Colombini, F., Nishitani, T., Orrù, N. and Pernazza, L.
Osaka J. Math.
48 (2011), 645–673

SOME WELL-POSED CAUCHY PROBLEM FOR SECOND ORDER HYPERBOLIC EQUATIONS WITH TWO INDEPENDENT VARIABLES

FERRUCCIO COLOMBINI, TATSUO NISHITANI, NICOLA ORRÙ and LUDOVICO PERNAZZA

(Received February 19, 2010)

Abstract

In this paper we discuss the C^∞ well-posedness for second order hyperbolic equations $Pu = \partial_t^2 u - a(t, x)\, \partial_x^2 u = f$ with two independent variables (t, x). Assuming that the C^∞ function $a(t, x) \geq 0$ verifies $\partial_t^p a(0, 0) \neq 0$ with some p and that the discriminant $\Delta(x)$ of $a(t, x)$ vanishes of finite order at $x = 0$, we prove that the Cauchy problem for P is C^∞ well-posed in a neighbourhood of the origin.

1. Introduction

In this paper we deal with the C^∞ well-posedness of the Cauchy problem for a second order hyperbolic operator with two independent variables $P = \partial_t^2 - a(t, x)\, \partial_x^2$, $(t, x) \in \mathbb{R}^2$:

$$
\begin{cases} Pu = \partial_t^2 u - a(t, x)\, \partial_x^2 u = f, \\ u(0, x) = u_0(x),\ \partial_t u(0, x) = u_1(x) \end{cases} \tag{1.1}
$$

near the origin of $\mathbb{R}^2$, where we always assume that $a(t, x) \geq 0$. In [11] and [12], assuming that $a(t, x)$ is real analytic in (t, x), it is proved that the Cauchy problem for P is C^∞ well-posed. On the other hand, in [4], the authors give a counterexample involving a function $a(t) \in C^\infty([0, T])$, positive for $t > 0$, such that the Cauchy problem for $P = \partial_t^2 - a(t)\, \partial_x^2$ is not C^∞ well-posed. The main feature of this $a(t)$ is that $da(t)/dt$ changes sign infinitely many times when $t \downarrow 0$. There are many works trying to extend the C^∞ well-posedness result in [11] without the analyticity assumptions on $a(t, x)$ (see for example, [1], [2], [3], [5], [8], [10], [13]).

In this paper we assume that $a(t,x)$ is of class C^∞ in (t,x) and essentially a polynomial in t and we discuss the C^∞ well-posedness question under this rather general assumption. If $a(0,0) \neq 0$ then P is strictly hyperbolic and if $a(0,0) = \partial_t a(0,0) = 0$ but $\partial_t^2 a(0, 0) \neq 0$ then P is effectively hyperbolic at $(0, 0)$ and hence the Cauchy problem is C^∞ well-posed for any lower order term (see [7], [11]). Thus we may assume that

2000 Mathematics Subject Classification. Primary 35L15; Secondary 35L80, 35B30.

$a(0,0) = \partial_t a(0,0) = \partial_t^2 a(0,0) = 0$ without restrictions as far as the C^∞ well-posedness is concerned. We assume that there is a $p \in \mathbb{N}$, $p \geq 3$ such that

$$\partial_t^p a(0,0) \neq 0. \tag{1.2}$$

Then applying the Malgrange preparation theorem we can write

$$a(t,x) = e(t,x)(t^p + a_1(x)t^{p-1} + \cdots + a_p(x)) \tag{1.3}$$

where $e, a_1, \ldots, a_p$ are of class C^∞ in a neighbourhood of the origin and $e(0,0) \neq 0$. Let $\Delta(x)$ be the discriminant of $a(t,x)/e(t,x)$ as a polynomial in t. We call $\Delta(x)$ the discriminant of $a(t,x)$. We now assume that there is $q \in \mathbb{N}$ such that

$$\left(\frac{d}{dx}\right)^q \Delta(0) \neq 0. \tag{1.4}$$

Then we have

Theorem 1.1. *Assume* (1.2) *and* (1.4). *Then the Cauchy problem* (1.1) *is C^∞ well-posed in a neighbourhood of the origin.*

One can easily generalize Theorem 1.1 a little bit as follows:

Theorem 1.1′. *Assume that $b_j(t,x)$, $j = 1, \ldots, r$ are functions of class C^∞ and verify the conditions* (1.2) *and* (1.4) *with some p_j, $q_j \in \mathbb{N}$ (the nonnegativity of $b_j(t,x)$ is not assumed) and that $a(t,x) = b_1(t,x)^{m_1} \cdots b_r(t,x)^{m_r}$ where $m_j \in \mathbb{N}$ and $B_j(t,x) = b_j(t,x)^{m_j} \geq 0$ near the origin. Then the assertion of* Theorem 1.1 *holds.*

In Section 2 we define a weighted energy and in Sections 3 and 4 we derive a priori estimates. In Section 5 we prove Theorem 1.1. Finally in Sections 6, 7 and 8 we construct the weight functions.

2. Energy

Throughout this paper an index x or t will denote respectively a space or time derivative, e.g. $u_x = \partial_x u$ and $k_{n,t} = \partial_t k_n$. As usual, we set $D = \partial_x / i$.

We prove Theorem 1.1 by deriving a priori estimates. Take $\chi(x) \in C_0^\infty(\mathbb{R})$ such that $\chi(x) = 1$ in a neighbourhood of the origin; $\chi(x)a(t,x)$ is then defined and of class C^∞ in $[-T,T] \times \mathbb{R}$.

Let us consider an energy

$$\mathcal{E}(t,u) = \sum_{n=0}^{\infty} e^{-ct} A(t)^n \int k_n(t,x)[|u_{n,t}|^2 + \chi(x)a(t,x)|\partial_x u_n|^2 + (n^2+1)|u_n|^2]\, dx$$

where $c > 0$, $A(t) = e^{a-bt}$ with $a, b > 0$ and

$$u_n = \frac{1}{n!} \log^n \langle D \rangle u, \quad \langle \xi \rangle^2 = \xi^2 + 1.$$

Here

$$\langle D \rangle^s u = e^{s \log\langle D \rangle} u = \sum_{n=0}^{\infty} \frac{s^n}{n!} \log^n \langle D \rangle u$$

has the role of a partition of unity. Although $(s^n/n!)\log^n \langle D \rangle$ does not localize the frequencies ξ so much (but see Lemma 3.1 below), it has the advantage that $\partial_\xi^\ell((s^n/n!) \log^n \langle \xi \rangle)$ conserves the same form up to factors $\xi^i \langle \xi \rangle^{-j}$. In order that this energy may work well to derive a priori estimates, the weight functions $k_n(t, x)$ are required to verify some suitable properties. For similar examples of energy see [8], [9] and [13]. Our main task in this paper is then to construct a sequence of weight functions $k_n(t, x)$ for $a(t, x)$ satisfying the properties listed in the next proposition:

Proposition 2.1. *Let $N > 1$ be a given constant and $a(t, x)$ be a nonnegative function of class C^∞ satisfying* (1.2) *and* (1.4). *One can find $T > 0$ and construct a sequence of weight functions $k_n(t, x)$ defined on $[-T, T] \times \mathbb{R}$ verifying the following properties*:

1) *$k_n(t, x)$ is a Lipschitz continuous function and*

$$C_1 2^{-C_2 n} \leq k_n(t, x) \leq 1.$$

2) $k_{n,t}(t, x) \geq -C_3 e^{C_4 n}$.
3) *We have that*

$$|k_{n,x}(t, x)| \sqrt{\chi(x) a(t, x)} \leq C_5(n + 1) k_n(t, x).$$

4) *We have that*

$$k_{n,t}(t, x) \leq -N \frac{|\chi(x) a_t(t, x)|}{\chi(x) a(t, x) + 2^{-2n}} k_n(t, x) + C_6(n + 1) k_n(t, x).$$

5) $k_{n+1}(t, x) \leq C_7 k_n(t, x)$.

The proof of Proposition 2.1 will be given in Sections 6, 7 and 8.

3. Energy estimate

In what follows we write simply $a(t, x)$ instead of $\chi(x)a(t, x)$ and assume that $u \in C^2([-T, T]; \mathcal{S}(\mathbb{R}))$ verifies

$$Pu = \partial_t^2 u - a(t, x)\, \partial_x^2 u = f.$$

Let us define

$$u_{\beta,s,j} = 2^{-n\beta} \frac{D^{\beta+j}}{\langle D\rangle^{s+j}} u \quad \text{and} \quad u_{n,\beta,s,j} = \frac{\log^n \langle D\rangle}{n!} u_{\beta,s,j}. \tag{3.1}$$

With these definitions, $u_{0,0,0} = u$ and $u_n = u_{n,0,0,0}$. We introduce the energy

$$\begin{aligned}
\mathcal{E}(t, u) &= \sum_{n=0}^{\infty} \sum_{\beta=0}^{p} \sum_{s=0}^{p+q} \sum_{j=0}^{1} e^{-ct} A^n(t) \int k_n(t, x)[|\partial_t u_{n,\beta,s,j}|^2 + a(t, x)|\partial_x u_{n,\beta,s,j}|^2 \\
&\qquad\qquad + (n^2 + 1)|u_{n,\beta,s,j}|^2]\, dx \\
&= \sum_{n=0}^{\infty} \sum_{\beta=0}^{p} \sum_{s=0}^{p+q} \sum_{j=0}^{1} E_n(t, u_{\beta,s,j})
\end{aligned}$$

where $k_n(t, x)$ is given by Proposition 2.1 (we will later determine the undefined quantities of this expression, namely a, b in the term $A(t)$, the coefficient c and the number of terms of the sum, that depends on $p, q \in \mathbb{N}$).

Performing the derivative of $E_n(t, u)$ with respect to t we have that

$$\begin{aligned}
\frac{d}{dt} E_n(t, u) &= -(c + nb)E_n(t, u) \\
&\quad + e^{-ct} A^n(t) \int k_{n,t}(t, x)[|u_{n,t}|^2 + a(t, x)|\partial_x u_n|^2 + (n^2 + 1)|u_n|^2]\, dx \\
&\quad + e^{-ct} A^n(t) \int k_n(t, x) 2\, \mathrm{Re}(u_{n,tt} \overline{u}_{n,t})\, dx \\
&\quad + e^{-ct} A^n(t) \int k_n(t, x) a_t(t, x)|\partial_x u_n|^2\, dx \\
&\quad + e^{-ct} A^n(t) \int k_n(t, x) a(t, x) 2\, \mathrm{Re}(\partial_x u_n \overline{u}_{n,xt})\, dx \\
&\quad + (n^2 + 1) e^{-ct} A^n(t) \int k_n(t, x) 2\, \mathrm{Re}(u_{n,t} \overline{u}_n)\, dx \\
&= -(c + nb)E_n(t, u) + I_2(u_n) + I_3(u_n) + I_4(u_n) + I_5(u_n) + I_6(u_n).
\end{aligned}$$

We then begin studying $I_6(u_n)$: note that

$$I_6(u_n) \le e^{-ct} A^n(t) \left[\int k_n(n|u_{n,t}|^2 + n^3|u_n|^2)\, dx + \int k_n(|u_{n,t}|^2 + |u_n|^2)\, dx \right],$$

therefore it is clear that $I_6(u_n)$ can be bounded by $CnE_n(t,u)$. Thus we have that

$$\sum_{n,\beta,s,j} I_6(u_{n,\beta,s,j}) \le C \sum_{n,\beta,s,j} nE_n(t, u_{\beta,s,j}) \tag{3.2}$$

where the sum is taken over $n \in \mathbb{N}$, $0 \le \beta \le p$, $0 \le s \le p+q$ and $j = 0, 1$.

