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Temperature Distribution along Thin Rod with Radiation Loss from Side Surface†

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Abstract

The paper describes a mathematical approach of solving the temperature distribution along a fine rod whoes one end is highly heated, taking into account the radiation loss from the peripheral surface. A non-linear differential equation of heat conduction has a particular solution only under the special boundary condition, but it has no analytical solution under the arbitrary conditions such as when a cooling terminal is placed at a finite length from the hot end. Then, the authors have applied a method of integration of series to solve the problems under question. Introducing a new parameter Θ_c that is related with the temperature gradient or heat flux density at the cooling terminal, one obtains the rigorous solution of the fundamental equation in the infinite series. The effect of cooling terminal is minor if Θ_c is less enough than 1, and in such cases the particular solution in analytical function is useful to estimate the thermal field near the hot end.

1. Introduction

It is one of the theoretical interests to analyze the temperature distribution along a rod in which the radiation loss from the surface must be taken into account. Examples are often met with in arc discharges with refractory electrodes, where the one end of a rod is heated to extremely high temperature by the arc, while the another is kept low by water or air cooling. Considering the heat balance at the electrode-arc interface, the higher heat conduction loss to solid will accompany the more increase in the electrode potential drop (anode or cathode drops) to maintain an arc. Those who conduct the calorimetric meausrements of the electrodes coolant will notice that the larger calory is absorbed by the coolant under the shorter electrode strick-out conditions, but they will hardly observe noticiable change in arc voltage in a constant current discharge if the stick-out is not too short. The above phenomena can not be explained by a simple heat flow model in which only the conduction loss is taken into account. Hence, it will be necessary to consider the effect of other types of heat loss, for example, radiation and convection losses, on the temperature distribution along an electrode.

A heat conduction equation including the radiation and convection terms is not analytically solved. If the temperature under question is very high, the convection term will be reasonably omitted, and then the mathematical treatment becomes simpler. In this paper will be described the method to obtain the solution of nonlinear differential equation of one dimensional heat flow under the arbitrary boundary condition and some examples of calculated results.

2. Heat flow model and analytical solution

In order to solve the temperature distribution along a thin rod when the radiation loss from the side surface is taken into account, the following assumptions are premitted in this analysis.

1) The end of a rod is kept at the constant temperature of T_0.
2) The rod diameter is small enough to neglect the radial temperature distribution.
3) The temperature under consideration is high, then the axial temperature distribution is govrened by the radiation loss from the side surface and axial heat conduction.
4) Physical properties such as heat conductivity and emissivity are constant.

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Then, the steady state equation of one-dimensional heat conduction is given as follow, taking the coordinate as shown in **Fig. 1**.

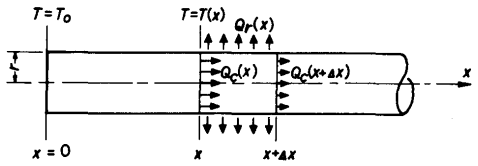

Fig. 1 Model of heat flow and coordinates

$$\frac{d^2T(x)}{dx^2}=\frac{2\varepsilon\sigma}{Kr}T(x)^4 \qquad (1)$$

where, K : heat conductivity (cal/cm·s·°K)
r : rod radius (cm)
ε : emissibity
σ : Stephan-Boltzmann constant ($\sigma=1.36\times10^{-12}$ cal/cm²s·°K⁴)

The equation (1) can be expressed in the non-dimensional form as follow;

$$\frac{d^2\Theta(\xi)}{d\xi^2}=\frac{10}{9}\alpha^2\Theta(\xi)^4 \qquad (2)$$

where, $\Theta=T/T_0$
$\xi=x/r$
$$\alpha^2=\frac{9\varepsilon\sigma rT_0{}^3}{5K}$$

