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FINITE GROUPS WHOSE ABELIAN SUBGROUPS HAVE CONSECUTIVE ORDERS

NAOKI CHIGIRA

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1. Introduction

Let G be a finite group and n be a positive integer. A group G is called an OC_n group if every element of G has order less than or equal to n and for each positive integer $m \leq n$ there exists an element of G of order m. B. H. Neumann [8] determined all OC_3 groups and R. Brandl and W. Shi [1] classified all OC_n groups. In recent years a number of papers have dealt with the question of characterizing groups G by the set of all orders of elements in G. See [1], [2] or [10].

Now we will consider the order of abelian subgroups of G instead of the order of elements of G. A group G is called an OA_n group if the order of any abelian subgroup of G is less than or equal to n and for any positive integer $m \leq n$ there exists an abelian subgroup of G of order m. For example, any abelian subgroup of the alternating group A_5 on 5 letters is isomorphic to one of the groups $\{1, Z_2, Z_3, Z_2 \times Z_2, Z_5\}$ where Z_m is a cyclic group of order m. Thus the alternating group A_5 is an OA_5 group. In this paper we will classify all OA_n groups applying the results of [6] and [14] which are proved by using the classification of finite simple groups.

Theorem. *Let G be an OA_n group. Then $n \leq 6$ and G is isomorphic to one of the symmetric groups $1, S_2, S_3, S_4, S_5$ or the alternating groups A_4, A_5.*

There are only seven isomorphism classes of OA_n groups although there are infinitely many isomorphism classes of OC_n groups.

2. Preliminaries

The prime graph $\Gamma(G)$ of G is a graph whose vertex set is the set of primes dividing $|G|$ and distinct two primes p and q are joined by an edge if there exists an element of G of order pq. Let $\nu(\Gamma(G))$ be the number of connected components of $\Gamma(G)$ and in the case where $|G|$ is even, let π_1 be the connected component containing 2. For any integer m, put $\pi(m)$ the set of all primes dividing m.

A finite group G is called a 2-Frobenius group if it has a chain $1 \subset H \subset K \subset G$

of normal subgroups, where K is a Frobenius group with Frobenius kernel H and G/H is a Frobenius group with Frobenius kernel K/H. A 2-Frobenius group is always solvable.

Theorem (Gruenberg-Kegel [7], [14]). *If $\nu(\Gamma(G)) \geq 2$, then one of the following holds.*
(1) *G is a Frobenius group or a 2-Frobenius group.*
(2) *G has normal subgroups N and G_0 with $N \subset G_0$ such that N is a nilpotent π_1-group, G_0/N is a simple group and G/G_0 is a solvable π_1-group.*
Especially if G is solvable, then $\nu(\Gamma(G)) \leq 2$.

The following theorem is well known.

Theorem (Bertrand's postulate [5, p.82]). *For any real number $t \geq 1$, there exists a prime p such that $t < p \leq 2t$.*

Let G be an OA_n group. Note that if $n \geq 2$, then $|G|$ is even and thus π_1 is not empty. The following lemma is fundamental.

Lemma 1. *Let G be an OA_n group and p be a prime.*
(1) *p divides $|G|$ if and only if $p \leq n$.*
(2) *p^2 divides $|G|$ if and only if $p^2 \leq n$.*
(3) *If $\sqrt{n} < p \leq n$, then a Sylow p-subgroup of G is cyclic of order p.*
(4) *Suppose that p is an odd prime. Then $p \leq n/2$ if and only if $p \in \pi_1$.*
(5) *If $n/2 < p \leq n$, then $\{p\}$ forms a connected component of $\Gamma(G)$ and a Sylow p-subgroup is cyclic of order p.*
(6) *Suppose that p is the largest prime dividing $|G|$. Then $n/2 < p \leq n$.*
(7) *$\nu(\Gamma(G)) \geq 2$ if $n \geq 3$.*

Proof. (1) If $|G|$ is divisible by p, then there exists a cyclic subgroup of order p. Then we have $p \leq n$. Conversely, if $p \leq n$ then there exists an abelian subgroup of order p in G by the definition of OA_n groups. This yields that p divides $|G|$.
(2) Since a group of order p^2 is abelian, we have the result by using similar arguments in the proof of (1).
(3) If $\sqrt{n} < p \leq n$, G does not have an abelian subgroup of order p^2 since $n < p^2$. This yields that a Sylow p-subgroup of G is cyclic of order p.
(4) If $p \leq n/2$, there exists an abelian subgroup of order $2p$ by the definition of OA_n groups. Hence $p \in \pi_1$. Conversely, if $p \in \pi_1$, there exists a prime $q \in \pi_1$ such that G has an element of order pq, that is, G has an abelian subgroup of order pq. Since G is an OA_n group, $pq \leq n$. Since $2p \leq pq$, we have $p \leq n/2$.
(5) If there exists a prime q such that G has an abelian subgroup of order pq, then