Next, let us consider $I_2(u_n)$ and $I_4(u_n)$ (the terms $I_3(u_n)$ and $I_5(u_n)$ will be estimated together in the next section). Note that

$$k_n a_t |\partial_x u_n|^2 \le k_n \frac{|a_t|}{a+2^{-2n}} a |\partial_x u_n|^2 + k_n \frac{|a_t|}{a+2^{-2n}} 2^{-2n} |\partial_x u_n|^2. \tag{3.3}$$

With a slight abuse of notation we will set $A = A(0)$ in what follows.

Lemma 3.1. *For every $t \in [-T, T]$ (for a suitably small T) and every fixed s, j, if p and A are large enough we have that*

$$\begin{aligned}&\sum_n A^n(t) \sum_{\beta=0}^{p} \int k_n \frac{|a_t|}{a+2^{-2n}} 2^{-2n} |\partial_x u_{n,\beta,s,j}|^2\, dx \\ &\le \sum_n A^n(t) \sum_{\beta=1}^{p} \int k_n \frac{|a_t|}{a+2^{-2n}} |u_{n,\beta,s,j}|^2\, dx + C \sum_n A^n(t) \int k_n |u_{n,0,s,j}|^2\, dx.\end{aligned}$$

Proof. Let us denote by $\|u\|$ the $L^2(\mathbb{R})$ norm of $u(t, \cdot)$. Obviously

$$k_n \frac{|a_t|}{a+2^{-2n}} 2^{-2n} |\partial_x u_{n,\beta,s,j}|^2 = k_n \frac{|a_t|}{a+2^{-2n}} |u_{n,\beta+1,s,j}|^2$$

if $0 \le \beta < p$. If $\beta = p$, noting that $|a_t| \le C$ and $k_n \le 1$ by Proposition 2.1 (and fixing s, j and setting $w = u_{0,s,j}$, $w_n = u_{n,0,s,j}$) we have that

$$\begin{aligned}&\sum_n A^n(t) 2^{-2n(p+1)} \int k_n \frac{|a_t|}{a+2^{-2n}} |D^{p+1} w_n|^2\, dx \\ &\le C_1 \sum_n A^n(t) 2^{-2np} \|\langle D\rangle^{p+1} w_n\|^2 \\ &\le C_1 \sum_n A^n(t) 2^{-2np} \left\| \sum_m (p+1)^m \frac{\log^{m+n}\langle D\rangle}{m!\, n!} w \right\|^2 \\ &\le C_2 \sum_{m,n} A^n(t) 2^{-2np} (m+1)^2 (p+1)^{2m} \left\| \frac{\log^{m+n}\langle D\rangle}{m!\, n!} w \right\|^2 \\ &\le C_2 \sum_{m,n} A(t)^{m+n} 2^{-2(m+n)p} A(t)^{-m} (m+1)^2 \\ &\qquad \times 2^{2mp} 2^{2m(p+1)} 2^{2(m+n)} \left\| \frac{\log^{m+n}\langle D\rangle}{(m+n)!} w \right\|^2.\end{aligned} \tag{3.4}$$

Set $\mu = m+n$; choosing p large enough, by Proposition 2.1 we can have that $k_\mu 2^{2\mu(p-1)} \geq C_3 > 0$. Observe that whatever the choice of b may be, we can suppose that $A(t) \geq A/2$ for $t \in [-T, T]$ simply decreasing T; on the other hand, we also choose A large with respect to $2^2 \cdot 2^{4p+2} \cdot 2$, so that (taking into account that $\sum_{m=0}^{\infty} 1/2^m = 2$), the last line in (3.4) can be bounded by

$$2C_2 \sum_\mu A^\mu 2^{-2\mu(p-1)} \|w_\mu\|^2 \leq C_4 \sum_\mu A^\mu \int k_\mu |w_\mu|^2 \, dx.$$

This ends the proof of Lemma 3.1. □

Recall now that by 4) of Proposition 2.1

$$k_n \frac{|a_t|}{a + 2^{-2n}} \leq -\frac{1}{N} k_{n,t} + \frac{C}{N}(n+1)k_n. \tag{3.5}$$

By Lemma 3.1 and (3.3), (3.5) we see that (for every fixed s and j)

$$\sum_{n,\beta} I_4(u_{n,\beta,s,j}) \leq -\frac{1}{N} \sum_{n,\beta} e^{-ct} A^n(t) \int k_{n,t}(a|\partial_x u_{n,\beta,s,j}|^2 + |u_{n,\beta,s,j}|^2) \, dx$$
$$+ C \sum_{n,\beta} n E_n(u_{\beta,s,j}).$$

From 4) of Proposition 2.1 we have that $k_{n,t} \leq C(n+1)k_n$, thus, since $1 - 1/N > 0$, we obtain that

$$\sum_{n,\beta} I_4(u_{n,\beta,s,j}) + \sum_{n,\beta} I_2(u_{n,\beta,s,j}) \leq C \sum_{n,\beta} n E_n(u_{\beta,s,j}). \tag{3.6}$$

4. Energy estimate (continued)

We turn to $I_5(u_n)$. Note that

$$\begin{aligned} I_5(u_n) &= 2e^{-ct} A^n(t) \int k_n a(t,x) \, \mathrm{Re}(u_{n,x} \overline{u}_{n,xt}) \, dx \\ &= -2e^{-ct} A^n(t) \int k_{n,x} a(t,x) \, \mathrm{Re}(u_{n,x} \overline{u}_{n,t}) \, dx \\ &\quad - 2e^{-ct} A^n(t) \int k_n a_x(t,x) \, \mathrm{Re}(u_{n,x} \overline{u}_{n,t}) \, dx \\ &\quad - 2e^{-ct} A^n(t) \int k_n a(t,x) \, \mathrm{Re}(u_{n,xx} \overline{u}_{n,t}) \, dx \\ &= J_1(u_n) + J_2(u_n) + J_3(u_n). \end{aligned}$$

By 3) of Proposition 2.1 we have

$$|J_1(u_n)| \leq Ce^{-ct} A^n(t) \int n k_n (|u_{n,t}|^2 + a(t,x)|u_{n,x}|^2) \, dx \leq CnE_n(u) \tag{4.1}$$

and from the Glaeser inequality, applied to $a \geq 0$, it follows that

$$|J_2(u_n)| \leq Ce^{-ct}A^n(t)\int k_n(|u_{n,t}|^2 + a(t,x)|u_{n,x}|^2)\,dx \leq CE_n(u). \tag{4.2}$$

We still have to estimate

$$J_3(u_{n,\beta,s,j}) = -2e^{-ct}A^n(t)\int k_n(t,x)a(t,x)\,\mathrm{Re}(\partial_x^2 u_{n,\beta,s,j}\ \partial_t\overline{u}_{n,\beta,s,j})\,dx;$$

but note that

$$\begin{aligned} &I_3(u_{n,\beta,s,j}) + J_3(u_{n,\beta,s,j}) \\ &= 2e^{-ct}A^n(t)\int k_n\,\mathrm{Re}\left(\left[\frac{\log^n\langle D\rangle}{n!}\frac{D^{\beta+j}}{\langle D\rangle^{s+j}}, a\right]\partial_x^2 u\cdot c_{n,\beta}\ \partial_t\overline{u}_{n,\beta,s,j}\right)dx \\ &\quad + 2e^{-ct}A^n(t)\int k_n(t,x)\,\mathrm{Re}(f_{n,\beta,s,j}\ \partial_t\bar{u}_{n,\beta,s,j})\,dx \end{aligned} \tag{4.3}$$

where $c_{n,\beta} = 2^{-n\beta}$ and $\beta = 0, 1, \ldots, p$, $s = 0, 1, \ldots, p+q$, $j = 0, 1$ and $f_{n,\beta,s,j}$ is defined as in (3.1).

We rewrite the commutator as

$$\begin{aligned} &\left[\frac{\log^n\langle D\rangle}{n!}\frac{D^{\beta+j}}{\langle D\rangle^{s+j}}, a(t,x)\right]\partial_x^2 u_{n,\beta,s,j}\cdot c_{n,\beta} \\ &= \sum_{1\leq l<p+q+2-s}\frac{(-i)^l}{l!}\ \partial_x^l a\Phi^{(l)}_{\beta,s,j}(D)\,\partial_x^2 u\cdot c_{n,\beta} + R(u_{n,\beta,s,j}) \end{aligned} \tag{4.4}$$

where

$$\Phi_{\beta,s,j}(\xi) = \frac{\log^n\langle\xi\rangle}{n!}\frac{\xi^{\beta+j}}{\langle\xi\rangle^{s+j}}$$

and

$$\begin{aligned} R(u_{n,\beta,s,j}) = \frac{-1}{(m-1)!}\iiint_0^1 &e^{ix\xi}\,\Phi^{(m)}_{\beta,s,j}(\eta+\theta(\xi-\eta)) \\ &\times(1-\theta)^{m-1}(\xi-\eta)^m\hat{a}(t,\xi-\eta)\eta^2\hat{u}(t,\eta)c_{n,\beta}\,d\theta\,d\eta\,d\xi \end{aligned}$$

with $m = p+q+2-s$. Here $\hat{a}(t,\xi)$ denotes the Fourier transform of $a(t,x)$ with respect to x.

As a consequence, writing $r = p + q$, we see that

$$
\begin{aligned}
&I_3(u_{n,\beta,s,j}) + J_3(u_{n,\beta,s,j})\\
&\le e^{-ct}\frac{1}{n+1}A^n(t)\int k_n\Big|\sum_{1\le l<m}\frac{(-i)^l}{l!}\,\partial_x^l a\Phi^{(l)}_{\beta,s,j}(D)\,\partial_x^2 uc_{n,\beta}\Big|^2\,dx\\
&\quad+e^{-ct}(n+1)A^n(t)\int k_n|\partial_t u_{n,\beta,s,j}|^2\,dx\\
&\quad+e^{-ct}\frac{1}{n+1}A^n(t)\int k_n|R(u_{n,\beta,s,j})|^2\,dx\\
&\quad+e^{-ct}(n+1)A^n(t)\int k_n|\partial_t u_{n,\beta,s,j}|^2\,dx\\
&\quad+e^{-ct}A^n(t)\int k_n(t,x)|f_{n,\beta,s,j}|^2\,dx+e^{-ct}A^n(t)\int k_n|\partial_t u_{n,\beta,s,j}|^2\,dx.
\end{aligned}
\tag{4.5}
$$

The second, fourth and sixth term are smaller than $CnE_n(u_{\beta,s,j})$ for some $C>0$. We keep the fifth one as it is and study the other two in the following two lemmas; we start with the first term.

