruling on V_0. Put $P_i:=l_i\cap \bar{M}_n$ for $i=1,2$ and $P_j:=l_j\cap M_n$ for $j=3,4$. Let $\mu_1\colon V_1\to V_0$ be the blowing-up with centers $P_1,\ldots,P_4$. Put $E_i:=\mu^{-1}(P_i)$ and $Q_i:=E_i\cap\mu_1'(l_i)$ ($i=1,\ldots,4$). Let $\mu_2: V_2\to V_1$ be the blowing-up with centers $Q_1,\ldots,Q_4$. Put $V:=V_2$ and

$$D:=\mu_2'\left(\sum_{i=1}^{4}E_i+\mu_1'\left(M_n+\bar{M}_n+\sum_{i=1}^{4}l_i\right)\right).$$

Then D consists of two connected components and each connected component of D is of type (X). We can easily see that $D^{\#}=(\mu_2\circ\mu_1)'(C_1+M_n)+(1/2)\mu_2'\big(\sum_{i=1}^{4}E_i+\mu_1'\big(\sum_{i=1}^{4}l_i\big)\big)$ and $D^{\#}+K_V\equiv 0$. So, $\bar{\kappa}(V-D)=\bar{p}_g(V-D)=0$ and $\bar{P}_2(V-D)=1$. We say that the pair (V,D) is of type $2X_n$.

The following theorem is the main result of the present article.

Theorem 3.6. *Let (W,C) be an almost minimal SNC-pair such that W is a rational surface, $\bar{\kappa}(W-C)=\bar{p}_g(W-C)=0$, $\bar{P}_2(W-C)>0$ and $\lfloor C^{\#}\rfloor\neq 0$. Let (V,D) be a strongly minimal model of (W,C). Then the pair (V,D) is one of the pairs enumerated in* Examples 3.1–3.5.

In what follows, we prove Theorem 3.6.

Let (V,D) be the same pair as in Theorem 3.6. By Lemma 1.8, we can decompose D as a sum of connected components

$$D=\sum_{i=1}^{r}D^{(i)}+\sum_{j=1}^{s}D^{(r+j)}+\sum_{k=1}^{t}D^{(r+s+k)}\quad (r,s,t\geq 0),$$

where $D^{(i)}$ ($1\leq i\leq r$) is a divisor of type (X) (if $r_1=1$) or type (H_{r_i}) (if $r_i\geq 2$), $D^{(r+j)}$ ($1\leq j\leq s$) is a (-2)-rod or a (-2)-fork, and $D^{(r+s+k)}$ ($1\leq k\leq t$) is a divisor of type (K_{n_k}). By the hypothesis $\lfloor C^{\#}\rfloor\neq 0$ and the construction of strongly minimal models (see Section 2), we know that $\lfloor D^{\#}\rfloor\neq 0$, i.e., $r>0$. For $1\leq i\leq r$, let

$$D^{(i)}=\sum_{i'=1}^{r_i}D_{i'}^{(i)}+\sum_{l=1}^{4}C_l^{(i)}$$

be the irreducible decomposition of $D^{(i)}$ such that the configuration of $D^{(i)}$ is given as in Fig. 2, where $r = r_i$, $D_{i'} = D_{i'}^{(i)}$ and $C_l = C_l^{(i)}$. Then,

$$D^{\#} = \sum_{i=1}^{r}\left(\sum_{i'=1}^{r_i} D_{i'}^{(i)} + \frac{1}{2}\sum_{l=1}^{4} C_l^{(i)}\right) + \frac{1}{2}\sum_{k=1}^{t} D^{(r+s+k)}.$$

Let $\mu_1\colon V' \to V$ be the blowing-up of all the singular points (the intersection points of the irreducible components) of $D^{(r+s+1)}, \dots, D^{(r+s+t)}$. Then $\mu_1'(D^{(r+s+k)})$ ($k = 1, \dots, t$) is a disjoint union of n_k (-4)-curves. Since $2(D^{\#} + K_V) \sim 0$ and $D^{\#} = \sum_{i=1}^{r}\bigl(\sum_{i'=1}^{r_i} D_{i'}^{(i)} + (1/2)\sum_{l=1}^{4} C_l^{(i)}\bigr) + (1/2)\sum_{k=1}^{t} D^{(r+s+k)}$, we have

$$\mu_1'\left(\sum_{i=1}^{r}\left(\sum_{l=1}^{4} C_l^{(i)}\right) + \sum_{k=1}^{t} D^{(r+s+k)}\right) \sim -2\left(\sum_{i=1}^{r}\left(\sum_{i'=1}^{r_i} \mu_1'(D_{i'}^{(i)})\right) + K_{V'}\right).$$

Hence, there exists a double covering $\mu_2\colon V'' \to V'$ with the branch locus $\mu_1'\bigl(\sum_{i=1}^{r}\bigl(\sum_{l=1}^{4} C_l^{(i)}\bigr) + \sum_{k=1}^{t} D^{(r+s+k)}\bigr)$, here V'' is a smooth projective surface.