$pq \le n$ because G is an OA_n group. We have $2p \le pq \le n$, a contradiction. Hence $\{p\}$ is a connected component of $\Gamma(G)$ and a Sylow p-subgroup is cyclic of order p.
(6) By Bertrand's postulate, there exists a prime r such that $n/2 < r \le n$. We see that r divides $|G|$ by (1). Since p is the largest prime dividing $|G|$, we have $r \le p$. This yields that $n/2 < p \le n$.
(7) Because there is a prime r such that $n/2 < r \le n$ by Bertrand's postulate, $\nu(\Gamma(G)) \ge 2$ if $n \ge 3$. □

Proposition 1. *If $n \ge 47$, then $\sharp\{p : prime | n/2 < p \le n\} \ge 6$.*

Proof. See [1, p.395] □

Theorem (Williams [14], Iiyori-Yamaki [6]). *For any finite group G, $\nu(\Gamma(G)) \le 6$.*

As a corollary, we have the following:

Corollary 1. *If G is an OA_n group, then $n \le 46$.*

Proof. Suppose that $n \ge 47$. Then Lemma 1 (5) and Proposition 1 imply that $\nu(\Gamma(G)) \ge 7$. This contradicts the theorem of Williams and Iiyori-Yamaki. □

3. The Proof of the Main Theorem

Proposition 2. *Let G be a solvable OA_n group. Then $G \simeq 1$, Z_2, S_3, A_4 or S_4.*

Proof. By Gruenberg-Kegel's theorem, if G is solvable then $\nu(\Gamma(G)) \le 2$. If $n \ne 1, 2, 3, 4, 6, 10$, then there exist primes p and q such that $n/2 < p < q \le n$ (See [1, p.396, TABLE I]). Then $\nu(\Gamma(G)) \ge 3$ by Lemma 1 (4). This is a contradiction. If $n = 10$, there exists a Hall $\{3, 5, 7\}$-subgroup H of G because G is solvable. Then $\nu(\Gamma(H)) = 3$. This is a contradiction. If $n = 6$, then $|G| = 2^a \cdot 3 \cdot 5$ for some integer a. A Hall $\{3, 5\}$-subgroup H is cyclic of order 15, a contradiction. Hence $n \le 4$. If $\nu(\Gamma(G)) = 1$, then $G \simeq Z_2$. If $\nu(\Gamma(G)) = 2$, again by Gruenberg-Kegel's theorem, G is a Frobenius group or a 2-Frobenius group. If G is Frobenius, then its Frobenius kernel N must be isomorphic to $Z_2 \times Z_2$ or Z_3. Then we have $G \simeq A_4$ or S_3. If G is 2-Frobenius, there exist normal subgroups K and H such that K is a Frobenius group with Frobenius kernel H and G/H is a Frobenius group with Frobenius kernel K/H. Then $H \simeq Z_2 \times Z_2$ or Z_3. Since K/H is a Frobenius kernel of G/H and it is also isomorphic to a Frobenius complement of K, K/H must be a cyclic subgroup of odd order. This yields that $H \simeq Z_2 \times Z_2$ and $K/H \simeq Z_3$. This implies that $G \simeq S_4$. □

Lemma 2. *Let G be a nonsolvable OA_n group. Then G is not a Frobenius group.*

Proof. By Lemma 1 (7), we see $\nu(\Gamma(G)) \geq 2$. Suppose that $G = NH$ is a nonsolvable Frobenius group with Frobenius kernel N and Frobenius complement H. Then H has a subgroup $H_0 \simeq SL(2,5) \times M$ with $(H : H_0) \leq 2$, where M is a group in which every Sylow subgroup is cyclic and $|M|$ is not divisible by 2, 3 and 5 (See [9, p.204]). Let p be the largest prime dividing $|G|$. Since $p \notin \pi_1$ by Lemma 1, p does not divide $|H|$. Therefore p divides $|N|$. If $|N|$ is divisible by a prime $q \neq p$, N has an abelian subgroup of order $pq \geq 2p > n$ because N is nilpotent. This is a contradiction. Hence N is a p-group and $N \simeq Z_p$ by Lemma 1. Since $|N|-1 \geq |H|$, we have $p \geq 121$. This contradicts Corollary 1 and completes the proof. □