Lemma 4.1. *We have that*

$$
\begin{aligned}
&e^{-ct}\sum_{n,\beta,s,j}\frac{1}{n+1}A^n(t)\int k_n\Big|\sum_{1\le l<m}\frac{(-i)^l}{l!}\,\partial_x^l a\Phi^{(l)}_{\beta,s,j}(D)\,\partial_x^2 uc_{n,\beta}\Big|^2\,dx\\
&\le C\sum_{n,\beta,s,j}(n+1)E_n(u_{\beta,s,j}).
\end{aligned}
$$

Proof. We write $r = p + q$ and let n stay fixed for the moment. The left-hand side can then be estimated by

$$
C(p,q)\sum_{\beta\le p,s\le r,j}\frac{1}{n+1}A^n(t)\int k_n\sum_{1\le l<m}\frac{1}{(l!)^2}\big|\partial_x^l a\Phi^{(l)}_{\beta,s,j}(D)\,\partial_x^2 uc_{n,\beta}\big|^2\,dx.
\tag{4.6}
$$

We first consider the term with $l = 1$ of this expression:

$$
\begin{aligned}
&|\partial_x a\Phi^{(1)}_{\beta,s,j}(D)\,\partial_x^2 uc_{n,\beta}|\\
&=\Big|\partial_x a\Big[\frac{\log^{n-1}\langle D\rangle}{(n-1)!}\frac{D^{\beta+j+1}}{\langle D\rangle^{s+j+2}}\\
&\qquad+\frac{\log^n\langle D\rangle}{n!}\Big(\frac{(\beta+j)D^{\beta+j-1}}{\langle D\rangle^{s+j}}-(s+j)\frac{D^{\beta+j+1}}{\langle D\rangle^{s+j+2}}\Big)\Big]\partial_x^2 uc_{n,\beta}\Big|
\end{aligned}
$$

$$\le C\sqrt{a}\biggl(\Bigl|\frac{D^{\beta+j+2}}{\langle D\rangle^{s+j+2}}\,\partial_x u_{n-1}\Bigr| + (p+1)\Bigl|\frac{D^{\beta+j}}{\langle D\rangle^{s+j}}\,\partial_x u_n\Bigr| + (s+1)\Bigl|\frac{D^{\beta+j+2}}{\langle D\rangle^{s+j+2}}\,\partial_x u_n\Bigr|\biggr)c_{n,\beta}$$
$$\le C_1\sqrt{a}\biggl(\Bigl|\frac{D^{\beta+j}}{\langle D\rangle^{s+j}}\,\partial_x u_{n-1}\Bigr|c_{n-1,\beta} + \Bigl|\frac{D^{\beta+j}}{\langle D\rangle^{s+j+2}}\,\partial_x u_{n-1}\Bigr|c_{n-1,\beta} + \Bigl|\frac{D^{\beta+j}}{\langle D\rangle^{s+j}}\,\partial_x u_n\Bigr|c_{n,\beta} + \Bigl|\frac{D^{\beta+j}}{\langle D\rangle^{s+j+2}}\,\partial_x u_n\Bigr|c_{n,\beta}\biggr).$$

Here we have used $D^2 = \langle D\rangle^2 - 1$ and

$$\frac{c_{n,\beta}}{c_{n',\beta'}} \le 1, \quad n' \le n,\ \beta' \le \beta. \tag{4.7}$$

Thus (4.6) with $l = 1$ can be estimated by

$$C\sum_{\beta\le p, s\le r, j}\frac{1}{n+1}A^n(t)\int k_n\biggl[a\Bigl|\frac{D^{\beta+j}}{\langle D\rangle^{s+j}}\,\partial_x u_{n-1}c_{n-1,\beta}\Bigr|^2 + a\Bigl|\frac{D^{\beta+j}}{\langle D\rangle^{s+j+2}}\,\partial_x u_{n-1}c_{n-1,\beta}\Bigr|^2 + a\Bigl|\frac{D^{\beta+j}}{\langle D\rangle^{s+j}}\,\partial_x u_n c_{n,\beta}\Bigr|^2 + a\Bigl|\frac{D^{\beta+j}}{\langle D\rangle^{s+j+2}}\,\partial_x u_n c_{n,\beta}\Bigr|^2\biggr]dx$$
$$\le C\frac{1}{n+1}\sum_{\beta\le p, s\le r, j}(E_{n-1}(u_{\beta,s,j}) + E_n(u_{\beta,s,j})) + C\frac{1}{n+1}\sum_{\beta\le p, r+1\le s\le r+2, j}(E_{n-1}(u_{\beta,s,j}) + E_n(u_{\beta,s,j}))$$

because $k_n \le Ck_{n-1}$ by 5) of Proposition 2.1 and $A^n(t) \le CA(t)^{n-1}$.

We next consider the terms with $l \ge 2$. Note that one can write

$$\biggl[\frac{\log^n\langle\xi\rangle}{n!}\,\frac{\xi^{\beta+j}}{\langle\xi\rangle^{s+j}}\biggr]^{(l)}\xi^2 = \sum_{h=0}^{\min\{l,n\}}\sum_{\substack{l_1\ge h, l_1+l_2=l\\ l_2\le\beta+2+j+l_1}} C_{h,l_1,l_2}\frac{\log^{n-h}\langle\xi\rangle}{(n-h)!}\,\frac{\xi^{\beta+2+j+l_1-l_2}}{\langle\xi\rangle^{s+j+2l_1}} \tag{4.8}$$

for some constants C_{h,l_1,l_2} whose absolute values are bounded by a constant depending on p and q, but not on n. If $2 + j + l_1 - l_2$ is even and nonnegative, then using $\xi^2 = \langle\xi\rangle^2 - 1$ the right-hand side can be written as

$$\sum_{h=0}^{\min\{l,n\}}\sum_{s\le s'\le s+2r+3}\sum_{\beta'\le\beta}\sum_{j=0}^{1} C_{h,\beta',s',j}\frac{\log^{n-h}\langle\xi\rangle}{(n-h)!}\,\frac{\xi^{\beta'+j}}{\langle\xi\rangle^{s'+j}} \tag{4.9}$$

(because $2+j+l_1-l_2 \le j+2l_1$ for $l \ge 2$) where $|C_{h,\beta',s',j}|$ is bounded by a constant independent of n. The same argument applied to the case in which $2+j+l_1-l_2$ is odd and nonnegative shows that the right-hand side can be written in the same form (4.9). Then (4.6) with $l \ge 2$ can be bounded by

$$C(p,q) \sum_{\substack{\beta \le p, j\\ s \le 3r+3}} \sum_{h=0}^{\min\{r+1-s,n\}} \frac{1}{n+1} A^n(t) \int k_n(t,x) \sum_{j=0}^{1} \left| \frac{D^{\beta+j}}{\langle D\rangle^{s+j}} u_{n-h} \right|^2 c_{n-h,\beta}^2 \, dx$$

because of (4.7). This is bounded by

$$C(p,q,A) \sum_{\beta \le p, s \le 3r+3, j} \sum_{h=n-r-1}^{n} \frac{1}{h+1} E_h(u_{\beta,s,j})$$

because we can suppose $A(t) \le 2A$. We now need to deal with the terms with $s > r$:

$$\sum_{\beta \le p, r < s \le 3r+3, j} \frac{1}{n+1} A^n(t) \int k_n(t,x) \sum_{j=0}^{1} \left| \frac{D^{\beta+j}}{\langle D\rangle^{s+j}} u_n \right|^2 c_{n,\beta}^2 \, dx.$$

But since $k_n \le 1$ by 1) of Proposition 2.1 and $\beta \le p$, $s \ge r = p+q$, we have

$$\begin{aligned}
&\sum_n \frac{1}{n+1} A^n(t) \int k_n(t,x) \sum_{j=0}^{1} \left| \frac{D^{\beta+j}}{\langle D\rangle^{s+j}} u_n \right|^2 c_{n,\beta}^2 \, dx \\
&\le C \sum_n A^n(t) \int |\langle D\rangle^{-q} u_n|^2 \, dx \le C \int \left(\sum_n A^{n/2}(t) \langle \xi \rangle^{-q} \frac{\log^n \langle \xi \rangle}{n!} \right)^2 |\hat{u}|^2 \, d\xi \\
&\le C \int (\langle \xi \rangle^{-q+\sqrt{A(t)}})^2 |\hat{u}|^2 \, d\xi \le C \int |u|^2 \, dx \le C_2 \int k_0(t,x) |u_0|^2 \, dx
\end{aligned}$$

provided $q > \sqrt{2A} > \sqrt{A(t)}$. □

It remains to estimate the third term of (4.5), the one containing $|R(u_{n,\beta,s,j})|^2$.

Lemma 4.2. *We have that*

$$\sum_{n,\beta,s,j} \frac{1}{n+1} A^n \int k_n |R(u_{n,\beta,s,j})|^2 \, dx \le C(p,q,A) \int k_0(t,x) |u_0|^2 \, dx \tag{4.10}$$

for large q.

Proof. Recall that the left-hand side of (4.10) is by definition

$$\sum_{n,\beta,s,j} A^n(t)\int k_n\left|\int e^{ix\xi}\left(\iint_0^1 \Phi^{(m)}_{\beta,s,j}(\eta+\theta(\xi-\eta))\frac{1}{(m-1)!}(1-\theta)^{m-1}\right.\right.$$
$$\left.\left.\times(\xi-\eta)^m\hat{a}(t,\xi-\eta)\eta^2\hat{u}(t,\eta)\,d\theta\,d\eta\right)d\xi\right|^2 c^2_{n,\beta}\,dx$$

which by Parseval's formula is bounded by

$$\sum_{n,\beta,s,j} A^n(t)\int\left|\iint_0^1 \Phi^{(m)}_{\beta,s,j}(\eta+\theta(\xi-\eta))\frac{1}{(m-1)!}(1-\theta)^{m-1}\right.$$
$$\left.\times(\xi-\eta)^m\hat{a}(t,\xi-\eta)\eta^2\hat{u}(t,\eta)\,d\theta\,d\eta\right|^2 d\xi$$

because $k_n\le 1$ and $c_{n,\beta}\le 1$. From (4.9) it is enough to estimate terms of the form

$$C(A,p,q)\sum_n A^n(t)\int\left|\iint_0^1 \frac{\log^n\langle\eta+\theta(\xi-\eta)\rangle}{n!}\frac{(\eta+\theta(\xi-\eta))^{\beta_1+j}}{\langle\eta+\theta(\xi-\eta)\rangle^{s_1+j}}\right.$$
$$\left.\times(\xi-\eta)^m\hat{a}(t,\xi-\eta)\eta^2\hat{u}(t,\eta)\,d\theta\,d\eta\right|^2 d\xi$$

with

$$s_1-\beta_1\ge s+m-p=q+2.$$

Applying the inequality $\langle\eta+\xi\rangle^s\le 2^{|s|}\langle\eta\rangle^s\langle\xi\rangle^{|s|}$ we see that this is bounded by (writing $\hat{u}(\eta)$ for $\hat{u}(t,\eta)$ and $\hat{a}(\eta)$ for $\hat{a}(t,\eta)$)

$$C(A,p,q)\sum_n A^n\int\left|\iint_0^1 \frac{\log^n\langle\eta+\theta(\xi-\eta)\rangle}{n!}\frac{1}{\langle\eta+\theta(\xi-\eta)\rangle^{q+2}}\,d\theta\right.$$
$$\left.\times|(\xi-\eta)^m\hat{a}(\xi-\eta)|\,|\eta^2\hat{u}(\eta)|\,d\theta\,d\eta\right|^2 d\xi$$
$$\le C\sum_n(3^2A)^n\int\left(\int\langle\xi-\eta\rangle^{m+q+2}|\hat{a}(\xi-\eta)|\frac{\log^n\langle\eta\rangle}{n!}\frac{1}{\langle\eta\rangle^q}|\hat{u}(\eta)|\,d\eta\right)^2 d\xi$$
$$+C\sum_n(3^2A)^n\int\left(\int\langle\xi-\eta\rangle^{m+q+2}\frac{\log^n\langle\xi-\eta\rangle}{n!}|\hat{a}(\xi-\eta)|\frac{1}{\langle\eta\rangle^q}|\hat{u}(\eta)|\,d\eta\right)^2 d\xi$$
$$+C\sum_n(3^2A)^n\int\left(\frac{\log^n 2}{n!}\int\langle\xi-\eta\rangle^{m+q+2}\,|\hat{a}(\xi-\eta)|\frac{1}{\langle\eta\rangle^q}|\hat{u}(\eta)|\,d\eta\right)^2 d\xi$$

with $C = 3C(A, p, q)$. By the Schwarz inequality the first integral is estimated by

$$
\begin{aligned}
&C_1(A, p, q) \sum_n A^n 3^{2n} \int \Big(\int \langle \xi - \eta_1 \rangle^{m+q+2} |\hat{a}(t, \xi - \eta_1)| \, d\eta_1 \\
&\qquad\qquad \times \int \langle \xi - \eta \rangle^{m+q+2} |\hat{a}(t, \xi - \eta)| \frac{|\hat{u}_n(\eta)|^2}{\langle \eta \rangle^{2q}} \, d\eta \Big) \, d\xi \\
&\le C_1(A, p, q) \Big(\int \langle \eta_1 \rangle^{m+q+2} |\hat{a}(t, \eta_1)| \, d\eta_1 \Big)^2 \sum_n A^n 3^{2n} \int \frac{|\hat{u}_n(\eta)|^2}{\langle \eta \rangle^{2q}} \, d\eta \\
&\le C_2(A, p, q) \int \Big(\sum_n A^{n/2} 3^n \frac{|\hat{u}_n(\eta)|}{\langle \eta \rangle^{q}} \Big)^2 \, d\eta \\
&\le C_2(A, p, q) \int \big| \langle \eta \rangle^{3\sqrt{A} - q} |\hat{u}(\eta)| \big|^2 d\eta \\
&\le C_2(A, p, q) \int |\hat{u}(\eta)|^2 \, d\eta \le C_3(A, p, q) \int k_0(t, x) |u_0|^2 \, dx.
\end{aligned}
$$

Here we choose first A large and then q so that $q > 3\sqrt{A}$.

