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Non-finitely generated monoids corresponding to finitely generated homogeneous subalgebras

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The goal of this paper is to study the possible monoids appearing as the associated monoids of the initial algebra of a finitely generated homogeneous $\Bbbk$-subalgebra of a polynomial ring $\Bbbk[x_1, \dots, x_n]$. Clearly, any affine monoid can be realized since the initial algebra of the affine monoid $\Bbbk$-algebra is itself. On the other hand, the initial algebra of a finitely generated homogeneous $\Bbbk$-algebra is not necessarily finitely generated. In this paper, we provide a new family of non-finitely generated monoids which can be realized as the initial algebras of finitely generated homogeneous $\Bbbk$-algebras. Moreover, we also provide an example of a non-finitely generated monoid which cannot be realized as the initial algebra of any finitely generated homogeneous $\Bbbk$-algebra.

1. Introduction

Let $\Bbbk$ be a field, $S = \Bbbk[x_1, \dots, x_n]$ the polynomial ring in n variables over $\Bbbk$, and $\preceq$ a monomial order on $(\mathbb{Z}_{\geq 0})^n$. We use an abbreviation of monomials $x_1^{u_1} \cdots x_n^{u_n}$ with $x^{\mathbf{u}}$ for $\mathbf{u} = (u_1, u_2, \dots, u_n) \in (\mathbb{Z}_{\geq 0})^n$. Given a non-zero polynomial $f = \sum c_{\mathbf{u}} x^{\mathbf{u}} \in S$ with $c_{\mathbf{u}} \in \Bbbk$, we define $\operatorname{supp} f := \{\mathbf{u} \in (\mathbb{Z}_{\geq 0})^n \mid c_{\mathbf{u}} \neq 0\}$, $\deg_{\preceq} f := \max_{\preceq}(\operatorname{supp} f)$, and $\operatorname{in}_{\preceq} f = c_{\deg_{\preceq} f} x^{\deg_{\preceq} f}$. Let R be a finitely generated $\Bbbk$-subalgebra of S. We define $\deg_{\preceq} R := \{\deg_{\preceq} f \mid f \in R \backslash \{0\}\}$ and $\operatorname{in}_{\preceq} R$ the $\Bbbk$-vector space spanned by $\{\operatorname{in}_{\preceq} f \mid f \in R\}$, called *initial algebra* of R. A subset $\mathcal{F}$ of R is said to be *SAGBI basis* of R if the $\Bbbk$-algebra generated by $\{\operatorname{in}_{\preceq} f \mid f \in \mathcal{F}\}$ is equal to $\operatorname{in}_{\preceq} R$. The word "SAGBI" is introduced by Robbiano and Sweedler [5] and stands for "Subalgebra Analog to Gröbner Bases for Ideal". Remark that $\operatorname{in}_{\preceq} R$ is not necessarily a finitely generated $\Bbbk$-algebra even if R is finitely generated, so R may have no finite SAGBI basis with respect to any monomial orders. The sufficient condition for R to have infinite SAGBI basis is not found yet as far

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as the authors know. In [4], it is claimed that it will be helpful for this problem to study the mechanism of non-finite generation of the initial algebras. The goal of this paper is to study non-finitely generated monoids appearing as non-finitely generated homogeneous monoid $\Bbbk$-algebras.

Let M be a monoid, a set with an operation $M \times M \to M$ that is associative and has the identity. For a monoid $M \subset (\mathbb{Z}_{\geq 0})^n$ and a field $\Bbbk$, $\Bbbk[M]$ is a $\Bbbk$-vector space with the base $\{x^{\mathbf{u}} \mid \mathbf{u} \in M\}$. Since both $\Bbbk[M]$ and $\text{in}_{\preceq} R$ are $\Bbbk$-subalgebras of S generated by monomials, for any initial algebra $\text{in}_{\preceq} R$, there exists a monoid M such that

$$\text{in}_{\preceq} R = \Bbbk[M].$$

Concretely, such a monoid M is $\deg_{\preceq} R$. Therefore, to check whether $\text{in}_{\preceq} R$ is finitely generated or not is equivalent to check whether $M(= \deg_{\preceq} R)$ is so. In that sense, the following question naturally arises.