Lemma 3. *Let G be a nonsolvable OA_n group. Then $F(G) = 1$, where $F(G)$ is the Fitting subgroup of G.*

Proof. By Lemma 1 (7), we see $\nu(\Gamma(G)) \geq 2$. By Gruenberg-Kegel's theorem, G has normal subgroups N and G_0 with $N \subset G_0$ such that N is a nilpotent π_1-group, G_0/N is a simple group and G/G_0 is a solvable π_1-group since G is not a Frobenius group by Lemma 2. We see that $N = F(G)$. Suppose that $N \neq 1$. Let N_0 be a minimal normal subgroup of G_0. Then N_0 is an elementary abelian p-group for some $p \in \pi_1$. Let q be the largest prime dividing $|G|$. Then we see that $q \geq 5$, $n/2 < q \leq n$ and q divides $|G_0|$ by Gruenberg-Kegel's theorem. By Lemma 1 (5), $\{q\}$ is a connected component of $\Gamma(G)$ and a Sylow q-subgroup is cyclic of order q. Then N_0Q is a Frobenius group for some $Q \in Syl_q(G)$ since $C_{N_0}(x) = 1$ for any $x \in Q - \{1\}$. Hence q divides $|N_0| - 1$. If p is odd, then $|N_0| - 1$ is even. We have $q \leq (|N_0| - 1)/2 \leq (n-1)/2 < n/2$, a contradiction. Hence we have $p = 2$. Then $|N_0| = 2$, 4, 8, 16 or 32 by Corollary 1. If $|N_0| = 32$, then $q = 31$. In this case, G has an abelian subgroup H of order 29 since $32 \leq n$. Since H can not act on N fixed point freely, N_0H has an element of order $58 > 46$, a contradiction. If $|N_0| = 16$, then $q = 5$ because $q \geq 5$. In this case, G has an abelian subgroup H of order 13 since $16 \leq n$. This contradicts the choice of q. If $|N_0| = 2$ or 4, then $q \leq |N_0|-1 \leq 3$, a contradiction. If $|N_0| = 8$ then $q = 7$. Since $q = 7$ is the largest prime dividing G and G has an abelian subgroup N_0 of order 8, we have $8 \leq n < 11$. Furthermore we have $5 \in \pi_1$, since a Sylow 5-subgroup of G does not act on N_0 fixed point freely. This implies that $n = 10$. In this case, $C_{G_0}(N_0)$ is a 2 group. In fact, if $C_{G_0}(N_0)$ has an element x of odd prime order, then $N_0\langle x\rangle$ is an abelian subgroup whose order is more than 24. This is a contradiction. Since G_0 has a nonsolvable simple factor and $G_0/C_{G_0}(N_0)$ is isomorphic to a subgroup of $GL(3,2)$, $G_0/C_{G_0}(N_0) \simeq GL(3,2)$ and $N \simeq C_{G_0}(N_0)$. We see that 5 does not divide $|G|$ since orders of $Aut(GL(3,2))$ and $C_{G_0}(N_0)$ are not divisible by 5. This is a contradiction. This completes the proof. □

The above lemma implies that if G is a nonsolvable OA_n group, then there exists a simple group G_0 such that $G_0 \subseteq G \subseteq Aut(G_0)$. We will use this notation in the following propositions.

Proposition 3. *Let G be a nonsolvable OA_n group.*

(1) *If G_0 is an alternating group A_m on m letters, then $m = 5$. Conversely, A_5 is an OA_5 group and S_5 is an OA_6 group.*

(2) *G_0 is not a sporadic simple group.*

Proof. (1) If $G_0 \simeq A_m$, $\nu(\Gamma(G_0)) \leq 3$ by [14]. Hence $2 \leq \nu(\Gamma(G)) \leq 3$. This yields that $5 \leq n \leq 16$, $n \neq 13$ by counting the number of primes p with $n/2 < p \leq n$. (See Lemma 1 (5) and [1, p.396, TABLE I].) If $G_0 \simeq A_5$, then $n < 7$ since 7 does not divide $|Aut(G_0)|$. Clearly A_5 is an OA_5 group and S_5 is an OA_6 group. If $G_0 \simeq A_6$, then $n < 7$. On the other hand, A_6 has an abelian subgroup of order 9. This is a contradiction. If $G_0 \simeq A_7$ or A_8, then $n < 11$. But $G_0 \supseteq A_7 \supset \langle (1,2)(3,4),(1,3)(2,4) \rangle \times \langle (5,6,7) \rangle$ which is abelian of order 12. If $G_0 \supseteq A_9$, then $A_9 \supset \langle (1,2)(3,4),(1,3)(2,4) \rangle \times \langle (5,6,7,8,9) \rangle$ which is abelian of order 20. This is a contradiction since $n \leq 16$.