The second term is bounded by

$$
\begin{aligned}
&C_4(A, p, q) \sum_n A^n 3^{2n} \Big(\int \langle \eta_1 \rangle^{m+q+2} \frac{\log^n \langle \eta_1 \rangle}{n!} |\hat{a}(t, \eta_1)| \, d\eta_1 \Big)^2 \int \frac{|\hat{u}(\eta)|^2}{\langle \eta \rangle^{2q}} \, d\eta \\
&\le C_5(A, p, q) \Big(\sum_n A^{n/2} 3^n \int \langle \eta_1 \rangle^{m+q+2} \frac{\log^n \langle \eta_1 \rangle}{n!} |\hat{a}(t, \eta_1)| \, d\eta_1 \Big)^2 \int |\hat{u}(\eta)|^2 d\eta \\
&\le C_6(A, p, q) \Big(\int \langle \eta_1 \rangle^{m+q+2+3\sqrt{A}} |\hat{a}(t, \eta_1)| \, d\eta_1 \Big)^2 \int |\hat{u}(\eta)|^2 \, d\eta \\
&\le C_7(A, p, q) \int |\hat{u}(\eta)|^2 \, d\eta.
\end{aligned}
$$

The last term can be estimated similarly and so we end the proof of Lemma 4.2. □

From (4.1), (4.2), (4.5), Lemma 4.1 and Lemma 4.2 it follows that

$$
\sum_{n,\beta,s,j} \{ I_3(u_{n,\beta,s,j}) + I_5(u_{n,\beta,s,j}) \} \le C \sum_{n,\beta,s,j} n E_n(u_{\beta,s,j}) + [f(t)]^2 \tag{4.11}
$$

where

$$
[f(t)]^2 = e^{-ct} \sum_{n,\beta,s,j} A^n(t) \int k_n(t, x) \left| \frac{\log^n \langle D \rangle}{n!} \frac{D^{\beta+j}}{\langle D \rangle^{s+j}} f(t, x) 2^{-n\beta} \right|^2 dx.
$$

5. Proof of Theorem 1.1

Summing up the estimates (3.2), (3.6) and (4.11) we have that

$$\frac{d}{dt}\mathcal{E}(t,u) \le [f(t)]^2$$

and hence

$$\mathcal{E}(t,u) \le \mathcal{E}(t_0,u) + \int_{t_0}^{t} [f(s)]^2\,ds \tag{5.1}$$

for $-T \le t_0 \le t \le T$. Let us denote by $\|u\|_r$ the standard norm in the Sobolev space $H^r(\mathbb{R})$. Then we have

Proposition 5.1. *There is $r_1 \in \mathbb{N}$ such that for any $r_2 \in \mathbb{R}$ we can find C such that*

$$\|u_t(t)\|_{r_2}^2 + \|u(t)\|_{r_2}^2 \le C\left(\|u_t(t_0)\|_{r_1+r_2}^2 + \|u(t_0)\|_{r_1+r_2+1}^2 + \int_{t_0}^{t} \|f(s,\,\cdot)\|_{r_1+r_2}^2\,ds\right)$$

for any $-T \le t_0 \le t \le T$ and for $u \in C^2([-T,T];\mathcal{S}(\mathbb{R}))$ verifying $Pu = f$.

Proof. It is clear that

$$[u(t)]^2 \ge e^{-ct}c_0 \int |u(t,x)|^2\,dx = c_0 e^{-ct}\|u\|^2$$

because $k_0(t,x) \ge c_0 > 0$ by 1) of Proposition 2.1 (the notation $[\,\cdot\,]$ is defined at the end of last section). This together with (5.1) shows that

$$\|u_t(t)\|^2 + \|u(t)\|^2 \le C\left(\mathcal{E}(t_0,u) + \int_{t_0}^{t} [f(s)]^2\,ds\right). \tag{5.2}$$

On the other hand we see that

$$\begin{aligned}
[u(t)]^2 &\le 2e^{-ct}\sum_{n,\beta,s} A^n(t)\|u_n\|_{\beta-s}^2 \le C_1 e^{-ct}\sum_n A^n(t)\|u_n\|_p^2 \\
&\le C_1 e^{-ct}\int \langle\xi\rangle^{2p}|\hat{u}|^2\left(\sum_n A(t)^{n/2}\frac{\log^n\langle\xi\rangle}{n!}\right)^2 d\xi \\
&\le C_1 e^{-ct}\int \langle\xi\rangle^{2p+2\sqrt{A(t)}}|\hat{u}|^2\,d\xi \le e^{-ct}\|u\|_{r_1}^2
\end{aligned}$$

with $r_1 = p + \sqrt{2A(0)}$ because we can suppose $A(t) \le 2A(0)$ for $-T \le t \le T$. Similarly, we have that

$$\begin{aligned}
&e^{-ct} \sum_{n,\beta,s,j} A^n(t) \int k_n(t,x) a(t,x) \left| \frac{D^{\beta+j}}{\langle D\rangle^{s+j}}\, \partial_x u_n(t,x) 2^{-n\beta} \right|^2 dx \\
&\le 2e^{-ct} \sum_{n,\beta,s} A^n(t) \|u_n\|^2_{\beta-s+1} \le C_2 e^{-ct} \sum_n A^n(t) \|u_n\|^2_{p+1} \\
&\le C_2 e^{-ct} \|u\|^2_{r_1+1}.
\end{aligned}$$

Taking (5.1) and (5.2) into account we get that

$$\|u_t(t)\|^2 + \|u(t)\|^2 \le C_3 \left(\|u_t(t_0)\|^2_{r_1} + \|u(t_0)\|^2_{r_1+1} + \int_{t_0}^t \|f(s)\|^2_{r_1}\, ds \right). \tag{5.3}$$

Repeating the same arguments as in Sections 3 and 4 for

$$u_{n,\beta,\gamma,s,j} = 2^{-n\beta} \frac{\log^n \langle D\rangle}{n!} \frac{D^{\beta+\gamma+j}}{\langle D\rangle^{s+j}} u$$

with $\gamma = 0, 1, \ldots, r_2$, we obtain the desired result. □

Proposition 5.2. *There is $r_1 \in \mathbb{N}$ such that for any $r_2 \in \mathbb{R}$ one can find C such that*

$$\|u_t(t)\|^2_{r_2} + \|u(t)\|^2_{r_2} \le C \left(\|u_t(t_0)\|^2_{r_1+r_2} + \|u(t_0)\|^2_{r_1+r_2+1} + \int_{t_0}^t \|f(s,\ \cdot)\|^2_{r_1+r_2}\, ds \right)$$

for any $-T \le t_0 \le t \le T$ and for any $u \in C^2([-T, T]; \mathcal{S}(\mathbb{R}))$ satisfying

$$P^* u = \partial_t^2 u - a(t,x)\, \partial_x^2 u - 2a_x(t,x)\, \partial_x u - a_{xx}(t,x) u = f.$$

Proof. To check the proposition it suffices to estimate

$$F(u_n) = 2e^{-ct} A^n(t) \int k_n(t,x)\, \mathrm{Re}\left[\frac{\log^n \langle D\rangle}{n!} (2a_x\, \partial_x u + a_{xx} u) \cdot \overline{u}_{n,t} \right] dx. \tag{5.4}$$

Since

$$\begin{aligned}
&\frac{\log^n \langle D\rangle}{n!} (2a_x\, \partial_x u + a_{xx} u) \\
&= 2a_x \partial_x u_n + a_{xx} u_n + 2\left[\frac{\log^n \langle D\rangle}{n!}, a_x \right] \partial_x u + \left[\frac{\log^n \langle D\rangle}{n!}, a_{xx} \right] u
\end{aligned}$$

repeating the same arguments as in Section 4 we get that

$$\sum_{n,\beta,s,j} F(u_{n,\beta,s,j}) \le C \sum_{n,\beta,s,j} E_n(u_{\beta,s,j}):$$

this proves the desired assertion. □

By Propositions 5.1 and 5.2, we can apply standard arguments of functional analysis to conclude Theorem 1.1 (see, for example, Section 23.2 in [6]).

To check Theorem 1.1′ we first note that if $k_{jn}(t, x)$, $n \in \mathbb{N}$ are weight functions for $B_j(t, x) \geq 0$ verifying Proposition 2.1 then

$$k_n(t, x) = \prod_{j=1}^{r} k_{jn}(t, x), \quad n \in \mathbb{N}$$

are weight functions for $\prod_{j=1}^{r} B_j(t, x)$ verifying Proposition 2.1. Thus to show Theorem 1.1′ we can assume that $r = 1$. Write $m = m_1$ and $B_1(t, x) = b(t, x)^m$. Note that if m is odd and hence $b(t, x) \geq 0$ near the origin then the proof is obvious because the weight functions for $b(t, x)$ given in Proposition 2.1 are also weight functions for $b(t, x)^m$. Let m be even and hence $b(t, x)^m = [b(t, x)^2]^{m/2}$. Repeating the same arguments as in Sections 6 and 7 with minor changes such as

$$k_{m,t_0(x_0)}(t, x) = \exp\left[N \int_{I_m(x) \cap [t_0(x_0), t]} \frac{|b_t(s, x)|}{|b(s, x)|}\, ds \right]$$

for $t > t_0(x_0)$ and $k_{m,t_0(x_0)}(t, x) = 1$ if $t \leq t_0(x_0)$ with $I_m(x) = \{s \mid 2^{-m} \leq |b(t, x)| \leq 2^{-m+2}\}$ we obtain the required weight functions for $b(t, x)^2$ which is also the required weight functions for $[b(t, x)^2]^{m/2}$.

6. Construction of the weight functions

To prove Proposition 2.1 it turns out that the notation is simpler if we construct the reciprocal functions $1/k_n(t, x)$; we will denote them again by k_n and list in the proposition below the analogous properties that they should enjoy.