Question 1.1. *Can we characterize non-finitely generated monoids arising from some finitely generated $\Bbbk$-subalgebras?*

Towards the solution of Question 1.1, in this paper, we concentrate on our discussion in the case of homogeneous subalgebras of $\Bbbk[x, y]$. In particular, we mainly study subalgebras generated by one homogeneous binomial $x^{\mathbf{v}_1} + x^{\mathbf{v}_2}$ and finitely many monomials $x^{\mathbf{u}_1}, x^{\mathbf{u}_2}, \dots, x^{\mathbf{u}_t}$.

There are two main results in this paper. The first main result is to provide a class of non-finitely generated monoids that correspond to some finitely generated subalgebras.

Theorem 3.4. *Let $\mathbf{v}_1, \mathbf{v}_2 \in (\mathbb{Z}_{\geq 0})^2$ be linearly independent over $\mathbb{Q}$. Let $\preceq$ be a monomial order with $\mathbf{v}_1 \succeq \mathbf{v}_2$ and let $C \subset (\mathbb{R}_{\geq 0})^2$ be the cone generated by $\mathbf{v}_1, \mathbf{v}_2$. We take $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_s$ from $(\mathbb{Z}_{\geq 0})^2 \cap C^\circ$, where C° denotes the interior of C. Let N be the monoid generated by $\mathbf{v}_2$, and let L be an N-module generated by $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_s$. We define M by setting the monoid generated by $\{\mathbf{v}_1\} \cup L$. If R is a $\Bbbk$-algebra generated by*

$$G := \{x^{\mathbf{v}_1} + x^{\mathbf{v}_2}\} \cup \{x^{\mathbf{u}} \mid \mathbf{u} \in L\},$$

then R is finitely generated. Moreover, for a monomial order $\preceq$ with $\mathbf{v}_1 \succeq \mathbf{v}_2$, we have $\text{in}_{\preceq} R = \Bbbk[M]$. In particular, G is an infinite SAGBI basis of R.

The following second main result is to show that submonoids of $(\mathbb{Z}_{\geq 0})^2$ do not necessarily correspond to some initial algebras of finitely generated homogeneous subalgebras.

Theorem 4.1. *Let M be a submonoid of $(\mathbb{Z}_{\geq 0})^2$ generated by infinitely many irreducible elements $\{(1, n^2) \mid n \in \mathbb{Z}_{\geq 0}\}$. Then, for any subalgebra R generated by finitely many homogeneous polynomials in $\Bbbk[x, y]$ and any monomial order $\preceq$ of $(\mathbb{Z}_{\geq 0})^2$, $\text{in}_{\preceq} R$ is never equal to $\Bbbk[M]$.*

This paper is organized as follows. In Section 2, we prepare the fundamental materials on SAGBI basis, monoids and cones. In Section 3, we enumerate examples of monoids and subalgebras, and show the first main result as a generalization of them. In Section 4, we give a proof of the other main result, Theorem 4.1. In Section 5, we display examples that do not suit the class in Section 3.

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2. SAGBI basis criterion, monoids and cones

In this section, we introduce the fundamental materials on SAGBI basis, monoids, and cones.

We first provide the SAGBI basis criterion. We use the notation used in [6, Chapter 11] on SAGBI basis. Algorithm 2.1 is a modification of [6, Algorithm 11.1].

Algorithm 2.1 The subduction algorithm.

```
Input: F = {f_1, f_2, ..., f_s} ⊂ S, f ∈ S
Output: q ∈ k[F], r ∈ S such that f = q + r
  q := 0; r := 0
  p := f
  while p ∉ k do
      find i_1, i_2, ..., i_s ∈ Z_{≥0} and c ∈ k\{0} such that

              in_≼ p = c · in_≼ f_1^{i_1} · in_≼ f_2^{i_2} ··· in_≼ f_s^{i_s}.        (*)
      if representation (*) exists then
          q := q + c · f_1^{i_1} · f_2^{i_2} ··· f_s^{i_s}
          p := p − c · f_1^{i_1} · f_2^{i_2} ··· f_s^{i_s}
      else
          r := r + in_≼ p
          p := p − in_≼ p
  return q, r
```