Put $\mu := \mu_2 \circ \mu_1$. Then $\mu'\bigl(\sum_{i=1}^{r}\bigl(\sum_{l=1}^{4} C_l^{(i)}\bigr)\bigr)$ (resp. $\mu'(D^{(r+s+k)})$ $(1 \le k \le t)$) is a disjoint union of $4r$ (-1)-curves (resp. n_k (-2)-curves). Further, for each i $(1 \le i \le r)$, $\mu'\bigl(\sum_{i'=1}^{r_i} D_{i'}^{(i)}\bigr)$ is a smooth elliptic curve (resp. a loop of $2(r_i - 1)$ smooth rational curves) if $r_i = 1$ (resp. $r_i \ge 2$). Let $\nu\colon V'' \to \tilde{V}$ be the contraction of the $4r$ (-1)-curves $\mu'(C_l^{(i)})$'s $(1 \le i \le r,\ 1 \le l \le 4)$. Put $D_{\tilde{V}} := \nu_*(\mu^{-1}(D))$. Then we can easily see that $D_{\tilde{V}}$ is an SNC-divisor, $D_{\tilde{V}}^{\#} = \nu_*\bigl(\mu'\bigl(\sum_{i=1}^{r}\bigl(\sum_{i'=1}^{r_i} D_{i'}^{(i)}\bigr)\bigr)\bigr)$ and $D_{\tilde{V}}^{\#} + K_{\tilde{V}} \sim 0$. In particular, $(\tilde{V}, D_{\tilde{V}})$ is an almost minimal SNC-pair with $\bar{\kappa}(\tilde{V} - D_{\tilde{V}}) = 0$ and $\bar{p}_g(\tilde{V} - D_{\tilde{V}}) = 1$.

Lemma 3.7. *With the same notations and assumptions as above, we have*:
(1) $r = 1$ *or* 2.
(2) *If* $r = 1$, *then* $\tilde{V}$ *is a rational surface. In particular, the pair* $(\tilde{V}, D_{\tilde{V}})$ *is an Iitaka surface* (*see* [24]).
(3) *If* $r = 2$, *then* $\tilde{V}$ *is an elliptic ruled surface and* $r_1 = r_2 = 1$.

Proof. By the hypothesis $r \ge 1$, $D_{\tilde{V}}$ contains either a smooth elliptic curve or a loop of smooth rational curves. We infer from Lemma 1.5 (2) that $\kappa(\tilde{V}) = -\infty$.

Assume that $\tilde{V}$ is a rational surface. Then the pair $(\tilde{V}, D_{\tilde{V}})$ is an Iitaka surface. It follows from [24, Lemma 1.5] that $\lfloor D_{\tilde{V}}^{\#} \rfloor$ is connected. Hence, $r = 1$.

Assume that $\tilde{V}$ is an irrational ruled surface. Then Lemma 1.6 (4) implies that $\tilde{V}$ is an elliptic ruled surface. Moreover, since $D_{\tilde{V}}^{\#} + K_{\tilde{V}} \sim 0$, $\lfloor D_{\tilde{V}}^{\#} \rfloor$ is a disjoint union of two elliptic curves by Lemma 1.6 (6). Hence, $r = 2$ and $r_1 = r_2 = 1$. □

Lemma 3.8. *The SNC-pair* $(V, D - \lfloor D^{\#} \rfloor)$ *is almost minimal and* $\bar{\kappa}(V - (D - \lfloor D^{\#} \rfloor)) = -\infty$.

Proof. Since (V, D) is a strongly minimal model of (W, C), the first assertion is clear. We prove the second assertion. We can easily see that $(D - \lfloor D^{\#} \rfloor)^{\#} = (1/2)\sum_{k=1}^{t} D^{(r+s+k)} = D^{\#} - \sum_{i=1}^{r}(D^{(i)})^{\#}$. Then

$$(D - \lfloor D^{\#} \rfloor)^{\#} + K_V = D^{\#} - \sum_{i=1}^{r}(D^{(i)})^{\#} + K_V \equiv -\sum_{i=1}^{r}(D^{(i)})^{\#}$$

and so $(D - \lfloor D^{\#} \rfloor)^{\#} + K_V$ is not nef. Hence, the second assertion follows from the first assertion and [15, Theorem 3.15.1 (p. 116)]. □

Lemma 3.9. *Let* $\pi\colon V \to \bar{V}$ *be the contraction of* $\mathrm{Supp}(D - \lfloor D^{\#} \rfloor)$ *to quotient singular points. Then, there exists a* $\mathbb{P}^1$*-fibration* $h\colon \bar{V} \to \mathbb{P}^1$ *such that every fiber of* h *is irreducible. In particular,*

$$\rho(V) = 2 + \#(D - \lfloor D^{\#} \rfloor).$$

Proof. As seen from the proof of Lemma 3.8, we know that $(D - \lfloor D^{\#} \rfloor)^{\#} + K_V$ is not nef. Since $(D - \lfloor D^{\#} \rfloor)^{\#} + K_V \equiv \pi^*(K_{\bar{V}})$, $K_{\bar{V}}$ is not nef, neither. Hence there exists an extremal rational curve $\bar{l}$ on $\bar{V}$. Let l be the proper transform of $\bar{l}$ on V. Since $(V, D - \lfloor D^{\#} \rfloor)$ is almost minimal, we infer from [15, Lemma 3.14.3 (p. 113)] that one of the following two cases takes place:
(a) The intersection matrix of $l + \mathrm{Bk}(D - \lfloor D^{\#} \rfloor)$ is negative semi-definite, but not negative definite. Furthermore, $(\bar{l})^2 = 0$.
(b) $\rho(\bar{V}) = 1$ and $-K_{\bar{V}}$ is ample. Namely, $\bar{V}$ is a rank one log del Pezzo surface (for the definition, see [11, Definition 1.1]).

Suppose that the case (b) takes place. By [1, Proposition 1] (see also [25, Lemma 1.8]), every singular point of $\bar{V}$ has index ≤ 2. So, $\bar{V}$ is a rank one log del Pezzo surface of index ≤ 2. On the other hand, since $r \geq 1$, $\bar{V}$ contains at least four rational double points of type A_1. This contradicts [17, Lemma 3] and [1] (for more details, see [2], [11, Theorem 1.1], [18]). Hence, the case (b) does not take place.