(2) See [4]. □

Proposition 4. *Let G be a nonsolvable OA_n group and G_0 a simple group of Lie type over the field of q elements. Then $G_0 \simeq A_1(4)$.*

Proof. Suppose that $\nu(\Gamma(G_0)) \geq 4$. By the classification of the prime graph components of finite simple groups, $G_0 \simeq E_8(q)$, $A_2(4)$, ${}^2B_2(q)$ or ${}^2E_6(2)$ (See [6, p.337, TABLE III] and [14, p.492, TABLE Ie]). The groups $A_2(4)$, ${}^2E_6(2)$ and their automorphism groups are not OA_n groups (See [4]). If $G_0 \simeq E_8(q)$, G_0 has a maximal torus of order $q^8 - q^4 + 1 \geq 2^8 - 2^4 + 1 > 46$, a contradiction (See [3]). Clearly $G_0 \not\simeq {}^2B_2(8)$ and $G_0 \not\simeq {}^2B_2(32)$ (See [4]). If $G_0 \simeq {}^2B_2(q)$ where $q = 2^{2m+1}$ and $m \geq 3$, then G_0 has a maximal torus of order $q + \sqrt{2q} + 1 \geq 2^7 + 2^4 + 1 > 46$, a contradiction (See [12]). Suppose that $\nu(\Gamma(G_0)) = 3$. This implies that $\nu(\Gamma(G)) \leq 3$ and therefore $5 \leq n \leq 16$, $n \neq 13$ (See [1, p.396, TABLE I]). If the characteristic is more than or equal to 5, then q is a prime because q divides $|G_0|$ and $n \leq 16$. Since q^2 does not divide $|G_0|$, $G_0 \simeq A_1(q)$. Clearly $G_0 \not\simeq A_1(7)$, $A_1(11)$, and $A_1(13)$ (See [4]). We have $G_0 \simeq A_1(5) \simeq A_5$. Suppose now that the characteristic is 3. If $n \leq 8$, in a similar way, we have $G_0 \simeq A_1(3)$, which is not simple. If $n \geq 9$, then G_0 is isomorphic to one of groups in [14, p.492, TABLE Id], that is, $G_0 \simeq A_1(q)$ $(q \equiv 1\ (4))$, $A_1(q)$ $(q \equiv -1\ (4))$, $E_7(3)$, $G_2(q)$ $(q \equiv 0\ (3))$, ${}^2G_2(q)$ $(q = 3^{2m+1},\ m \geq 1)$, or ${}^2D_p(3)$ $(p = 2^n+1,\ n \geq 2)$. Clearly $G_0 \not\simeq E_7(3)$ (See [14]). If