Let $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_s \in (\mathbb{Z}_{\geq 0})^n$ with $\mathrm{in}_\preceq f_i = x^{\mathbf{u}_i}$, $\mathcal{A} = (\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_s)$ the $n \times s$-matrix whose columns are $\mathbf{u}_i$'s, and let $I_\mathcal{A}$ be the toric ideal of $\mathcal{A}$, i.e. the kernel of the $\Bbbk$-algebra homomorphism

$$\Bbbk[X_1, X_2, \dots, X_s] \to \Bbbk[x_1, x_2, \dots, x_n], \quad X_i \mapsto x^{\mathbf{u}_i}.$$

Thanks to Proposition 2.1, we can determine whether $\mathcal{F}$ is a SAGBI basis.

Proposition 2.1 *([6, Corollary 11.5]). Let $\{p_1, p_2, \dots, p_t\}$ be generators of the toric ideal $I_\mathcal{A}$. Then $\mathcal{F}$ is a SAGBI basis if and only if Algorithm 2.1 subduces $p_i(f_1, f_2, \dots, f_s)$ to an element of $\Bbbk$ for all i.*

We use the notation used in [1, Chapters 1 and 2]. In this paper, let M be a submonoid of $(\mathbb{Z}_{\geq 0})^n$. For $\mathbf{x} \in M$, we call $\mathbf{x}$ *irreducible* on M if there are $\mathbf{y}, \mathbf{z} \in M$ with $\mathbf{x} = \mathbf{y} + \mathbf{z}$, then either $\mathbf{y}$ or $\mathbf{z}$ must be $\mathbf{0}$. A monoid M is non-finitely generated if and only if M has infinitely many irreducible elements on M. A set N with an operation $M \times N \to N$ is called an *M-module* if

$$(\mathbf{u} + \mathbf{v}) + \mathbf{x} = \mathbf{u} + (\mathbf{v} + \mathbf{x}) \quad \text{and} \quad \mathbf{0} + \mathbf{x} = \mathbf{x} \qquad \text{for any } \mathbf{u}, \mathbf{v} \in M \text{ and } \mathbf{x} \in N.$$

It is convenient to observe $\mathbb{R}_{\geq 0}M := \{\sum_{i=1}^s a_i \mathbf{x}_i \mid \mathbf{x}_i \in M, a_i \in \mathbb{R}_{\geq 0}, s \in \mathbb{Z}_{\geq 0}\}$ for determining if M is finitely generated. For $i = 1, 2, \dots, t$, let σ_i be a linear form on $\mathbb{R}^n$ and let H_i, H_i^+ be linear hyperplanes and linear closed halfspaces such that

$$H_i := \{\mathbf{x} \in \mathbb{R}^n \mid \sigma_i(\mathbf{x}) = 0\} \ \text{ and } \ H_i^+ := \{\mathbf{x} \in \mathbb{R}^n \mid \sigma_i(\mathbf{x}) \geq 0\},$$

respectively. Note that we also often use $\mathbf{w} \in \mathbb{R}^n$ to describe each linear form σ, i.e., $\sigma(\mathbf{x}) = \langle \mathbf{x}, \mathbf{w} \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product of $\mathbb{R}^n$. A *polyhedral cone* is defined to be an intersection of finitely many linear closed halfspaces, i.e., C is written as $C = \bigcap_{i=1}^t H_i^+$. Moreover, by using Proposition 2.2, we can determine if a given monoid M is finitely generated.