By [15, Lemma 3.14.4 (p. 114)], for a sufficiently large integer n, the complete linear system $|n\bar{l}|$ defines a $\mathbb{P}^1$-fibration $h\colon \bar{V} \to \mathbb{P}^1$. Since the SNC-pair $(V, D - \lfloor D^{\#} \rfloor)$ is almost minimal, $\bar{V}$ is relatively minimal, i.e., there exist no irreducible curves $\bar{C}$ on $\bar{V}$ with $(\bar{C})^2 < 0$ and $\bar{C} \cdot K_{\bar{V}} < 0$ (cf. [6, p. 469], [15, Chapter 2, Section 4]). Hence, every fiber of h is irreducible. This proves the first assertion. Since $\rho(\bar{V}) = 2$, the second assertion is clear. □

Now, let $\Phi = h \circ \pi\colon V \to \mathbb{P}^1$. Then Φ is a $\mathbb{P}^1$-fibration. Let F be a fiber of Φ. We infer from Lemma 3.9 that F is a singular fiber of Φ if and only if $\pi(F) \cap \mathrm{Sing}\bar{V} \neq \emptyset$.

We prove the following lemma.

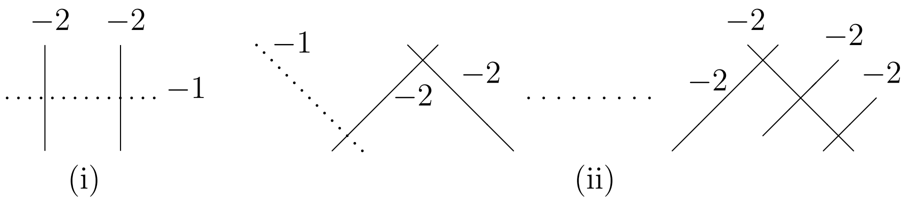

Fig. 3.

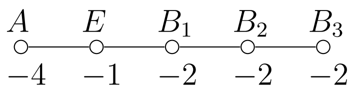

Fig. 4.

Lemma 3.10. *Let F be a singular fiber of Φ. Then F consists only of a* (−1)-*curve and* (−2)-*curves. Moreover, the configuration of F_{red} is given as one of* (i) *and* (ii) *in* Fig. 3.

Proof. By Lemma 3.9, we know that $\operatorname{Supp} F$ consists of a (−1)-curve and some connected components of $D - \lfloor D^{\#} \rfloor$. Let E be the unique (−1)-curve contained in $\operatorname{Supp} F$. Note that each connected component of $F_{\mathrm{red}} - E$ is a (−2)-rod, a (−2)-fork or a divisor of type (K_n).

If every connected component of $F_{\mathrm{red}} - E$ is a (−2)-rod or a (−2)-fork, then we can easily see that the configuration of F_{red} is given as one of (i) and (ii) in Fig. 3 (cf. [9, Lemma 5.5]).

Suppose that $F_{\mathrm{red}} - E$ contains divisors of type (K_n). Then, since F can be contracted to a smooth rational curve with self-intersection number zero, we know that the weighted dual graph of F_{red} is given as in Fig. 4, where $F_{\mathrm{red}} = A + E + B_1 + B_2 + B_3$. Both of A and $B_1 + B_2 + B_3$ are connected components of D. In particular, $B_1 + B_2 + B_3 = D^{(r+j)}$ for some j, $1 \leq j \leq s$. Since

$$(D - \lfloor D^{\#} \rfloor) \cdot E = E \cdot (A + B_1) = 2$$

and $D^{\#} + K_V \equiv 0$, we have

$$0 = E \cdot (D^{\#} + K_V) = E \cdot (\lfloor D^{\#} \rfloor + K_V) + \frac{1}{2} E \cdot A = E \cdot \lfloor D^{\#} \rfloor + \frac{1}{2},$$

which is a contradiction. □

As a consequence of Lemma 3.10, we obtain the following lemma.

Lemma 3.11. $t = 0$. *Namely, D contains no divisors of type (K_n).*

By Lemma 3.7, $r = 1$ or 2. In the following lemma, we consider the case $r = 2$.

Lemma 3.12. *Assume that* $r = 2$. *Then the pair* (V, D) *can be constructed as in* Example 3.5. *In particular,* $s = 0$.

Proof. We infer from Lemmas 3.7 (1) and 3.11 that

$$D^{\#} = D_1^{(1)} + D_2^{(2)} + \frac{1}{2}\left(\sum_{i=1}^{4} C_i^{(1)} + \sum_{i=1}^{4} C_i^{(2)}\right).$$

So $D-(D_1^{(1)}+D_2^{(2)}) = D-\lfloor D^{\#}\rfloor$ is contained in fibers of Φ. We note that neither $D_1^{(1)}$ nor $D_1^{(2)}$ is a fiber component of Φ. Indeed, if $D_1^{(i)}$ ($i = 1$ or 2) is a fiber component of Φ, then the divisor $D^{(i)}$ is contained in a fiber of Φ, which contradicts Lemma 3.10. Let F_i ($i = 1, 2, 3, 4$) be the fiber of Φ containing $C_i^{(1)}$. By Lemma 3.10, $F_1 = C_1^{(1)} + 2E_1 + B'$, where E_1 is a (-1)-curve, B' is a (-2)-curve and $E_1 \cdot C_1^{(1)} = E_1 \cdot B' = 1$.

Claim 1. $B' = C_j^{(2)}$ *for some* j, $1 \le j \le 4$.

Proof. Suppose that $B' \not\subset \mathrm{Supp}(D^{\#})$. Then the coefficient of B' in $D^{\#}$ is zero and $E_1 \cdot \left(\sum_{i=1}^4 C_i^{(1)} + \sum_{i=1}^4 C_i^{(2)}\right) = E_1 \cdot C_1^{(1)} = 1$. Since $D^{\#} + K_V \equiv 0$, we have

$$\begin{aligned} 0 &= E_1 \cdot (D^{\#} + K_V) \\ &= E_1 \cdot \left(D_1^{(1)} + D_1^{(2)} + \frac{1}{2}C_1^{(1)} + K_V\right) \\ &= E_1 \cdot (D_1^{(1)} + D_1^{(2)}) - \frac{1}{2}, \end{aligned}$$

which is a contradiction. Hence, $B' \subset \mathrm{Supp}(D^{\#})$. Since neither $D_1^{(1)}$ nor $D_1^{(2)}$ is a fiber component of Φ, $B' \neq D_1^{(1)}, D_1^{(2)}$.