$G_0 \simeq A_1(q)$, then we have $q = 3$ or 9 since a Sylow q-subgroup of G_0 is abelian and $n \leq 16$. Since G_0 is simple, $G \not\simeq A_1(3)$ and we see $G_0 \not\simeq A_1(9) \simeq A_6$ by Proposition 3. Clearly $G_0 \not\simeq G_2(3)$ (See [4]). If $G_0 \simeq G_2(q)$ $(q \equiv 0(3))$ and $q \geq 3^2$ then G_0 has a maximal torus of order $q^2+q+1 \geq 3^4+3^2+1 > 16$, a contradiction (See [3]). If $G_0 \simeq {}^2G_2(q)$ $(q = 3^{2m+1},\ m \geq 1)$, G_0 has a maximal torus of order $q+\sqrt{3q}+1 > 16$, a contradiction (See [13]). If $G_0 \simeq {}^2D_p(3)$ $(p = 2^n+1$ is a prime, $n \geq 2)$, then G_0 has a maximal torus of order $(3^p+1)/4 > 16$, a contradiction (See [12] or [14]). Suppose now that the characteristic is 2. Then $G_0 \simeq A_1(q)$, $A_2(2)$, ${}^2A_5(2)$, $E_7(2)$, ${}^2F_4(q)$ or $F_4(q)$ by [6, p.336, TABLE II]. Clearly $G_0 \not\simeq A_2(2)$, ${}^2A_5(2)$, $E_7(2)$, $A_1(8)$ and $A_1(16)$ (See [4]). If $G_0 \simeq A_1(q)$, we have $q \leq 16$ since a Sylow 2-subgroup of G_0 is abelian. We have $G_0 \simeq A_1(4) \simeq A_5$ (See [4]). If $G_0 \simeq {}^2F_4(q)$ $(q = 2^{2m+1},\ m \geq 1)$, then G_0 has a maximal torus of order $q^2+\sqrt{2q^3}+q+\sqrt{2q}+1 > 16$, a contradiction (See [11]). Clearly $G_0 \not\simeq F_4(2)$ (See [4]). If $G_0 \simeq F_4(q)$, then G_0 has a maximal torus of order $q^4+1 > 16$, a contradiction (See [3]). This completes the case where $\nu(\Gamma(G_0)) = 3$. Suppose that $\nu(\Gamma(G_0)) = 2$. Then $n = 6$ or 10 (See [1, p.396, TABLE I]). If the characteristic is more than or equal to 5, we have $G_0 \simeq A_1(q)$, a contradiction since $\nu(\Gamma(A_1(q))) = 3$. We have that the characteristic is 2 or 3. Suppose now that the characteristic is 3. By an argument similar to that in the case where $\nu(\Gamma(G_0)) = 3$, we see $n = 10$. Notice that G_0 has prime graph components $\pi_1 = \{2,3,5\}$ and $\{7\}$. And G_0 is isomorphic to one of groups in [14, p.490, TABLE Ib, p.491, TABLE Ic] whose characteristic is 3. We see that there exist no groups satisfying our condition in this case. Suppose now that the characteristic is 2. Then $n = 6$ and the connected components are $\pi_1 = \{2,3\}$ and $\{5\}$ or $n = 10$ and the connected components are $\pi_1 = \{2,3,5\}$ and $\{7\}$. And G_0 is isomorphic to one of groups in [6, p.336, TABLE Ia, Ib]. We see that only ${}^2A_3(2)$ has the connected components $\pi_1 = \{2,3\}$ and $\{5\}$. However we see $G_0 \not\simeq {}^2A_3(2)$ by [4]. Also we see that only $A_3(2)$, $C_3(2)$ and $D_4(2)$ have the connected components $\pi_1 = \{2,3,5\}$ and $\{7\}$. However we see that $G_0 \not\simeq A_3(2)$, $C_3(2)$ and $D_4(2)$ by [4]. This yields that there exist no groups satisfying our conditions in this case. This completes the proof. □

Proof of Theorem. Straightforward from Propositions 2, 3 and 4. □

References

[1] R. Brandl and W. Shi: *Finite groups whose element orders are consecutive integers*, J. Algebra, **143** (1991), 388–400.

[2] R. Brandl and W. Shi: *The characterization of* $PSL(2,q)$ *by its element orders*, J. Algebra, **163** (1994), 109–114.

[3] R. Carter: *Conjugacy classes in the Weyl group*, Compositio Math. **25** (1972), 1–59.

[4] J. Conway, R. Curtis, S. Norton, R. Parker and R. Wilson: Atlas of Finite Groups, Clarendon Press, Oxford, 1985.
[5] L.K. Hua: Introduction to Number Theory, Springer-Verlag, Berlin Heidelberg, 1982.
[6] N. Iiyori and H. Yamaki: *Prime graph components of the simple groups of Lie type over the field of even characteristic*, J. Algebra, **155** (1993), 335–343; Corrigendum, ibid. **181** (1996), 659.
[7] N. Iiyori and H. Yamaki: *A problem of Frobenius*, Canad. Math. Soc. Conf. Proc. **14** (1993), 237–244.
[8] B.H. Neumann: *Groups whose elements have bounded orders*, J. London Math. Soc. **12** (1937), 195–198.
[9] D.S. Passman: Permutation Groups, Benjamin, New York, 1968.
[10] C.E. Praeger and W. Shi: *A characterization of some alternating and symmetric groups*, Comm. Algebra, **22** (1994), 1507–1530.
[11] K. Shinoda: *The conjugacy classes of the finite Ree groups of type* (F_4), J. Fac. Sci. Univ. Tokyo, **22** (1975), 133-159.
[12] T.A. Springer and R. Steinberg: *Conjugacy classes*, Seminar on Algebraic Groups and Related Finite Groups, Lecture Notes in Mathematics 131, Springer-Verlag, Berlin-Heidelberg-New York, 1970.
[13] H.N. Ward: *On Ree's series of simple groups*, Trans. Amer. Math. Soc. **121** (1966), 62–89.
[14] J.S. Williams: *Prime graph components of finite groups*, J. Algebra, **69** (1981), 487–513.

Department of Mathematical Sciences
Muroran Institute of Technology
Hokkaido 050, Japan
e-mail: chigira@muroran-it.ac.jp