A non-linear differential equation such as (2) is proved not to integrate analytically under the arbitrary boundary conditions as $\Theta=1$ at $\xi=0$ and $\Theta=\Theta_w$ at $\xi=\xi_w$.[1] The equation (2) has, however, the following particular solutions only under the boundary condition of

$$\begin{cases}\xi=0\ ;\Theta=1\\ \xi=\infty;\Theta=0\end{cases} \qquad (3).$$

That is,

$$\Theta(\xi)=(1\pm\alpha\xi)^{-2/3}.$$

Here, we consider no heat source along the rod except the end surface, which is equivalent that $(d\Theta/d\xi)$ is always negative at any position. Then, a reasonable solution is

$$\Theta(\xi)=(1+\alpha\xi)^{-2/3} \qquad (4).$$

Fig. 2 is an example of calculated temperature distribution for physical constants represented in the figure supposing a graphite arc electrode. The temperature decreases greatly in the short range of ξ, reaching to the half value of T_0 at $\xi=5$ ($x=15$ mm), but its gradient becomes lower in larger values of ξ. This is naturally due to the radiation loss, which is clearly understood by introducing a new quantity, radiation loss factor ρ.

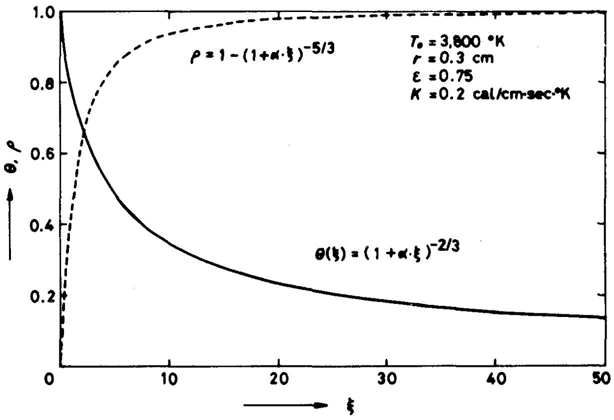

Fig. 2 Axial temperature distribution and radiation loss factor under the boundary condition of (3)

Here, denoting the heat input at the edge ($\xi=0$) by q_0 $(=-d\Theta(0)/d\xi)$ and heat output at $\xi=\xi$ by $q_\xi(=-d\Theta(\xi)/d\xi)$, the total radiation loss q_ρ from the surface between $\xi=0\sim\xi$ is equal to the amount (q_0-q_ξ), then ρ is expressed as follows.

$$\begin{aligned}\rho(\xi)&=1-(q_\xi/q_0)\\&=1-[d\Theta(\xi)/d\xi]/[d\Theta(0)/d\xi]\\&=1-(1+\alpha\xi)^{-5/3}\end{aligned} \qquad (5)$$

In Fig. 2 is shown the calculated result of $\rho(\xi)$ by the dotted curve, where ρ increases rapidly with the increase of ξ, and 90% of the total heat input at the edge q_0 is lost by the radiation in the range of $\xi=0\sim7$ ($x=0\sim21$ mm).

3. Solution by a series expansion under the arbitrary boundary condition

In the previous section was described the special case which satisfies the boundary condition (3). In the followings will be discussed the more general cases that there exists a cooling terminal at the finite length of $\xi=\xi_w$.

When the both ends of a finite rod are fixed in constant temperatures, i.e. the boundary condition of

$$\begin{cases}\xi=0\ ;\ \Theta=1\\ \xi=\xi_w\ ;\ \Theta=\Theta_w(\Theta_w<1)\end{cases} \qquad (6),$$

the basic equation (2) is impossible to solve analytically. Then, the authors applied a method of seriese expansion to obtain the solution which satisfies the condition (6).

Taking

$$\frac{2\alpha}{3}\xi=\zeta \qquad (7),$$

then, the equation (2) and boundary condition (6) are modified as follows.

$$2\frac{d^2\Theta}{d\zeta^2}=5\Theta^4$$

$$\therefore \left(\frac{d\Theta}{d\zeta}\right)^2=\Theta^5+\text{const.} \tag{8}$$

$$\begin{cases}\zeta=0;\ \Theta=1\\ \zeta=\zeta_w(=2\alpha\xi_w/3);\ \Theta=\Theta_w\end{cases} \tag{9}$$