Suppose that $B' \subset \mathrm{Supp}(D^{(1)})$, i.e., $B' = C_j^{(1)}$ ($2 \le j \le 4$). It then follows from $D^{\#} + K_V \equiv 0$ that $E_1 \cdot D = E_1 \cdot \left(\sum_{i=1}^4 C_i^{(1)}\right) = 2$. So, $F_1 \cdot D^{(2)} = 0$, i.e., $\mathrm{Supp}(D^{(2)})$ is contained in a fiber of Φ. This is a contradiction because $D_1^{(2)}$ is not a fiber component of Φ. Hence, $B' \subset \mathrm{Supp}(D^{(2)})$. □

By Claim 1, we may assume that

$$F_i = C_i^{(1)} + 2E_i + C_i^{(2)},$$

where E_i is a (-1)-curve with $E_i \cdot C_i^{(1)} = E_i \cdot C_i^{(2)} = 1$, for $i = 1, \ldots, 4$. Then $D_1^{(1)}$ and $D_1^{(2)}$ are sections of Φ.

Claim 2. *$F_1, \ldots, F_4$ exhaust all the singular fibers of Φ. In particular, $\rho(V) = 10$.*

Proof. Suppose to the contrary that Φ has a singular fiber G other than $F_1, \ldots, F_4$. Then, by Lemma 3.10, $\operatorname{Supp} G$ has a unique (-1)-curve E' and $\operatorname{Supp}(G_{\mathrm{red}} - E') \subset \operatorname{Supp}(D - (D^{(1)} + D^{(2)}))$. Since $1 = E' \cdot D^{\#} = E' \cdot (D_1^{(1)} + D_1^{(2)})$, we have $E' \cdot D_1^{(i)} = 1$ for $i = 1$ or 2. However, this is a contradiction because the coefficient of E' in G is equal to two and $D_1^{(1)}$ and $D_1^{(2)}$ are sections of Φ

Therefore, Φ has no singular fibers other than $F_1, \ldots, F_4$. It is then clear that $\rho(V) = 2 + \sum_{i=1}^{4}(\#(F_i) - 1) = 10$. □

By Claims 1 and 2, we can easily see that the pair (V, D) can be constructed as in Example 3.5. □

In the subsequent argument, we consider the case $r = 1$. We put $D_i := D_i^{(1)}$ $(1 \le i \le r_1)$ and $C_j := C_j^{(1)}$ $(1 \le j \le 4)$. Then

$$D^{\#} = \sum_{i=1}^{r_1} D_i + \frac{1}{2}\sum_{j=1}^{4} C_j.$$

Lemma 3.13. *Assume that $r = 1$ and $s = 0$, i.e., D is connected. Then the pair (V, D) is of type $X[2]$ (cf.* Example 3.1), *$H[-1, 0, -1]$ (cf.* Example 3.2) *or $H[n, -n]$ (cf.* Example 3.3).

Proof. Since D is connected and the pair (V, D) is a strongly minimal model of (W, C), we have $D \cdot E \ge 2$ for any (-1)-curve E. So the pair (V, D) is strongly minimal in the sense of [9, Section 2] (see also [10]). Hence, the assertion follows from [9, Theorem 4.5] (see also [10, Theorem 2.10]). □

From now on, we assume further that $s > 0$, i.e., D is not connected. Let F_1 be the fiber of Φ containing C_1. We prove the following lemma.

Lemma 3.14. *With the same notation and assumptions as above, we have*:
(1) *$F_1 = C_1 + 2E_1 + B'$, where E_1 is a (-1)-curve, B' is a (-2)-curve and $E_1 \cdot C_1 = E_1 \cdot B' = 1$.*
(2) *$B' = C_j$ for some j, $2 \le j \le 4$.*
(3) *If $r_1 \ge 2$, then $B' = C_2$.*

Proof. (1) If D_1 is not a fiber component of Φ, then the assertion follows from Lemma 3.10. We assume that D_1 is a fiber component of Φ. Then D_1 and C_2 are contained in $\operatorname{Supp}(F_1)$. If D_1 is a (-1)-curve, then $F_1 = 2D_1 + C_1 + C_2$, which proves the assertion. Suppose that $(D_1)^2 \leq -2$. By virtue of Lemma 3.10, D_r is not a fiber component of Φ (see the proof of Lemma 3.12). Since (V, D) is a strongly minimal model of (W, C), none of $D_2, \dots, D_{r-1}$ are (-1)-curves. It then follows from Lemma 3.10 that $F_1 = 2(E_1 + D_s + D_{s-1} + \cdots + D_1) + C_1 + C_2$, where $1 \leq s \leq r-1$ and E_1 is a (-1)-curve with $E_1 \cdot D_s = 1$. Since $D - \lfloor D^\# \rfloor = D - (D_1 + \cdots + D_r)$ is contained in fibers of Φ and $D^\# + K_V \equiv 0$, we know that $E_1 \cdot D = E_1 \cdot D_s = 1$. This is a contradiction because (V, D) is a strongly minimal model of (W, C). The assertion is thus verified.