Proposition 2.2 *([1, Corollary 2.10]). A monoid M is finitely generated if and only if $\mathbb{R}_{\geq 0}M$ is a polyhedral cone.*

A *face* F of C is a non-empty intersection of a linear hyperplane $H = \{\mathbf{x} \in \mathbb{R}^n \mid \sigma(\mathbf{x}) = 0\}$ and C satisfying $C \subset H^+$. Namely,

$$F := H \cap C = \{\mathbf{x} \in C \mid \sigma(\mathbf{x}) = 0\} \neq \emptyset.$$

For a polyhedral cone $C = \bigcap_{i=1}^t H_i^+$, we define

$$C^\circ := \{\mathbf{x} \in C \mid \sigma_i(\mathbf{x}) > 0 \text{ for each } i\}.$$

Note that for all $\mathbf{x} \in C$ and $\mathbf{x}' \in C^\circ$, we have $\mathbf{x} + \mathbf{x}' \in C^\circ$.

3. Examples of monoids and subalgebras

In this section, various non-finitely generated monoids are generalized (Lemma 3.3) and we construct finitely generated subalgebras that correspond to monoids (Theorem 3.4).

We found these examples through computational experiments by using the package `"SubalgebraBases" [2] on Macaulay2` [3]. On our experiments, we focused on subalgebras generated by one homogeneous binomial $x^{\mathbf{v}_1} + x^{\mathbf{v}_2}$ and t monomials $x^{\mathbf{u}_1}, x^{\mathbf{u}_2}, \dots, x^{\mathbf{u}_t}$. First, we discuss the case $t = 1$.

Proposition 3.1. *Let $\mathbf{v}_1, \mathbf{v}_2 \in (\mathbb{Z}_{\geq 0})^2$ be linearly independent. Assume that the binomial $x^{\mathbf{v}_1} + x^{\mathbf{v}_2}$ is homogeneous. For any $\mathbf{u} \in (\mathbb{Z}_{\geq 0})^2$, the $\mathbb{k}$-subalgebra $R = \mathbb{k}[x^{\mathbf{v}_1} + x^{\mathbf{v}_2}, x^{\mathbf{u}}]$ has a finite SAGBI basis.*

Proof. We may assume $\mathbf{v}_1 \succeq \mathbf{v}_2$ without loss of generality.

Let $\mathbf{v}_1, \mathbf{u}$ be linearly independent over $\mathbb{Q}$. We consider a linear relation

$$a_1\mathbf{v}_1 + a_2\mathbf{u} = b_1\mathbf{v}_1 + b_2\mathbf{u},$$

where $a_1, a_2, b_1, b_2 \in \mathbb{Z}_{\geq 0}$. Since $\mathbf{v}_1, \mathbf{u}$ are linearly independent, we have $a_1 = b_1$ and $a_2 = b_2$. Thus, any cancellation of initial terms in R cannot occur and we obtain that $\{x^{\mathbf{v}_1} + x^{\mathbf{v}_2}, x^{\mathbf{u}}\}$ is a SAGBI basis of R.

Let $m\mathbf{v}_1 = \ell\mathbf{u}$ with some coprime positive integers m, ℓ. Then, we can obtain a polynomial in R as follows:

$$\begin{aligned} f &:= \frac{1}{\binom{m}{1}}((x^{\mathbf{v}_1} + x^{\mathbf{v}_2})^m - (x^{\mathbf{u}})^\ell) \\ &= x^{(m-1)\mathbf{v}_1+\mathbf{v}_2} + \frac{\binom{m}{2}}{\binom{m}{1}} x^{(m-2)\mathbf{v}_1+2\mathbf{v}_2} + \cdots + \frac{1}{\binom{m}{1}} x^{m\mathbf{v}_2}. \end{aligned}$$

Now, we prove $\{x^{\mathbf{v}_1} + x^{\mathbf{v}_2}, x^{\mathbf{u}}, f\}$ is a SAGBI basis of R. Similarly to the previous case, we consider the equality

$$a_1\mathbf{v}_1 + a_2\mathbf{u} + a_3((m-1)\mathbf{v}_1 + \mathbf{v}_2) = b_1\mathbf{v}_1 + b_2\mathbf{u} + b_3((m-1)\mathbf{v}_1 + \mathbf{v}_2).$$

Since $\mathbf{v}_1$ and $\mathbf{u}$ are linearly dependent while $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, we obtain $a_3 = b_3$. Therefore, it is sufficient to check cancellations involving only $x^{\mathbf{v}_1} + x^{\mathbf{v}_2}$ and $x^{\mathbf{u}}$. Since we have f, the cancellations involving $x^{\mathbf{v}_1} + x^{\mathbf{v}_2}$ and $x^{\mathbf{u}}$ have been also already discussed. □

We can similarly prove the following.