Defining the temperature gradient ($-d\Theta/d\zeta$) at the position $\zeta=\zeta_w$ where $\Theta=\Theta_w$,

$$-\left(\frac{d\Theta}{d\zeta}\right)_{\Theta_w}\equiv\Theta_c^{5/2}\ (\Theta_c\geq 0)\ . \tag{10},$$

the integration constant of equation (8) is Θ_c^5. As is evident from the above definition, the Θ_c is a quantity which is related to the temperature gradient or cooling rate at $\zeta=\zeta_w$. Reminding that $d\Theta/d\zeta$ is always negative, the equation becomes

$$\frac{d\Theta}{d\zeta}=-(\Theta^5+\Theta_c^5)^{\frac{1}{2}} \tag{11}.$$

Since $\Theta=1$ at $\zeta=0$, the above relation is rewritten in the form of integral equation.

$$\int_{\Theta}^{1}(\Theta^5+\Theta_c^5)^{-\frac{1}{2}}d\Theta=\zeta \tag{12}$$

or

$$\int_{\Theta_w}^{\Theta}(\Theta^5+\Theta_c^5)^{-\frac{1}{2}}d\Theta=\zeta_w-\zeta \tag{13}$$

The equation (12) is approximated as equations (14) and (15) depending on the amount of Θ_c.

If $\Theta_c\gg\Theta$, the temperature distribution is linear as shown in the next equation.

$$\int_{\Theta}^{1}(\Theta^5+\Theta_c^5)^{-\frac{1}{2}}d\Theta\simeq\Theta_c^{-\frac{5}{2}}(1-\Theta)$$

$$\therefore\ \Theta\simeq 1-\Theta_c^{\frac{5}{2}}\cdot\zeta=1-\frac{2\alpha}{3}\Theta_c^{\frac{5}{2}}\cdot\xi \tag{14}$$

When $\Theta_c\ll\Theta$, the result coinsides with the equation (4) which is a particular solution under the boundary condition of (3).

$$\int_{\Theta}^{1}(\Theta^5+\Theta_c^5)^{-\frac{1}{2}}d\Theta\simeq-\frac{2}{3}(1-\Theta^{-\frac{3}{2}})$$

$$\therefore\ \Theta\simeq\left(1+\frac{3}{2}\zeta\right)^{-\frac{2}{3}}=(1+\alpha\xi)^{-\frac{2}{3}} \tag{15}$$

Under the condition of arbitrary value of Θ_c, solution in series expansion will be described hereinafter.

In case of $\Theta>\Theta_c$;

$$(\Theta^5+\Theta_c^5)^{-\frac{1}{2}}=\Theta^{-\frac{5}{2}}\left[1+\left(\frac{\Theta_c}{\Theta}\right)^5\right]^{-\frac{1}{2}}$$

$$=\Theta^{-\frac{5}{2}}\sum_{n=0}^{\infty}(-1)^n\frac{\prod_{m=1}^{n}(2m-1)}{n!\ 2^n}\left(\frac{\Theta_c}{\Theta}\right)^{5n}$$

$$\int_{\Theta}^{1}\Theta^{-5n-\frac{5}{2}}d\Theta=-\frac{1}{\left(5n+\frac{3}{2}\right)}(1-\Theta^{-5n-\frac{3}{2}})$$

Therefore, the equation (13) is expressed by series as the next form.

$$\zeta=\int_{\Theta}^{1}(\Theta^5+\Theta_c^5)^{-\frac{1}{2}}d\Theta$$

$$=-\sum_{n=0}^{\infty}(-1)^n\frac{\prod_{m=1}^{n}(2m-1)}{n!\ 2^n\left(5n+\frac{3}{2}\right)}\Theta_c^{5n}(1-\Theta^{-5n-\frac{3}{2}}) \tag{16}$$

While in case of $\Theta_w\leq\Theta<\Theta_c$;

$$(\Theta^5+\Theta_c^5)^{-\frac{1}{2}}=\Theta_c^{-\frac{5}{2}}\cdot\sum_{n=0}^{\infty}(-1)^n\frac{\prod_{m=1}^{n}(2m-1)}{n!\ 2^n}\left(\frac{\Theta}{\Theta_c}\right)^{5n}$$

$$\int_{\Theta_w}^{\Theta}\Theta^{5n}d\Theta=\frac{1}{(5n+1)}(\Theta^{5n+1}-\Theta_w^{5n+1})$$