(2) If either B' is not a component of $\mathrm{Supp}(D^\#)$ or $B' \subset \mathrm{Supp}\left(\sum_{i=1}^{r_1} D_i\right)$, then $E_1 \cdot (C_1 + C_2 + C_3 + C_4) = E_1 \cdot C_1 = 1$ because C_j $(j = 2, 3, 4)$ is contained in a fiber of Φ different from F_1. Since $D^\# + K_V \equiv 0$, we have

$$
\begin{aligned}
0 &= E_1 \cdot (D^\# + K_V) \\
&= E_1 \cdot \left(\sum_{i=1}^{r_1} D_i + \frac{1}{2} C_1 + K_V \right) \\
&= E_1 \cdot \left(\sum_{i=1}^{r_1} D_i \right) - \frac{1}{2},
\end{aligned}
$$

which is a contradiction. Hence, $B' = C_j$ for some j, $2 \leq j \leq 4$.

(3) Suppose that $B' \neq C_2$. We may assume that $B' = C_3$. Then $F_1 = C_1 + 2E_1 + C_3$ and D_1 and D_{r_1} are sections of Φ. Let F_2 be the fiber of Φ containing C_2. Then we can easily see that $F_2 = C_2 + 2E_2 + C_4$, where E_2 is a (-1)-curve with $E_2 \cdot C_2 = E_2 \cdot C_4 = 1$. By the assumption $s > 0$, Φ has a singular fiber F other than F_1 and F_2. By Lemma 3.10, $\mathrm{Supp}\, F$ has a unique (-1)-curve, say E, and the coefficient of E in F is equal to two. If $\mathrm{Supp}(F_{\mathrm{red}} - E) \cap \mathrm{Supp}(D^{(1)}) \neq \emptyset$, then we infer from Lemma 3.10 that $F = 2E + F_1 + F_2$, where F_1 and F_2 are (-2)-curves, F_1 is a (-2)-rod in D and F_2 is a component of $D_1 + \cdots + D_r = D^{(1)} - (C_1 + C_2 + C_3 + C_4)$. This is a contradiction because (V, D) is a strongly minimal model of (W, C) (see Lemma 2.1 (4), (4-i)). Hence $\mathrm{Supp}(F_{\mathrm{red}} - E) \cap \mathrm{Supp}(D^{(1)}) = \emptyset$. Then E must meet both of D_1 and D_{r_1}. However, this a contradiction because D_1 and D_{r_1} are sections of Φ. Therefore, $B' = C_2$. □

In the following lemma, we consider the case $r_1 = 1$.

Lemma 3.15. *With the same notation as above, assume further that* $r_1 = 1$. *Then the pair* (V, D) *can be constructed as in* Example 3.1.

Proof. By Lemma 3.14 (2), we may assume that $F_1 = C_1 + 2E_1 + C_3$. Let F_2 be the fiber of Φ containing C_2. Then we can easily see that $F_2 = C_2 + 2E_2 + C_4$, where E_2 is a (-1)-curve. Note that D_1 is a 2-section of Φ. Let P_1 and P_2 be the two ramification points of a double covering $\Phi|_{D_1} \colon D_1 \to \mathbb{P}^1$.

Since Supp D contains (-2)-curves other than $C_1, \dots, C_4$, Φ has singular fibers other than F_1 and F_2. Let $F_3, \dots, F_{2+j}$ $(j \geq 1)$ exhaust the singular fibers of Φ other than F_1 and F_2. Lemma 3.10 then implies that each $\mathrm{Supp}(F_{2+i})$ $(1 \leq i \leq j)$ consists only of a (-1)-curve, say E_{2+i}, and (-2)-curves. Since $\mathrm{Supp}((F_{2+i})_{\mathrm{red}} - E_{2+i}) \subset \mathrm{Supp}(D - D^{(1)})$ for any $i = 1, \dots, j$ and D_1 is a 2-section of Φ, we know that $E_{2+i} \cdot D_1 = 1$. So, the point $\mathrm{Supp}(F_{2+i}) \cap D_1$ $(1 \leq i \leq j)$ is a ramification point of $\Phi|_{D_1} \colon D_1 \to \mathbb{P}^1$. In particular, $j = 1$ or 2.

Let $\mu \colon V \to V'$ be the successive contraction of the (-1)-curves $E_3, \dots, E_{2+j}$ and consecutively (smoothly contractible) curves in the fibers $F_3, \dots, F_{2+j}$. Then, $\rho(V') = 6$, $\mu_*(D^\# + K_V) = \mu_*((D^{(1)})^\# + K_V) = \mu_*(D^{(1)})^\# + K_{V'} \equiv 0$, $2((\mu_*(D^{(1)})^\# + K_{V'}) \sim 0$ and $\mu_*(D^{(1)})$ is connected. So the pair $(V', \mu_*(D^{(1)}))$ is of type $X[2]$ in Example 3.1. Therefore, the pair (V, D) can be constructed as in Example 3.1. □

Finally, we consider the case $r_1 \geq 2$.

Lemma 3.16. *With the same notation and assumptions as above, assume further that* $r_1 \geq 2$. *Then the following assertions hold true*:
(1) $r_1 = 2$.
(2) *One of* D_1 *and* D_2 *is a* (-1)*-curve.*

Proof. By Lemma 3.14 (3), $F_1 = C_1 + 2E_1 + C_2$. We consider the following two cases separately.