Proposition 6.1. *Let $N > 0$ be a given constant. Then there is $T > 0$, a sequence of weight functions $k_n(t, x) \in W^{1,\infty}((-T, T) \times \mathbb{R})$ and some positive constants $C_1, \ldots, C_8$ (all depending on N except C_6) such that*

1) $1 \leq k_n(t, x) \leq C_1 e^{C_2 n}$,
2) $0 \leq \partial_t k_n(t, x) \leq C_3 e^{C_4 n}$,
3) *in a neighbourhood of the origin we have*

$$|\partial_x k_n(t, x)| \sqrt{a(t, x)} \leq C_5 n k_n(t, x),$$

4) *in a neighbourhood of the origin we have*

$$\frac{\partial_t k_n(t, x)}{k_n(t, x)} \geq \frac{N}{C_6} \frac{|a_t(t, x)|}{a(t, x) + 2^{-2n}} - C_7 n,$$

5) $k_{n-1} \leq C_8 k_n$.

Proof. The proof is fairly long: we need several steps and we will finish it in the last section. Recall that one can write

$$a(t, x) = e(t, x)(t^p + a_1(x)t^{p-1} + \cdots + a_p(x))$$

in a neighbourhood U of the origin and that, changing the scale of the t coordinate if necessary and using Glaeser's inequality, we may assume that, in U, $0 \leq a(t,x) \leq 1$ and

$$|\partial_x \sqrt{a(t, x)}| \leq L = \frac{1}{320(p+1)}.$$

Let ϵ be a positive number. Since the functions

$$a(t, x) - \epsilon, \quad a(t, x) - 16\epsilon$$

are regular in t, we can write also them as a non-zero function multiplied by a Weierstrass polynomial in a neighbourhood of (0, 0). Let $\Delta_1(x, \epsilon)$ be the discriminant of $a(t, x) - \epsilon$ and $\Delta_2(x, \epsilon)$ the discriminant of $a(t, x) - 16\epsilon$. We observe that up to maybe changing T the equations $a(t, x) - \epsilon = 0$, $a(t, x) - 16\epsilon = 0$, $t + T = 0$ and $t - T = 0$ have mutually distinct solutions in t for small x and $\epsilon > 0$.

Let $\Delta(x, \epsilon) = \Delta_1(x, \epsilon)\Delta_2(x, \epsilon)$; since $\Delta(x, 0)$ vanishes of order $2q$ at $x = 0$ by hypothesis (1.4) we can write, for d sufficiently small,

$$\Delta(x, \epsilon) = c(x, \epsilon)(x^{2q} + c_1(\epsilon)x^{2q-1} + \cdots + c_{2q}(\epsilon))$$

for $|x| < d$ and $|\epsilon| < \epsilon_0$. For $\epsilon > 0$ fixed ($\epsilon < \epsilon_0$), $\Delta(\,\cdot\,, \epsilon)$ has at most $2q$ real zeros for $|x| < d$:

$$x_1(\epsilon) \leq x_2(\epsilon) \leq \cdots \leq x_{q_1-1}(\epsilon)$$

where $q_1 - 1$ is the number of real zeros, in x, of $\Delta(x, \epsilon)$ and depends on ϵ. Taking $\epsilon_0 > 0$ and $\delta > 0$ ($\delta \ll d$) small we may assume that $-d + \delta < x_1(\epsilon)$ and $x_{q_1-1}(\epsilon) < d - \delta$ for $|\epsilon| < \epsilon_0$.

Let us call J_δ the interval $(-d + \delta, d - \delta)$; we can assume that $U = [-T, T] \times J_\delta$.

We now divide the interval J_δ into q_1 subintervals $A_j(\epsilon) = (x_{j-1}(\epsilon), x_j(\epsilon))$, $j = 1, \ldots, q_1$, where $x_0(\epsilon) = -d + \delta$, $x_{q_1}(\epsilon) = d - \delta$. For $x \in A_j(\epsilon)$ we can define p_j real functions

$$-T = t_{j1}(x, \epsilon) < \cdots < t_{jp_j}(x, \epsilon) = T$$

which are the roots in t of

$$(a(t, x) - \epsilon)(a(t, x) - 16\epsilon)(t + T)(t - T)$$

contained in the interval $[-T, T]$ and are continuous in $x \in A_j(\epsilon)$. In general p_j depends on j and ϵ; nevertheless, we always have $2 \leq p_j \leq 2p + 2$. We will at times make the dependence on ϵ implicit to simplify the notation.

Let us fix an integer m and put $\epsilon = 2^{-2m}$. We suppose that $2^{-2m} < \epsilon_0$, that is $m > m_0$; later we will deal with the case $m \leq m_0$. We choose one $A_j(2^{-2m})$ and one of the functions $t_{jl}(x, 2^{-2m})$ defined on it and denote it by $t_0(x, 2^{-2m})$ (or $t_0(x)$) for the time being, to avoid clumsiness (we will need to revert to the usual notation from Lemma 6.2 on). Note that either $t_0(x, 2^{-2m}) = \pm T$, or $a(t_0(x, 2^{-2m}), x) = 2^{-2m}$ or $a(t_0(x, 2^{-2m}), x) = 2^{-2m+4}$ in $A_j(2^{-2m})$. Define $b_{t_0}(t, x)$ by

$$b_{t_0}(t, x) = \sqrt{a(t_0(x), x)}$$

if $t \leq t_0(x)$ and

$$b_{t_0}(t, x) = \sqrt{a(t_0(x), x)} + \int_{t_0(x)}^{t} |\partial_s \sqrt{a(s, x)}|\, ds$$

if $t > t_0(x)$. Note that $b_{t_0}(t, x)$ is nondecreasing in t and $b_{t_0}(t, x) \geq \sqrt{a(t, x)}$ for $t > t_0(x)$. Define

$$Q_h = (h2^{-m} - 2^{-m-1}, h2^{-m} + 2^{-m-1})$$

for $h \in \mathbb{Z}$. We choose $x_h \in Q_h \cap A_j(2^{-2m})$ (if this set is not empty) and set $x'_h = x_h + 2^{-m}$. For m large, $2^{-m} < \delta$ and $x_h \in A_j(2^{-2m})$ implies $x'_h \in (-d, d)$ (here x_h and x'_h depend on j).

Let us put

$$\phi_{h,t_0}(t, x) = \left(\left(4 - \frac{|x - x_h|}{b_{t_0}(t, x_h)}\right) \vee 0\right) \wedge 1$$

and define

$$k_{m,t_0(x_0)}(t, x) = \exp\left[N \int_{I_m(x)\cap[t_0(x_0),t]} \frac{|a_t(s, x)|}{a(s, x)}\, ds\right] \tag{6.1}$$

if $t > t_0(x_0)$ and $k_{m,t_0(x_0)}(t, x) = 1$ if $t \leq t_0(x_0)$. Here N is a positive number, $x_0 \in A_j(2^{-2m})$ and

$$I_m(x) = \{s \mid 2^{-2m} \leq a(s, x) \leq 2^{-2m+4}\}.$$

We now set

$$\tilde{k}_{m,t_0}(t, x) = \sup_h [k_{m,t_0(x_h)}(t, x_h) k_{m,t_0(x_h)}(t, x'_h) \phi_{h,t_0}(t, x)] \vee 1$$

where the supremum is taken over all h such that $Q_h \cap A_j(2^{-2m}) \neq \emptyset$ (therefore it is indeed a maximum over a finite set). Products of functions $\tilde{k}_{m,t_0}(t, x)$ as t_0 varies among all the possible choices will be factors in the desired weight function $k_n(t, x)$.

Lemma 6.1. *We have*

1) $1 \le \tilde{k}_{m,t_0}(t,x) \le \exp[2N(p+1)\log 2^4]$,
2) $\partial_t \tilde{k}_{m,t_0}(t,x) \ge 0$,
3) $\partial_t \tilde{k}_{m,t_0}(t,x) \le C_9 2^m \tilde{k}_{m,t_0}(t,x)$,
4) $|\partial_x \tilde{k}_{m,t_0}(t,x)|\sqrt{a(t,x)} \le 2\exp[2N(p+1)\log 2^4]\tilde{k}_{m,t_0}(t,x)$.

Proof. Since $a(t,x)$ is a polynomial in t of degree p, 1) is easily checked. From

$$\partial_t k_{m,t_0(x_h)}(t,x_h) \ge 0,\ \partial_t k_{m,t_0(x_h)}(t,x_h') \ge 0,\ \partial_t \phi_{h,t_0}(t,x) \ge 0 \tag{6.2}$$

it follows that $\partial_t \tilde{k}_{m,t_0}(t,x) \ge 0$.

To prove 3) note that

$$\begin{aligned}
&\partial_t k_{m,t_0(x_h)}(t,x_h) \le N\frac{|a_t|}{a} k_{m,t_0(x_h)}(t,x_h) \le NC2^m k_{m,t_0(x_h)}(t,x_h),\\
&\partial_t k_{m,t_0(x_h)}(t,x_h') \le N\frac{|a_t|}{a} k_{m,t_0(x_h)}(t,x_h') \le NC2^m k_{m,t_0(x_h)}(t,x_h'),\\
&\partial_t \phi_{h,t_0} \le \frac{|x-x_h|}{b_{t_0}(t,x_h)}\frac{|\partial_t b_{t_0}(t,x_h)|}{b_{t_0}(t,x_h)} \le 4\frac{C}{2^{-m}} = 4C2^m.
\end{aligned}$$

Thus we see that

$$\begin{aligned}
&\partial_t[k_{m,t_0(x_h)}(t,x_h)k_{m,t_0(x_h)}(t,x_h')\phi_{h,t_0}(t,x)]\\
&\le 2NC2^m[k_{m,t_0(x_h)}(t,x_h)k_{m,t_0(x_h)}(t,x_h')\phi_{h,t_0}(t,x)]\\
&\quad + 4C2^m \exp[2N(p+1)\log 2^4]\\
&\le \{2NC2^m + 4C2^m \exp[2N(p+1)\log 2^4]\}\tilde{k}_{m,t_0}(t,x)
\end{aligned}$$

which shows that

$$\partial_t \tilde{k}_{m,t_0}(t,x) \le C_9 2^m \tilde{k}_{m,t_0}(t,x).$$

We turn to assertion 4). If $\tilde{k}_{m,t_0}(t,x) = 1$ then $\partial_x \tilde{k}_{m,t_0} = 0$ and hence the assertion clearly holds. If $\tilde{k}_{m,t_0}(t,x) > 1$, let the supremum in the definition of $\tilde{k}_{m,t_0}$ be attained for a certain index $\bar{h}$. Then it is clear that we have $t > t_0(x_{\bar{h}})$ and $\phi_{\bar{h},t_0}(t,x) > 0$. Thus $|x - x_{\bar{h}}| \le 4b_{t_0}(t,x_{\bar{h}})$, so that

$$|\sqrt{a(t,x)} - \sqrt{a(t,x_{\bar{h}})}| \le \frac{1}{4}|x - x_{\bar{h}}| \le b_{t_0}(t,x_{\bar{h}})$$

and hence

$$\sqrt{a(t,x)} \le \sqrt{a(t,x_{\bar{h}})} + b_{t_0}(t,x_{\bar{h}}) \le 2b_{t_0}(t,x_{\bar{h}})$$

because $b_{t_0}(t,x) \geq \sqrt{a(t,x)}$ for $t > t_0(x)$. Now we have that

$$|\partial_x \phi_{\bar{h},t_0}(t,x)| \sqrt{a(t,x)} \leq \frac{\sqrt{a(t,x)}}{b_{t_0}(t,x_{\bar{h}})} \leq 2$$

so that

$$\begin{aligned} |\partial_x \tilde{k}_{m,t_0}(t,x)| \sqrt{a(t,x)} &\leq 2 \exp[2N(p+1) \log 2^4] \\ &\leq 2 \exp[2N(p+1) \log 2^4] \tilde{k}_{m,t_0}(t,x) \end{aligned}$$

and hence 4). □

Lemma 6.2. *Let* $(t,x) \in U$ *be a point such that* $x \in A_j(2^{-2m})$, $t_{jl}(x, 2^{-2m}) < t < t_{jl+1}(x, 2^{-2m})$ *and* $2^{-2m+1} \leq a(t,x) \leq 2^{-2m+3}$. *If*

$$\tilde{k}_{m,t_{jl}}(t,x) = \left[k_{m,t_{jl}(x_{\bar{h}})}(t,x_{\bar{h}}) \cdot k_{m,t_{jl}(x_{\bar{h}})}\big(t,x'_{\bar{h}}\big) \cdot \phi_{\bar{h},t_{jl}}(t,x)\right]$$

(that is, the supremum in the definition of $\tilde{k}_{m,t_{jl}}$ *is attained at index* $\bar{h}$*), then* $|x - x_{\bar{h}}| \leq 160(p+1)/9 \cdot 2^{-m}$.