Then, the equation (14) becomes

$$\zeta_w-\zeta=\int_{\Theta_w}^{\Theta}(\Theta^5+\Theta_c^5)^{-\frac{1}{2}}d\Theta$$

$$=\Theta_c^{-\frac{5}{2}}\cdot\sum_{n=0}^{\infty}(-1)^n\frac{\prod_{m=1}^{n}(2m-1)}{n!\ 2^n(5n+1)}\left[\frac{\Theta^{5n+1}}{\Theta_c^{5n}}-\frac{\Theta_w^{5n+1}}{\Theta_c^{5n}}\right] \tag{17}$$

Here, denoting $\zeta=\zeta_c$ at $\Theta=\Theta_c$, it is clear from the equation (16) that

$$\zeta_c=-\sum_{n=0}^{\infty}(-1)^n\frac{\prod_{m=1}^{n}(2m-1)}{n!\ 2^n\left(5n+\frac{3}{2}\right)}\Theta_c^{5n}(1-\Theta_c^{-5n-\frac{3}{2}}) \tag{18}$$

While, from the equation (17);

$$\zeta_w-\zeta_c=\Theta_c^{-\frac{5}{2}}\cdot\sum_{n=0}^{\infty}(-1)^n\frac{\prod_{m=1}^{n}(2m-1)}{n!\ 2^n(5n+1)}\left[\Theta_c-\frac{\Theta_w^{5n+1}}{\Theta_c^{5n}}\right]$$

$$=\Theta_c^{-\frac{3}{2}}\cdot\sum_{n=0}^{\infty}(-1)^n\frac{\prod_{m=1}^{n}(2m-1)}{n!\ 2^n(5n+1)}\left[1-\left(\frac{\Theta_w}{\Theta}\right)^{5n+1}\right] \tag{19}$$

Eliminating ζ_c from equations (18) and (19), one obtains

$$\zeta_w=-\sum_{n=0}^{\infty}(-1)^n\frac{\prod_{m=1}^{n}(2m-1)}{n!\ 2^n\left(5n+\frac{3}{2}\right)}\Theta_c^{5n}$$

$$+\Theta_c^{-\frac{3}{2}}\sum_{n=0}^{\infty}(-1)^n\frac{\left(10n+\frac{5}{2}\right)\prod_{m=1}^{n}(2m-1)}{n!\ 2^n(5n+1)\left(5n+\frac{3}{2}\right)}$$

$$-\Theta_c^{-\frac{5}{2}}\cdot\Theta_w\sum_{n=0}^{\infty}(-1)^n\frac{\prod_{m=1}^{n}(2m-1)}{n!\ 2^n(5n+1)}\left(\frac{\Theta_w}{\Theta}\right)^{5n} \tag{20}$$

As described above, the solution of differential equation (8) that satisfies the boundary condition of (9) is thus rigorously expressed as relations (16), (17) and (20) by using a cooling parameter Θ_c. The euqaution (18) is the application limit of the both (16) and (17). Namely, the equation (16) must be applied for the range of $\zeta=0\sim\zeta_c(\Theta=1\sim\Theta_c)$, while the relation (17) for $\zeta=\zeta_c\sim\zeta_w(\Theta=\Theta_c\sim\Theta_w)$.