CASE 1: $E_1 = D_1$. Then D_2 is a 2-section of Φ. Suppose that $r_1 \geq 3$. Then $D_3 + \cdots + D_{r_1} + C_3 + C_4$ is contained in a (singular) fiber F_2 of Φ. Since each D_i $(3 \leq i \leq r_1)$ is not π-exceptional, it follows from Lemma 3.10 that D_3 is a (-1)-curve and $F_2 = 2D_3 + C_3 + C_4$. In particular, $r_1 = 3$. Then $D_2 \cap \mathrm{Supp}(F_1)$ and $D_2 \cap \mathrm{Supp}(F_2)$ exhaust the ramification points of a double covering $\Phi|_{D_2} \colon D_2 \to \mathbb{P}^1$. Since D is not connected, there exists another singular fiber, say F_3, of Φ. It then follows from Lemma 3.10 that F_3 contains a unique (-1)-curve E_3 and $\mathrm{Supp}((F_3)_{\mathrm{red}} - E_3) \subset \mathrm{Supp}(D - D^{(1)})$. Since $F_3 \cdot D^{(1)} = F_3 \cdot D_2 = 2$ and the coefficient of E_3 in F_3 is equal to two, $E_3 \cdot D_2 = 1$. So, $D_2 \cap \mathrm{Supp}(F_3)$ becomes a ramification point of $\Phi|_{D_2}$. This is a contradiction. Therefore, $r_1 = 2$. In this case, the assertion (2) is clear.

CASE 2: $E_1 \neq D_1$. In this case, $0 = E_1 \cdot (D^\# + K_V) = E_1 \cdot (1/2)(C_1 + C_2) + E_1 \cdot (D^\# - (1/2)(C_1 + C_2)) + E_1 \cdot K_V = E_1 \cdot (D^\# - (1/2)(C_1 + C_2))$. So, $E_1 \cdot D_1 = 0$. We know that D_1 is a 2-section of Φ and $D_2 + \cdots + D_{r_1} + C_3 + C_4$ is contained in a fiber F_2 of Φ. By using the same argument as in Case 1, we know that $r_1 = 2$, D_2 is a (-1)-curve and $F_2 = 2D_2 + C_3 + C_4$. Thus, in this case, the assertions (1) and (2) are verified. □

From Lemma 3.16 (2), we may assume that D_1 is a (-1)-curve and $F_1 = 2D_1 + C_1 + C_2$. Then D_2 is a 2-section of Φ. Moreover, $D_2 \cap \mathrm{Supp}(F_1)$ is a ramification point of a double covering $\Phi|_{D_2} \colon D_2 \to \mathbb{P}^1$. Let F_2 be the fiber of Φ containing C_3. Then we see that $F_2 = C_3 + C_4 + 2E_2$, where E_2 is a (-1)-curve, by using an argument similar to the proof of Lemma 3.14 (2).

Since D is not connected, by an argument similar to the proof of Lemma 3.15, we obtain the unique singular fiber F_3 of Φ other than F_1 and F_2. Then $D_2 \cap \mathrm{Supp}(F_3)$ is a ramification point of the double covering $\Phi|_{D_2}\colon D_2 \to \mathbb{P}^1$. By Lemma 3.10, F_3 consists of a unique (-1)-curve, say E_3, and (-2)-curves. Let $\mu\colon V \to V'$ be the successive contraction of the (-1)-curve E_3 and consecutively (smoothly contractible) curves in the fiber F_3. Then, $\mu_*(D) = \mu_*(D^{(1)})$ is a connected SNC-divisor, $\mu_*(D^\# + K_V) = (\mu_*(D^{(1)}))^\# + K_{V'} \equiv 0$, $2((\mu_*(D^{(1)}))^\# + K_{V'}) \sim 0$, $\mu_*(D_1)^2 = -1$ and $\rho(V') = 6$. By using the same argument as in the proof of [10, Theorem 2.10], we know that the pair $(V', \mu_*(D^{(1)}))$ is of type $H[1,-1]$ in Example 3.3, here we note that $\mu_*(D_2)^2 = 1$ since $\mu_*(D_1)^2 = -1$.

Therefore, we obtain the following result.

Lemma 3.17. *Assume that $r = 1$ and $r_1 \geq 2$. Then the pair (V, D) can be constructed as in* Example 3.4.

The proof of Theorem 3.6 is thus completed.

4. Log affine surfaces with $\bar{\kappa} = \bar{p}_g = 0$ and $\bar{P}_2 > 0$

In this section, we study log affine surfaces with $\bar{\kappa} = \bar{p}_g = 0$ and $\bar{P}_2 > 0$ by using the results in the previous sections.

A log affine surface is, by definition, a normal affine surface with at most quotient singular points. Let S be a log affine surface and put $S^0 := S - \mathrm{Sing}(S)$. Then we can consider the logarithmic n-genus $\bar{P}_n(S^0)$ (resp. the logarithmic Kodaira dimension $\bar{\kappa}(S^0)$) and call it the logarithmic n-genus (resp. the logarithmic Kodaira dimension) of S. We write $\bar{p}_g(S)$, $\bar{P}_n(S)$ and $\bar{\kappa}(S)$ instead of $\bar{p}_g(S^0)$, $\bar{P}_n(S^0)$ and $\bar{\kappa}(S^0)$, respectively.

Let $\bar{X}$ be a normal projective surface such that S is an affine open subset of $\bar{X}$, $\bar{X}$ is smooth along $\bar{B} = \bar{X} - S$ and $\bar{B}$ is an SNC-divisor on $\bar{X}$. Let $\pi\colon X \to \bar{X}$ be the minimal resolution of singularities on $\bar{X}$. Then $\tilde{S} := \pi^{-1}(S)$ is an Zariski open subset of X. Since $\bar{X}$ is smooth along $\bar{B}$, we can identify the divisor $\bar{B}$ on $\bar{X}$ with the divisor $\pi^{-1}(\bar{B})$ on X. Put $\Delta := \pi^{-1}(\mathrm{Sing}(S))$ and $B := \bar{B} + \Delta$. Then the pair (X, B) is an SNC-completion of S^0. Let (W, C) be an almost minimal model of (X, B). Then there exists a birational morphism $\mu\colon X \to W$ such that $C = \mu_*(B)$. Let $\pi^{(1)}\colon W \to \bar{W}$ be the contraction of $\mathrm{Supp}(\mathrm{Bk}(C))$ to quotient singular points and put $\bar{C} := \pi^{(1)}(C)$. Then we call the surface $S^{(1)} := \bar{W} - \bar{C}$ an *almost minimal model* of S. We say that S is almost minimal if it can be an almost minimal model of itself. Throughout the present section, we retain this situation.