Proof. We consider the interval Q_i that contains x. Let $x_i \in Q_i \cap A_j(2^{-2m})$: $|x - x_i| \leq 2^{-m}$ and $x'_i = x_i + 2^{-m}$ (it may happen that $x'_i \notin A_j(2^{-2m})$). For y between x and x_i we have $|\sqrt{a(t,y)} - \sqrt{a(t,x)}| \leq 2^{-m-2}$ so that

$$2^{-2m} < a(t,y) < 2^{-2m+4}$$

and $t_{jl}(y, 2^{-2m}) < t < t_{jl+1}(y, 2^{-2m})$. So we see that

$$2^{-2m} < a(t,x_i) < 2^{-2m+4}. \tag{6.3}$$

Suppose $k_{m,t_{jl}(x_i)}(t,x_i) = 1$: it follows that $a_t(s,x_i) = 0$ for all s such that $t_{jl}(x_i, 2^{-2m}) < s < t$, so that

$$a(t,x_i) = a(t_{jl}(x_i), x_i) = 2^{-2m} \quad \text{or} \quad 2^{-2m+4}$$

which contradicts (6.3). Thus we have $k_{m,t_{jl}(x_i)}(t,x_i) > 1$ and hence also

$$k_{m,t_{jl}(x_i)}(t,x_i) k_{m,t_{jl}(x_i)}(t,x'_i) > 1.$$

Since

$$\phi_{i,t_{jl}}(t,x) \geq \left(\left(4 - \frac{2^{-m}}{b_{t_{jl}}(t,x_i)}\right) \vee 0\right) \wedge 1 = 1$$

because $b_{t_{jl}}(t,x_i) \geq \sqrt{a(t_{jl}(x_i), x_i)} \geq 2^{-m}$, we see that

$$\tilde{k}_{m,t_{jl}}(t,x) = \sup_h [k_{m,t_{jl}(x_h)}(t,x_h) k_{m,t_{jl}(x_h)}(t,x'_h) \phi_{h,t_{jl}}(t,x)] > 1.$$

Assume now that when the index is $\bar{h}$ the supremum is attained. Then

$$|x - x_{\bar{h}}| \le 4b_{t_{jl}}(t, x_{\bar{h}})$$

and $t > t_{jl}(x_{\bar{h}})$ (since $k_{m,t_{jl}(x_{\bar{h}})}(t, x_{\bar{h}})k_{m,t_{jl}(x_{\bar{h}})}\big(t, x'_{\bar{h}}\big) > 1$). Consider the smallest value $\bar{t}$ such that

$$\sqrt{a(\bar{t}, x_{\bar{h}})} = \sup_{t_{jl}(x_{\bar{h}})\le r\le t} \sqrt{a(r, x_{\bar{h}})};$$

noting that $b_{t_{jl}}(t, x_{\bar{h}})$ is nondecreasing in t, it is easy to see that

$$\sqrt{a(\bar{t}, x_{\bar{h}})} \le b_{t_{jl}}(t, x_{\bar{h}}) \le (p+1)\sqrt{a(\bar{t}, x_{\bar{h}})}.$$

We first consider the case in which $t_{jl}(x) < \bar{t}$ $(\le t < t_{jl+1}(x))$. We observe that

$$\sqrt{a(\bar{t}, x)} = \alpha 2^{-m}$$

with α between 1 and 4; then

$$\begin{aligned}\left|\sqrt{a(\bar{t}, x_{\bar{h}})} - \alpha 2^{-m}\right| &\le L|x - x_{\bar{h}}| \le 4Lb_{t_{jl}}(t, x_{\bar{h}})\\ &\le 4L(p+1)\sqrt{a(\bar{t}, x_{\bar{h}})} \le \frac{1}{10}\sqrt{a(\bar{t}, x_{\bar{h}})}.\end{aligned}$$

We obtain that $(10/11)\alpha 2^{-m} \le \sqrt{a(\bar{t}, x_{\bar{h}})} \le (10/9)\alpha 2^{-m}$ and hence that

$$|x - x_{\bar{h}}| \le 4(p+1)\frac{10}{9}\alpha 2^{-m}.$$

We consider now the other case, i.e. when $t_{jl}(x) \ge \bar{t}$. Since $t_{jl}(x_{\bar{h}}) \le \bar{t}$ and $t_{jl}(x) \ge \bar{t}$, there exists some ξ between x and $x_{\bar{h}}$ such that $t_{jl}(\xi) = \bar{t}$ and hence

$$\sqrt{a(\bar{t}, \xi)} = 2^{-m} \quad \text{or} \quad \sqrt{a(\bar{t}, \xi)} = 2^{-m+2}.$$

Noting that

$$\begin{aligned}\left|\sqrt{a(\bar{t}, x_{\bar{h}})} - \sqrt{a(\bar{t}, \xi)}\right| &\le L|\xi - x_{\bar{h}}| \le 4Lb_{t_{jl}}(t, x_{\bar{h}})\\ &\le 4L(p+1)\sqrt{a(\bar{t}, x_{\bar{h}})} \le \frac{1}{10}\sqrt{a(\bar{t}, x_{\bar{h}})}\end{aligned}$$

we conclude as before that

$$\frac{10}{11}\alpha 2^{-m} \le \sqrt{a(\bar{t}, x_{\bar{h}})} \le \frac{10}{9}\alpha 2^{-m}, \quad |x - x_{\bar{h}}| \le 4(p+1)\frac{10}{9}\alpha 2^{-m}$$

where $\alpha = 1$ or 4. Thus we have $|x - x_{\bar{h}}| \le (160/9)\cdot(p+1)2^{-m}$ which ends the proof. $\square$

Lemma 6.3. *Let $(t, x) \in U$ be a point such that*

$$2^{-2m+1} \leq a(t, x) \leq 2^{-2m+3} :$$

there exist j and l such that

$$\partial_t \tilde{k}_{m,t_{jl}} \geq \frac{N}{C_{11}} \frac{|a_t(t, x)|}{a(t, x)} \tilde{k}_{m,t_{jl}} - C_{12} \tilde{k}_{m,t_{jl}}.$$

Proof. We choose j, l such that

$$x \in A_j(2^{-2m}), \quad t_{jl}(x, 2^{-2m}) < t < t_{jl+1}(x, 2^{-2m}).$$

Applying Lemma 6.2 and keeping the same notations, we have that

$$|\sqrt{a(t, x_{\bar{h}})} - \sqrt{a(t, x)}| \leq L|x_{\bar{h}} - x| \leq \frac{1}{18} \cdot 2^{-m}$$

so that $2^{-2m} < a(t, x_{\bar{h}}) < 2^{-2m+4}$. The same inequality holds for $a\left(t, x'_{\bar{h}}\right)$. This shows that

$$t \in I_m(x_{\bar{h}}) \cap I_m\left(x'_{\bar{h}}\right).$$

Then we have that

$$\partial_t\left[k_{m,t_{jl}(x_{\bar{h}})}(t, x_{\bar{h}}) k_{m,t_{jl}(x_{\bar{h}})}\left(t, x'_{\bar{h}}\right)\right] \phi_{\bar{h},t_{jl}}(t, x) \geq N\left[\frac{|a_t(t, x_{\bar{h}})|}{a(t, x_{\bar{h}})} + \frac{\left|a_t\left(t, x'_{\bar{h}}\right)\right|}{a\left(t, x'_{\bar{h}}\right)}\right] \tilde{k}_{m,t_{jl}}(t, x).$$

Note that by Taylor's formula

$$\begin{aligned} a_t(t, x) &= a_t(t, x_{\bar{h}}) + a_{tx}(t, x_{\bar{h}})(x - x_{\bar{h}}) + R_2(x - x_{\bar{h}}), \\ a_t\left(t, x'_{\bar{h}}\right) &= a_t(t, x_{\bar{h}}) + a_{tx}(t, x_{\bar{h}}) 2^{-m} + R_2(2^{-m}) \end{aligned}$$

where R_2 is the remainder of second order, which proves that

$$\begin{aligned} |a_t(t, x)| &\leq |a_t(t, x_{\bar{h}})| + \frac{160}{9} \cdot (p+1)\left(|a_t(t, x_{\bar{h}})| + \left|a_t\left(t, x'_{\bar{h}}\right)\right|\right) + C_{10} 2^{-2m} \\ &\leq \left(\frac{160}{9} \cdot (p+1) + 1\right)|a_t(t, x_{\bar{h}})| + \frac{160}{9} \cdot (p+1)\left|a_t\left(t, x'_{\bar{h}}\right)\right| + C_{10} 2^{-2m}. \end{aligned}$$

Thus one has that

$$\begin{aligned} \frac{|a_t(t, x)|}{a(t, x)} &\leq \left(\frac{160}{9} \cdot (p+1) + 1\right)\left(\frac{|a_t(t, x_{\bar{h}})|}{a(t, x)} + \frac{\left|a_t\left(t, x'_{\bar{h}}\right)\right|}{a(t, x)}\right) + C_{10} \\ &\leq C_{11}\left(\frac{|a_t(t, x_{\bar{h}})|}{a(t, x_{\bar{h}})} + \frac{\left|a_t\left(t, x'_{\bar{h}}\right)\right|}{a\left(t, x'_{\bar{h}}\right)}\right) + C_{10} \end{aligned}$$

where $C_{11} = 16((160/9) \cdot (p+1) + 1)$. These prove that

$$\partial_t \tilde{k}_{m,t_{jl}}(t, x) \geq \frac{N}{C_{11}} \frac{|a_t(t, x)|}{a(t, x)} \tilde{k}_{m,t_{jl}}(t, x) - \frac{C_{10}}{C_{11}} N \tilde{k}_{m,t_{jl}}(t, x)$$

which is the desired assertion. □

7. Construction of the weight functions (continued)

We now construct the second kind of factor $\tilde{k}'_{n,t_0}(t,x)$ which appears in the weight functions $k_n(t,x)$. The construction is largely analogous to what was done above for factors of the first kind.

Let ϵ be a positive number. Since the function

$$a(t,x) - 16\epsilon$$

is regular in t, then we can write it as a non-zero function multiplied by a Weierstrass polynomial in a neighbourhood of $(0,0)$. Let $\Delta(x,\epsilon)$ be the discriminant. Since $\Delta(x,0)$ vanishes of order q at $x = 0$, from the assumption (1.4) we can write

$$\Delta(x,\epsilon) = c(x,\epsilon)(x^q + c_1(\epsilon)x^{q-1} + \cdots + c_q(\epsilon))$$

for $|x| < d$ and $|\epsilon| < \epsilon_0$. For $\epsilon > 0$ fixed ($\epsilon < \epsilon_0$), $\Delta(\cdot,\epsilon)$ has at most q real zeros for $|x| < d$;

$$x_1(\epsilon) \le x_2(\epsilon) \le \cdots \le x_{q_1-1}(\epsilon).$$

As in Section 6, we may assume that $-d+\delta < x_1(\epsilon)$, $x_{q_1-1}(\epsilon) < d-\delta$ for $|\epsilon| < \epsilon_0$. We divide the interval $J'_\delta = (-d+\delta, d-\delta)$ into q_1 subintervals $A'_j(\epsilon) = (x_{j-1}(\epsilon), x_j(\epsilon))$, where $x_0(\epsilon) = -d+\delta$, $x_{q_1}(\epsilon) = d-\delta$. For $x \in A'_j(\epsilon)$ we can define p_j real functions $(0 \le p_j \le p+2)$

$$-T = t_{j1}(x,\epsilon) < \cdots < t_{jp_j}(x,\epsilon) = T$$

which are the roots of

$$(a(t,x) - 16\epsilon)(t+T)(t-T) = 0$$

contained in the interval $[-T, T]$ and are continuous in $x \in A'_j(\epsilon)$.