The next will be discussed the effect of cooling terminal on the thermal field just in the vicinty of heating end. It is evident from the equation (12) that the temperature gradient at the hot end is

$$\left(\frac{d\Theta}{d\xi}\right)_{\xi=0}=\frac{2\alpha}{3}\left(\frac{d\Theta}{d\zeta}\right)_{\zeta=0}=-\frac{2\alpha}{3}(1+\Theta_c{}^5)^{1/2} \tag{21}$$

since $\Theta=1$ at $\zeta=0$. One should note here that the value of $(d\Theta/d\xi)_{\xi=0}$ in case of $\Theta_c=0$ coincides with that obtained from the solution (4). The relation (21) indicates that the effect of Θ_c on the temperature gradient at the heating end is minor when Θ_c is less than the unity. For example, the value of $(d\Theta/d\xi)_{\xi=0}$ at $\Theta_c=0.73$ is within the error of 10% of that at $\Theta_c=0$. This leads one to a conclusion that the required heat flux density to maintain the constant temperature at the hot end does not change much even there exists a cooling terminal at a finite length, since the non-dimensional input heat flux density q_0 is given by the following relation including the radiation loss from the end surface to $-\xi-$ $(-x-)$ direction.

$$q_0=\frac{2\alpha}{3}\left(\frac{d\Theta}{d\zeta}\right)_{\zeta=0}+\beta$$
$$=\frac{2\alpha}{3}(1+\Theta_c{}^5)^{1/2}+\frac{5}{9}\alpha^2 \tag{22}$$

where, $\beta=r\varepsilon\sigma T_0{}^3/K=5\alpha^2/9$.

4. Calculated results of temperature distribution along a graphite rod

As described in the previous section, the solution of basic differential equation (2) (or (8)) under the boundary condition (6) (or (9)) can be obtained in the mathematically rigorous expressions. These infinite series are approximated by the finite series as described in the followings. Taking $n=0\sim6$, the equations (16), (17) and (18) are expanded;

$$\begin{aligned}\zeta=&-0.667(1-\Theta^{-\frac{3}{2}})+0.077\Theta_c{}^5(1-\Theta^{-\frac{13}{2}})\\&-0.033\Theta_c{}^{10}(1-\Theta^{-\frac{23}{2}})+0.019\Theta_c{}^{15}(1-\Theta^{-\frac{33}{2}})\\&-0.014\Theta_c{}^{20}(1-\Theta^{-\frac{43}{2}})+0.009\Theta_c{}^{25}(1-\Theta^{-\frac{53}{2}})\\&-0.007\Theta_c{}^{30}(1-\Theta^{-\frac{63}{2}})\end{aligned} \tag{23}$$

$$\begin{aligned}\zeta_w-\zeta=&\Theta_c{}^{-\frac{5}{2}}\cdot\Theta\Big[1-0.083\Big(\frac{\Theta}{\Theta_c}\Big)^5+0.034\Big(\frac{\Theta}{\Theta_c}\Big)^{10}\\&-0.020\Big(\frac{\Theta}{\Theta_c}\Big)^{15}+0.014\Big(\frac{\Theta}{\Theta_c}\Big)^{20}\\&-0.009\Big(\frac{\Theta}{\Theta_c}\Big)^{25}+0.007\Big(\frac{\Theta}{\Theta_c}\Big)^{30}\Big]\\&-\Theta_c{}^{-\frac{5}{2}}\cdot\Theta_w\Big[1-0.083\Big(\frac{\Theta_w}{\Theta_c}\Big)^5\\&+0.034\Big(\frac{\Theta_w}{\Theta_c}\Big)^{10}-0.020\Big(\frac{\Theta_w}{\Theta_c}\Big)^{15}\\&+0.014\Big(\frac{\Theta_w}{\Theta_c}\Big)^{20}-0.090\Big(\frac{\Theta_w}{\Theta_c}\Big)^{25}\\&+0.007\Big(\frac{\Theta_w}{\Theta_c}\Big)^{30}\Big]\end{aligned} \tag{24}$$

$$\begin{aligned}\zeta_c=&-0.667+0.077\Theta_c{}^2-0.033\Theta_c{}^{10}+0.019\Theta_c{}^{15}\\&-0.014\Theta_c{}^{20}+0.009\Theta_c{}^{30}+0.616\Theta_c{}^{-\frac{3}{2}}\end{aligned} \tag{25}$$