Lemma 4.1. *With the same notation as above, assume that $h^1(X, \mathcal{O}_X) = 0$ or $\bar{\kappa}(S) = -\infty$. Then either $S = S^{(1)}$ or $S \supset S^{(1)}$ and $S - S^{(1)}$ is a disjoint union of* *topologically contractible curves.*

Proof. See [6, Lemma 4 and Corollary 5]. □

Theorem 4.2. *Every log affine surface with logarithmic Kodaira dimension zero is a rational surface.*

Proof. Suppose that the above surface S has logarithmic Kodaira dimension zero and is not a rational surface. Since (W, C) is an almost minimal SNC-pair with $\bar{\kappa}(W - C) = 0$, we infer from Lemmas 1.5 and 1.6 that the pair (W, C) satisfies one of the following:
(a) W is a minimal surface with $\kappa(W) = 0$ and each connected component of C is a (-2)-rod or a (-2)-fork provided $C \neq 0$.
(b) W is an elliptic ruled surface with the ruling $p\colon W \to E$ over an elliptic curve E. Moreover, $C^{\#} = \lfloor C^{\#} \rfloor$, $\lfloor C^{\#} \rfloor$ is either a smooth elliptic curve or disjoint union of two smooth elliptic curves, and $2(\lfloor C^{\#} \rfloor + K_W) \sim 0$.

Since S is affine, B is a big divisor. Then $C = \mu_*(B)$ is also big. Moreover, since $\lfloor B^{\#} \rfloor = \lfloor \bar{B}^{\#} \rfloor$ is connected, so is $\lfloor C^{\#} \rfloor$. Here, we note that $\lfloor \bar{C}^{\#} \rfloor \neq 0$ because C is big. Hence, the pair (W, C) satisfies the condition (b) and $\lfloor C^{\#} \rfloor$ is a smooth elliptic curve.

On the other hand, since $(\lfloor C^{\#} \rfloor)^2 = (-K_W)^2 \leq 0$ and $\lfloor C^{\#} \rfloor$ is a connected component of C (see Lemma 1.6), the divisor C cannot be big. This is a contradiction. □

From now on, we assume further that $\bar{\kappa}(S) = \bar{p}_g(S) = 0$ and $\bar{P}_2(S) > 0$. Then, there exists a birational morphism $\nu\colon W \to V$ such that (V, D) $(D = \nu_*(C))$ is a strongly minimal model of (W, C). Let $\pi^{(2)}\colon V \to \bar{V}$ be the contraction of $\mathrm{Supp}(\mathrm{Bk}(D))$ to quotient singular points and put $\bar{D} := \pi^{(2)}(D)$ and $S^{(2)} := \bar{V} - \bar{D}$.

Lemma 4.3. *The surface $S^{(2)}$ is an affine open subset of S. Further, if $S \neq S^{(2)}$, then $S - S^{(2)}$ is a disjoint union of topologically contractible curves.*

Proof. Suppose that the pair (W, C) is not strongly minimal, i.e., the SNC-pair $(W, C - \lfloor C^{\#} \rfloor)$ is not almost minimal. Then Lemma 2.1 implies that there exists a (-1)-curve E such that either $E \subset \mathrm{Supp}\, C$ or $E \not\subset \mathrm{Supp}\, C$ and $E \cdot C \leq 2$. Moreover, by Lemma 2.1, if $E \not\subset \mathrm{Supp}\, C$ then $\pi^{(1)}(E) - (\bar{C} \cap \pi^{(1)}(E))$ is a topologically contractible curve. Thus, we know that $S^{(2)}$ can be obtained from $S^{(1)}$ by deleting off topologically contractible curves. By virtue of [5, Theorem 2], we know that $S^{(2)}$ is an affine open subset of S, here we note that S, $S^{(1)}$ and $S^{(2)}$ has at most quotient singular points. □

We call the surface $S^{(2)}$ a *strongly minimal model* of S and say that S is strongly minimal if it can be a strongly minimal model of itself.

Since S is affine, we have $\lfloor D^{\#} \rfloor \neq 0$. It then follows from Theorem 3.6 that the pair (V, D) is one of the pairs enumerated in Examples 3.1–3.5. We call the surface

Table 1.

Type	$e(S)$	Sing$S$	for details, see:
$H[-1, 0, -1]$	0	smooth	Example 3.2
$H[n, -n]$	1	smooth	Example 3.3
$H[-1, -1] + 2A_1$	1	$2A_1$	Example 3.4
$H[-2, -1] + A_3$	1	A_3	Example 3.4
$H[1 - r, -1] + D_r$ $(r \geq 4)$	1	D_r	Example 3.4
$X[2]$	2	smooth	Example 3.1
$X[0] + 2A_1$	2	$2A_1$	Example 3.1
$X[-1] + A_3$	2	A_3	Example 3.1

$S^{(2)}$ $X[4-(r_1+r_2)]+F_1+F_2$ (resp. $H[-1,0,-1]$, $H[n,-n]$ $(n \geq 0)$, $H[1-s,-1]+F$, $2X_n$) if (V, D) is of type $X[4-(r_1+r_2)]+F_1+F_2$ (resp. $H[-1, 0, -1]$, $H[n, -n]$ $(n \geq 0)$, $H[1-s,-1]+F$, $2X_n$). See Example 3.2 (resp. Example 3.4) for the notations F_1 and F_2 (resp. the notation F). We obtain the following result.