Let us fix an integer n and put $\epsilon = 2^{-2n}$. Take $A'_j(2^{-2n})$ and call $t_0(x, 2^{-2n})$ one of the functions defined on it. Note that either $t_0 = \pm T$ or $a(t_0(x,2^{-2n}),x) = 2^{-2n+4}$ in $A'_j(2^{-2n})$. Define $b'_{t_0}(t,x)$ by

$$b'_{t_0}(t,x) = \sqrt{a(t_0(x),x)} + 2^{-n}$$

if $t > t_0(x)$ and

$$b'_{t_0}(t,x) = \sqrt{a(t_0(x),x)} + \int_{t_0(x)}^{t} |\partial_s \sqrt{a(s,x)}|\, ds + 2^{-n}$$

if $t > t_0(x)$. Note that $b'_{t_0}(t,x)$ is nondecreasing in t and $b'_{t_0}(t,x) \ge \sqrt{a(t,x)} + 2^{-n}$ for $t > t_0(x)$. We then define

$$Q_h = (h2^{-n} - 2^{-n-1}, h2^{-n} + 2^{-n-1})$$

for $h \in \mathbb{Z}$; we choose $x_h \in Q_h \cap A'_j(2^{-2n})$ (if this set is not empty) and set $x'_h = x_h + 2^{-n}$. For n large, $x_h \in A'_j(2^{-2n})$ implies $x'_h \in (-d, d)$. Put

$$\phi'_{h,t_0}(t,x) = \left(\left(4 - \frac{|x - x_h|}{b'_{t_0}(t, x_h)}\right) \vee 0\right) \wedge 1$$

and define (since $x_0 \in A'_j(2^{-2n})$) $k'_{n,t_0(x_0)}(t,x) = 1$ if $t \le t_0(x_0)$ and

$$k'_{n,t_0(x_0)}(t,x) = \exp\left[N \int_{I'_n(x)\cap[t_0(x_0),t]} \frac{|a_t(s,x)|}{2^{-2n}}\, ds\right]$$

if $t > t_0(x_0)$. Here N is the positive constant given in the definition (6.1) of $k_{m,t_0(x_0)}(t,x)$ and

$$I'_n(x) = \{s \mid a(s,x) \le 2^{-2n+4}\}.$$

We now define $\tilde{k}'_{n,t_0}(t,x)$ by

$$\tilde{k}'_{n,t_0}(t,x) = \sup_h [k'_{n,t_0(x_h)}(t,x_h) k'_{n,t_0(x_h)}(t,x'_h) \phi'_{h,t_0}(t,x)] \vee 1$$

where the supremum is taken over all h such that $Q_h \cap A'_j(2^{-2n}) \ne \emptyset$.

This $\tilde{k}'_{n,t_0}(t,x)$ enjoys analogous properties as $\tilde{k}_{m,t_0}(t,x)$ listed in Lemma 6.1.

Lemma 7.1. *We have*
1) $1 \le \tilde{k}'_{n,t_0}(t,x) \le \exp[2N(p+1)2^4]$,
2) $\partial_t \tilde{k}'_{n,t_0}(t,x) \ge 0$,
3) $\partial_t \tilde{k}'_{n,t_0}(t,x) \le C_1 2^n \tilde{k}'_{n,t_0}(t,x)$,
4) $|\partial_x \tilde{k}'_{n,t_0}(t,x)| \sqrt{a(t,x)} \le 2\exp[2N(p+1)2^4] \tilde{k}'_{n,t_0}(t,x)$.

Proof. To check 2) it is enough to observe that

$$\partial_t k'_{n,t_0(x_h)}(t,x_h) \ge 0, \quad \partial_t k'_{n,t_0(x_h)}(t,x'_h) \ge 0, \quad \partial_t \phi'_{h,t_0}(t,x) \ge 0. \tag{7.1}$$

To see 3) note that

$$\partial_t k'_{n,t_0(x_h)}(t,x_h) \le N \frac{|a_t|}{2^{-2n}} k'_{n,t_0(x_h)}(t,x_h) \le N C_2 2^n k'_{n,t_0(x_h)}(t,x_h),$$
$$\partial_t k'_{n,t_0(x_h)}(t,x'_h) \le N \frac{|a_t|}{2^{-2n}} k'_{n,t_0(x_h)}(t,x'_h) \le N C_2 2^n k'_{n,t_0(x_h)}(t,x'_h).$$

On the other hand we have that

$$\partial_t \phi'_{h,t_0} \le \frac{|x-x_h|}{b'_{t_0}(t,x_h)} \frac{|\partial_t b'_{t_0}(t,x_h)|}{b'_{t_0}(t,x_h)} \le 4\frac{C_3}{2^{-n}} = 4C_3 2^n$$

and hence that

$$\begin{aligned}
&\partial_t[k'_{n,t_0(x_h)}(t,x_h)k'_{n,t_0(x_h)}(t,x'_h)\phi'_{h,t_0}(t,x)] \\
&\le 2NC_2 2^n[k'_{n,t_0(x_h)}(t,x_h)k'_{n,t_0(x_h)}(t,x'_h)\phi'_{h,t_0}(t,x)] \\
&\quad + 4C_3 2^n \exp[2N(p+1)2^4] \\
&\le \{2NC_2 2^n + 4C_3 2^n \exp[2N(p+1)2^4]\}\tilde{k}'_{n,t_0}(t,x)
\end{aligned}$$

which implies that

$$\partial_t \tilde{k}'_{n,t_0}(t,x) \le C_4 2^n \tilde{k}'_{n,t_0}(t,x).$$

We turn to the proof of 4). If $\tilde{k}'_{n,t_0}(t,x) = 1$ then $\partial_x \tilde{k}'_{n,t_0} = 0$ and nothing is to be proved. Assume that this is not the case. Let $\bar{h}$ be an index such that the supremum in the definition of $\tilde{k}'_{n,t_0}$ is attained for that index. We have $k'_{n,t_0(x_{\bar{h}})}(t,x_{\bar{h}})k'_{n,t_0(x_{\bar{h}})}\big(t,x'_{\bar{h}}\big)\phi'_{\bar{h},t_0}(t,x) > 1$, $t > t_0(x_{\bar{h}})$ and $\phi'_{\bar{h},t_0}(t,x) > 0$. We have thus $|x - x_{\bar{h}}| \le 4b'_{t_0}(t,x_{\bar{h}})$, so that

$$|\sqrt{a(t,x)} - \sqrt{a(t,x_{\bar{h}})}| \le \frac{1}{4}|x-x_{\bar{h}}| \le b'_{t_0}(t,x_{\bar{h}})$$

and hence

$$\sqrt{a(t,x)} \le \sqrt{a(t,x_{\bar{h}})} + b'_{t_0}(t,x_{\bar{h}}) \le 2b'_{t_0}(t,x_{\bar{h}}).$$

From this it follows that

$$|\partial_x \phi'_{\bar{h},t_0}(t,x)|\sqrt{a(t,x)} \le \frac{\sqrt{a(t,x)}}{b'_{t_0}(t,x_{\bar{h}})} \le 2$$

so that

$$|\partial_x \tilde{k}'_{n,t_0}(t,x)|\sqrt{a(t,x)} \le 2\exp[2N(p+1)2^4] \le 2\exp[2N(p+1)2^4]\tilde{k}'_{n,t_0}(t,x)$$

which shows 4). □

Lemma 7.2. *Let (t,x) be in $[-T,T]\times J'_\delta$ be a point such that $a(t,x) \le 2^{-2n+3}$, $x \in A'_j(2^{-2n})$ and $t_{jl}(x,2^{-2n}) < t < t_{jl+1}(x,2^{-2n})$. If the supremum of*

$$k'_{n,t_{jl}(x_h)}(t,x_h)\cdot k'_{n,t_{jl}(x_h)}(t,x'_h)\cdot \phi_{h,t_{jl}}(t,x)$$

on the set of indices h such that $Q_h \cap A'_j(2^{-2n}) \ne \emptyset$ is attained for index $\bar{h}$, then $|x - x_{\bar{h}}| \le (200(p+1)/9)\cdot 2^{-n}$.

Proof. We follow the proof of Lemma 6.2. We consider the interval Q_i that contains x. Let $x_i \in Q_i \cap A'_j(2^{-2n})$: $|x - x_i| \leq 2^{-n}$ and $x'_i = x_i + 2^{-n}$ (x'_i may not belong to $A'_j(2^{-2n})$). For y between x and x_i we have $|\sqrt{a(t, y)} - \sqrt{a(t, x)}| \leq 2^{-n-2}$ so that

$$a(t, y) < 2^{-2n+4}$$

and $t_{jl}(y, 2^{-2n}) < t < t_{jl+1}(y, 2^{-2n})$. So we see that

$$a(t, x_i) < 2^{-2n+4}.$$

If $k'_{n,t_{jl}(x_i)}(t, x_i) = 1$ it follows that $a_t(s, x_i) = 0$ for $t_{jl}(x_i, 2^{-2n}) < s < t$ so that

$$a(t, x_i) = a(t_{jl}(x_i), x_i) = 2^{-2n+4}$$

which is a contradiction. Thus we have that $k'_{n,t_{jl}(x_i)}(t, x_i) > 1$ and hence

$$k'_{n,t_{jl}(x_i)}(t, x_i) \cdot k'_{n,t_{jl}(x_i)}(t, x'_i) > 1.$$

Note that

$$\phi'_{i,t_{jl}}(t, x) \geq \left(\left(4 - \frac{2^{-n}}{b'_{t_{jl}}(t, x_i)}\right) \vee 0\right) \wedge 1 = 1$$

since $b'_{t_{jl}}(t, x_i) \geq 2^{-n}$. So we see that

$$\sup_h [k'_{n,t_{jl}(x_h)}(t, x_h) k'_{n,t_{jl}(x_h)}(t, x'_h) \phi'_{h,t_{jl}}(t, x)] > 1.$$

Suppose that the supremum is attained for a certain index $\bar{h}$. Then

$$|x - x_{\bar{h}}| \leq 4b'_{t_{jl}}(t, x_{\bar{h}})$$

and $t > t_{jl}(x_{\bar{h}})$ (since $k'_{n,t_{jl}(x_{\bar{h}})}(t, x_{\bar{h}}) k'_{n,t_{jl}(x_{\bar{h}})}(t, x'_{\bar{h}}) > 1$). Consider the first value $\bar{t}$ at which

$$\sqrt{a(\bar{t}, x_{\bar{h}})} = \sup_{t_{jl}(x_{\bar{h}}) \leq r \leq t} \sqrt{a(r, x_{\bar{h}})}$$

then we see as before that

$$\sqrt{a(\bar{t}, x_{\bar{h}})} + 2^{-n} \leq b'_{t_{jl}}(t, x_{\bar{h}}) \leq (p+1)\left(\sqrt{a(\bar{t}, x_{\bar{h}})} + 2^{-n}\right).$$