$$\begin{aligned}\zeta_w=&-0.667+1.559\Theta_c{}^{-\frac{3}{2}}+0.077\Theta_c{}^5-0.033\Theta_c{}^{10}\\&+0.019\Theta_c{}^{15}-0.014\Theta_c{}^{20}+0.009\Theta_c{}^{30}\\&-\Theta_c{}^{-\frac{5}{2}}\cdot\Theta_w\Big[1-0.083\Big(\frac{\Theta_w}{\Theta_c}\Big)^5+0.034\Big(\frac{\Theta_w}{\Theta_c}\Big)^{10}\\&-0.020\Big(\frac{\Theta_w}{\Theta_c}\Big)^{15}+0.014\Big(\frac{\Theta_w}{\Theta_c}\Big)^{20}\\&-0.009\Big(\frac{\Theta_w}{\Theta_c}\Big)^{25}+0.007\Big(\frac{\Theta_w}{\Theta_c}\Big)^{30}\Big]\end{aligned} \tag{26}$$

Fig. 3 shows an example of the temperature distribution along a fine graphite rod using the above relations, where the both ends are fixed T_0=3,800 °K $(\Theta_0=1)$ and T_w=300 °K $(\Theta_w$=300/3,800=0.08) respectively. In the curve of $\Theta_c=0.3$, for instance, the distances ζ_w and ζ_c which correspond with points B and C are computed from the equations (25) and (26). Therefore, the curves A$\sim$B and B$\sim$C are determined by relations (23) and (24). As is seen in the figure, the temperature in case of small Θ_c changes almost in the same manner with that of a semi-infinite case $(\Theta_c=0)$ in the short ranges of ζ. While, the difference

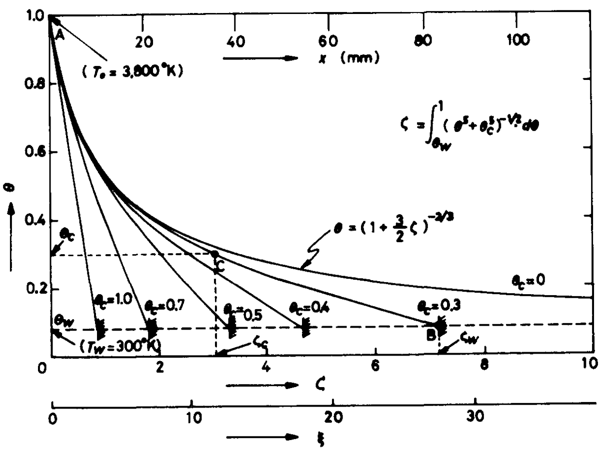

Fig. 3 Axial temperature distribution under the boundary condition of (6) or (9) (Physical properties are the same with Fig. 2.)

becomes larger in the longer ζ and the temperature varies almost linear in the vicinity of cooling terminal. In case that the cooling terminal is vrey near to the hot end, on the other hand, the temperature distribution is linear in the whole range as it was predicted by the previous equation (14). It is evident from the calculated results that the thermal field near the hot end is not affected greatly by the existence of a cooling terminal, if the another end of a graphite rod ($d=$ 6 mmϕ) is forced-cooled in longer distances than 40 mm from the hot end.

5. Conclusion

Some mathematical studies were conducted to solve a non-linear differential equation of heat conduction (One dimentional heat flow) which includes a radiation loss term. The results obtained are as follows;

(1) The fundamental equation has an analytical solution only under the boundary condition of $\Theta_0=1$ at $\xi=0$, and $\Theta=0$ at $\xi=\infty$.

(2) When the both ends of a fine rod are fixed in their temperatures respectively, rigorous solutions are obtained by the integration of series, using a new parameter Θ_c which is a measure of the temperature gradient or heat flux density at the cooling end.

(3) If Θ_c is small enough compared with 1, the temperature in the vicinity of the hot end is hardly affected by the existence of a cooling terminal. Namely, the temperature near the hot end is reasonably estimated by the analytical solution which is equal to the solution in series at $\Theta_c=0$.

(4) When Θ_c is large, the temperature distribution along a rod is linear like as a solution of one-dimensional heat conduction without radiation loss in steady state.

(5) Solution in infinite series were approximated by finite series and the results were represented in case of a graphite rod whose both ends were kept in constant temperatures.

Reference

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