Theorem 4.4. *Let S be a log affine surface with $\bar{\kappa}(S) = \bar{p}_g(S) = 0$ and $\bar{P}_2(S) > 0$. Assume that S is strongly minimal. Then S is one of the surfaces $H[-1, 0, -1]$, $H[n,-n]$ $(n \geq 0)$, $H[1-s,-1]+F$ $(s \geq 2)$, $X[2]$, $X[0]+2A_1$, $X[-1]+A_3$. Moreover, we have* Table 1, *where $e(S)$ denotes the topological Euler number of S.*

Proof. Since S is affine, $D = (\nu \circ \mu)_*(B)$ is big. Moreover, since S has only quotient singular points, $\lfloor D^\# \rfloor$ is connected. Hence, the first assertion follows from Theorem 3.6. The second assertion can be verified easily. □

ACKNOWLEDGEMENTS. The author would like to express his gratitude to the referee for pointing out errors and for giving useful comments and suggestions which improve the paper. The author was supported by Grant-in-Aid for Scientific Research (No. 20740006) from JSPS.

References

[1] V.A. Alekseev and V.V. Nikulin: *Classification of del Pezzo surfaces with log-terminal singularities of index ≤ 2, and involutions on $K3$ surfaces*, Soviet. Math. Dokl. **39** (1989), 507–511.
[2] V.A. Alexeev and V.V. Nikulin: Del Pezzo and $K3$ Surfaces, MSJ Memoirs **15**, Math. Soc. Japan, Tokyo, 2006.
[3] R. Blache: *The structure of l.c. surfaces of Kodaira dimension zero*, I, J. Algebraic Geom. **4** (1995), 137–179.
[4] T. Fujita: *On the topology of noncomplete algebraic surfaces*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **29** (1982), 503–566.

[5] J.E. Goodman: *Affine open subsets of algebraic varieties and ample divisors*, Ann. of Math. (2) **89** (1969), 160–183.
[6] R.V. Gurjar and M. Miyanishi: *Affine lines on logarithmic* **Q**-*homology planes*, Math. Ann. **294** (1992), 463–482.
[7] S. Iitaka: *On logarithmic $K3$ surfaces*, Osaka J. Math. **16** (1979), 675–705.
[8] Y. Kawamata: *On the classification of noncomplete algebraic surfaces*; in Algebraic Geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978), Lecture Notes in Math. **732**, Springer, Berlin, 215–232, 1978.
[9] H. Kojima: *Open rational surfaces with logarithmic Kodaira dimension zero*, Internat. J. Math. **10** (1999), 619–642.
[10] H. Kojima: *Open surfaces of logarithmic Kodaira dimension zero in arbitrary characteristic*, J. Math. Soc. Japan **53** (2001), 933–955.
[11] H. Kojima: *Rank one log del Pezzo surfaces of index two*, J. Math. Kyoto Univ. **43** (2003), 101–123.
[12] S.A. Kudryavtsev, *Classification of logarithmic Enriques surfaces with $\delta = 2$*, Math. Notes **72** (2002), 660–666.
[13] S.A. Kudryavtsev: *Classification of Enriques log surfaces with $\delta = 1$*, Math. Notes **76** (2004), 81–89.
[14] M. Miyanishi: Noncomplete Algebraic Surfaces, Lecture Notes in Mathematics **857**, Springer, Berlin, 1981.
[15] M. Miyanishi: Open Algebraic Surfaces, CRM Monograph Series **12**, Amer. Math. Soc., Providence, RI, 2001.
[16] M. Miyanishi and S. Tsunoda: *Noncomplete algebraic surfaces with logarithmic Kodaira dimension $-\infty$ and with nonconnected boundaries at infinity*, Japan. J. Math. (N.S.) **10** (1984), 195–242.
[17] M. Miyanishi and D.-Q. Zhang: *Gorenstein log del Pezzo surfaces of rank one*, J. Algebra **118** (1988), 63–84.
[18] N. Nakayama: *Classification of log del Pezzo surfaces of index two*, J. Math. Sci. Univ. Tokyo **14** (2007), 293–498.
[19] K. Oguiso and D.-Q. Zhang: *On the most algebraic $K3$ surfaces and the most extremal log Enriques surfaces*, Amer. J. Math. **118** (1996), 1277–1297.
[20] K. Oguiso and D.-Q. Zhang: *On extremal log Enriques surfaces*, II, Tohoku Math. J. (2) **50** (1998), 419–436.
[21] K. Oguiso and D.-Q. Zhang: *On the complete classification of extremal log Enriques surfaces*, Math. Z. **231** (1999), 23–50.
[22] F. Sakai: *Semistable curves on algebraic surfaces and logarithmic pluricanonical maps*, Math. Ann. **254** (1980), 89–120.
[23] S. Tsunoda: *Structure of open algebraic surfaces*, I, J. Math. Kyoto Univ. **23** (1983), 95–125.
[24] D.-Q. Zhang: *On Iitaka surfaces*, Osaka J. Math. **24** (1987), 417–460.
[25] D.-Q. Zhang: *Logarithmic Enriques surfaces*, J. Math. Kyoto Univ. **31** (1991), 419–466.
[26] D.-Q. Zhang: *Logarithmic Enriques surfaces*, II, J. Math. Kyoto Univ. **33** (1993), 357–397.
[27] D.-Q. Zhang: *Normal algebraic surfaces with trivial tricanonical divisors*, Publ. Res. Inst. Math. Sci. **33** (1997), 427–442.
[28] D.-Q. Zhang: *Quotients of $K3$ surfaces modulo involutions*, Japan. J. Math. (N.S.) **24** (1998), 335–366.
[29] D.-Q. Zhang: *Normal algebraic surfaces with trivial two or four times of the canonical divisor*, Internat. J. Math. **9** (1998), 377–406.

Mathematics Division
Faculty of Engineering
Niigata University
Niigata 950-2181
Japan
e-mail: kojima@ie.niigata-u.ac.jp