We first treat the case in which $t_{jl}(x) < \bar{t}$ ($\leq t < t_{jl+1}(x)$). Note that

$$\sqrt{a(\bar{t}, x)} + 2^{-n} = \alpha 2^{-n}$$

with α between 1 and 5. Thus one has

$$\begin{aligned}\left|\sqrt{a(\bar t, x_{\bar h})}+2^{-n}-\alpha 2^{-n}\right| &\le L|x-x_{\bar h}| \le 4Lb'_{t_{jl}}(t,x_{\bar h})\\ &\le 4L(p+1)\Big(\sqrt{a(\bar t, x_{\bar h})}+2^{-n}\Big) \le \frac{1}{10}\Big(\sqrt{a(\bar t, x_{\bar h})}+2^{-n}\Big).\end{aligned}$$

Then $(10/11)\alpha 2^{-n} \le \sqrt{a(\bar t, x_{\bar h})}+2^{-n} \le (10/9)\alpha 2^{-n}$ and hence

$$|x-x_{\bar h}| \le 4(p+1)\frac{10}{9}\alpha 2^{-n}.$$

We turn to the other case, i.e., if $t_{jl}(x) \ge \bar t$. Since $t_{jl}(x_{\bar h}) \le \bar t$ and $t_{jl}(x) \ge \bar t$ there exists ξ between x and $x_{\bar h}$ such that $t_{jl}(\xi)=\bar t$. That is

$$\sqrt{a(\bar t,\xi)} = 2^{-n+2}$$

and then

$$\begin{aligned}\left|\sqrt{a(\bar t, x_{\bar h})}+2^{-n}-\sqrt{a(\bar t,\xi)}-2^{-n}\right| &\le L|\xi-x_{\bar h}| \le 4Lb'_{t_{jl}}(t,x_{\bar h})\\ &\le 4L(p+1)\Big(\sqrt{a(\bar t, x_{\bar h})}+2^{-n}\Big)\\ &\le \frac{1}{10}\Big(\sqrt{a(\bar t, x_{\bar h})}+2^{-n}\Big).\end{aligned}$$

We conclude as before that

$$\frac{10}{11}\alpha 2^{-n} \le \sqrt{a(\bar t, x_{\bar h})}+2^{-n} \le \frac{10}{9}\alpha 2^{-n}, \quad |x-x_{\bar h}| \le 4(p+1)\frac{10}{9}\alpha 2^{-n}$$

where $\alpha=5$. This gives $|x-x_{\bar h}| \le (200/9)\cdot(p+1)2^{-n}$ and hence the assertion. □

Lemma 7.3. *Let* $(t,x)\in[-T,T]\times J'_\delta$ *with*

$$a(t,x) \le 2^{-2n+3}:$$

there exists j, l *such that*

$$\partial_t \tilde k'_{n,t_{jl}}(t,x) \ge \frac{N}{C_6}\frac{|a_t(t,x)|}{a(t,x)+2^{-2n}}\tilde k'_{n,t_{jl}}(t,x) - C_7\tilde k'_{n,t_{jl}}(t,x).$$

Proof. We choose j and l so that $x\in A'_j(2^{-2n})$ and $t_{jl}(x,2^{-2n}) < t < t_{jl+1}(x,2^{-2n})$. By Lemma 7.2 (using again $\bar h$ for a maximal index) we have that

$$|\sqrt{a(t,x_{\bar h})}-\sqrt{a(t,x)}| \le L|x_{\bar h}-x| \le \frac{5}{72}\cdot 2^{-n}$$

so that $a(t, x_{\bar{h}}) < 2^{-2n+4}$. We have the same inequality for $a\big(t, x'_{\bar{h}}\big)$ and hence

$$t \in I'_n(x_{\bar{h}}) \cap I'_n\big(x'_{\bar{h}}\big).$$

Therefore we have

$$\begin{aligned}
&\partial_t[k'_{n,t_{jl}(x_{\bar{h}})}(t, x_{\bar{h}})k'_{n,t_{jl}(x_{\bar{h}})}\big(t, x'_{\bar{h}}\big)]\phi'_{\bar{h},t_{jl}}(t, x)\\
&\geq N\left[\frac{|a_t(t, x_{\bar{h}})|}{2^{-2n}} + \frac{\big|a_t\big(t, x'_{\bar{h}}\big)\big|}{2^{-2n}}\right]\tilde{k}'_{m,t_{jl}}(t, x).
\end{aligned}$$

Note that again by Taylor's formula

$$\begin{aligned}
a_t(t, x) &= a_t(t, x_{\bar{h}}) + a_{tx}(t, x_{\bar{h}})(x - x_{\bar{h}}) + R_2(x - x_{\bar{h}}),\\
a_t\big(t, x'_{\bar{h}}\big) &= a_t(t, x_{\bar{h}}) + a_{tx}(t, x_{\bar{h}})2^{-n} + R_2(2^{-n}).
\end{aligned}$$

From this we get

$$\begin{aligned}
|a_t(t, x)| &\leq |a_t(t, x_{\bar{h}})| + \frac{200}{9}\cdot(p+1)\Big(|a_t(t, x_{\bar{h}})| + \big|a_t\big(t, x'_{\bar{h}}\big)\big|\Big) + C_5 2^{-2n}\\
&\leq \left(\frac{200}{9}\cdot(p+1)+1\right)|a_t(t, x_{\bar{h}})| + \frac{200}{9}\cdot(p+1)\big|a_t\big(t, x'_{\bar{h}}\big)\big| + C_5 2^{-2n}
\end{aligned}$$

so that

$$\begin{aligned}
\frac{|a_t(t, x)|}{a(t, x) + 2^{-2n}} &\leq \left(\frac{200}{9}\cdot(p+1)+1\right)\left(\frac{|a_t(t, x_{\bar{h}})|}{a(t, x) + 2^{-2n}} + \frac{\big|a_t\big(t, x'_{\bar{h}}\big)\big|}{a(t, x) + 2^{-2n}}\right) + C_5\\
&\leq C_6\left(\frac{|a_t(t, x_{\bar{h}})|}{2^{-2n}} + \frac{\big|a_t\big(t, x'_{\bar{h}}\big)\big|}{2^{-2n}}\right) + C_5
\end{aligned}$$

where $C_6 = ((200/9)\cdot(p+1)+1)$. Thus we conclude

$$\partial_t\tilde{k}'_{n,t_{jl}}(t, x) \geq \frac{N}{C_6}\frac{|a_t(t, x)|}{a(t, x) + 2^{-2n}}\tilde{k}'_{n,t_{jl}}(t, x) - \frac{C_5}{C_6}N\tilde{k}'_{n,t_l}(t, x)$$

and so Lemma 7.3 is proved. □

8. Proof of Proposition 6.1

Let $n \in \mathbb{N}$ be such that $n \geq m_0 + 1$. We set

$$\tilde{k}_m = \prod_{j,l}\tilde{k}_{m,t_{jl}}, \quad m = m_0, m_0+1, \ldots, n-1$$

and

$$\tilde{k}'_n = \prod_{j,l}\tilde{k}'_{n,t_{jl}}$$

where the product is taken over $j = 1, \dots, q_1$, $l = 0, 1, \dots, p_j$. For $0 \le m \le m_0 - 1$ we choose $\tilde{k}_m = 1$ and for $0 \le n \le m_0$ we also choose $\tilde{k}'_n = 1$. We finally define

$$k_n(t, x) = \tilde{k}_1 \cdot \tilde{k}_2 \cdot \dots \cdot \tilde{k}_{n-1} \cdot \tilde{k}'_n.$$

Then properties 1)–4) follow from Lemmas 6.1, 6.3, 7.1, 7.3. We now check 5). Since

$$\begin{aligned} k_{n-1} &= \tilde{k}_1 \tilde{k}_2 \cdots \tilde{k}_{n-2} \tilde{k}'_{n-1}, \\ k_n &= \tilde{k}_1 \tilde{k}_2 \cdots \tilde{k}_{n-1} \tilde{k}'_n \end{aligned}$$

hence

$$\frac{k_{n-1}}{k_n} = \frac{\tilde{k}'_{n-1}}{\tilde{k}_{n-1} \tilde{k}'_n}.$$

Here note that $\tilde{k}_{n-1} \ge 1$ since $\tilde{k}_{n-1} = \prod_{j,l} \tilde{k}_{m,t_{jl}}$ and $\tilde{k}_{m,t_{jl}}(t, x) \ge 1$ for any possible value of j and l. Similarly we have $\tilde{k}'_n \ge 1$. On the other hand we have that

$$\tilde{k}'_{n-1} = \prod_{j,l} \tilde{k}'_{m,t_{jl}} \le \exp[2N(2p + 2)2^4(p + 2)(q + 1)]:$$

in fact there are at most $(p + 2)(q + 1)$ functions in the product. This indeed proves

$$\frac{k_{n-1}}{k_n} \le C. \qquad \square$$

References

[1] F. Colombini, H. Ishida and N. Orrú: *On the Cauchy problem for finitely degenerate hyperbolic equations of second order*, Ark. Mat. **38** (2000), 223–230.

[2] F. Colombini, E. Jannelli and S. Spagnolo: *Well-posedness in the Gevrey classes of the Cauchy problem for a nonstrictly hyperbolic equation with coefficients depending on time*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **10** (1983), 291–312.

[3] F. Colombini and T. Nishitani: *On finitely degenerate hyperbolic operators of second order*, Osaka J. Math. **41** (2004), 933–947.

[4] F. Colombini and S. Spagnolo: *An example of a weakly hyperbolic Cauchy problem not well posed in C^∞*, Acta Math. **148** (1982), 243–253.

[5] P. D'Ancona: *Well posedness in C^∞ for a weakly hyperbolic second order equation*, Rend. Sem. Mat. Univ. Padova **91** (1994), 65–83.

[6] L. Hörmander: The Analysis of Linear Partial Differential Operators, III, Springer, Berlin, 1985.

[7] V.Ja. Ivrii: *Sufficient conditions for regular and completely regular hyperbolicity*, Trans. Moscow Math. Soc. **1** (1978), 1–65.

[8] E. Jannelli: *Gevrey well-posedness for a class of weakly hyperbolic equations*, J. Math. Kyoto Univ. **24** (1984), 763–778.

[9] K. Kajitani: The Well Posed Cauchy Problem for Hyperbolic Operators, Exposé au Séminaire Vaillant (1989).

[10] T. Kinoshita and S. Spagnolo: *Hyperbolic equations with non-analytic coefficients*, Math. Ann. **336** (2006), 551–569.

[11] T. Nishitani: *The Cauchy problem for weakly hyperbolic equations of second order*, Comm. Partial Differential Equations **5** (1980), 1273–1296.

[12] T. Nishitani: *A necessary and sufficient condition for the hyperbolicity of second order equations with two independent variables*, J. Math. Kyoto Univ. **24** (1984), 91–104.

[13] T. Nishitani: *On the Cauchy problem for $D_t^2 - D_x a(t,x) D_x$ in the Gevrey class of order $s > 2$*, Comm. Partial Differential Equations **31** (2006), 1289–1319.

Ferruccio Colombini
Dipartimento di Matematica
Università di Pisa
Largo B. Pontecorvo 5
56127 Pisa
Italy

Tatsuo Nishitani
Department of Mathematics
Osaka University
Machikaneyama 1-1
Toyonaka, 560-0043, Osaka
Japan

Nicola Orrù
Via D. Cimarosa 56
09128 Cagliari
Italy

Ludovico Pernazza
Dipartimento di Matematica
Università di Pavia
Via A. Ferrata 1, 27100 Pavia
Italy