Proposition 3.2. *Let $\mathbf{v}_1, \mathbf{v}_2 \in (\mathbb{Z}_{\geq 0})^2$ be linearly independent and let $\mathbf{u}_1, \mathbf{u}_2 \in (\mathbb{Z}_{\geq 0})^2$ be linearly independent. Assume that the binomials $x^{\mathbf{v}_1}+x^{\mathbf{v}_2}, x^{\mathbf{u}_1}+x^{\mathbf{u}_2}$ are homogeneous. Then the $\mathbb{k}$-subalgebra $R = \mathbb{k}[x^{\mathbf{v}_1}+x^{\mathbf{v}_2}, x^{\mathbf{u}_1}+x^{\mathbf{u}_2}]$ has a finite SAGBI basis.*

Proof. We may assume $\mathbf{v}_1 \succeq \mathbf{v}_2$ and $\mathbf{u}_1 \succeq \mathbf{u}_2$ without loss of generality. If $\mathbf{v}_1, \mathbf{u}_1$ are linearly independent, then the proof is the same as in Proposition 3.1.

Next, we assume that $m\mathbf{v}_1 = \ell\mathbf{u}_1$ with some positive integers m, ℓ. If $m = \ell = 1$ and $\mathbf{v}_2 = \mathbf{u}_2$, then $\{x^{\mathbf{v}_1} + x^{\mathbf{v}_2}\}$ is a SAGBI basis of R. We set $\mathbf{v}_2 \neq \mathbf{u}_2$, and let

$$\begin{aligned} f &\coloneqq (x^{\mathbf{v}_1} + x^{\mathbf{v}_2})^m - (x^{\mathbf{u}_1} + x^{\mathbf{u}_2})^\ell \\ &= \binom{m}{1} x^{(m-1)\mathbf{v}_1+\mathbf{v}_2} + \cdots + x^{m\mathbf{v}_2} - \left(\binom{\ell}{1} x^{(\ell-1)\mathbf{u}_1+\mathbf{u}_2} + \cdots + x^{\ell\mathbf{u}_2}\right). \end{aligned}$$

Then, $\mathrm{in}_{\preceq} f$ is $\binom{m}{1} x^{(m-1)\mathbf{v}_1+\mathbf{v}_2}$ or $-\binom{\ell}{1} x^{(\ell-1)\mathbf{u}_1+\mathbf{u}_2}$. In both cases, $\{x^{\mathbf{v}_1}+x^{\mathbf{v}_2}, x^{\mathbf{u}_1}+x^{\mathbf{u}_2}, f\}$ is a SAGBI basis of R. The proof is the same as Proposition 3.1 since both $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{u}_1, \mathbf{u}_2$ are linearly independent. □

We enumerate examples found in our experiments. These examples can be regarded as a generalization of [5, 1.20]. All monoids corresponding to these examples are non-finitely generated monoids by Lemma 3.3.

Lemma 3.3. *Let $\mathbf{v}_1, \mathbf{v}_2 \in (\mathbb{Z}_{\geq 0})^2$ be linearly independent and let $\preceq$ be a monomial order with $\mathbf{v}_1 \succeq \mathbf{v}_2$. Let C be the cone generated by $\mathbf{v}_1, \mathbf{v}_2$. Fix $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_s \in (\mathbb{Z}_{\geq 0})^2 \cap C^\circ$, let N be the monoid generated by $\mathbf{v}_2$, and let L be the N-module generated by $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_s$. We define M by setting the monoid generated by $\{\mathbf{v}_1\} \cup L$. Then M is not finitely generated.*

Proof. Let $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{R}^2$ such that

$$\begin{aligned} &\langle \mathbf{w}_1, \mathbf{v}_1 \rangle = 0 \text{ and } \langle \mathbf{w}_1, \mathbf{x} \rangle \geq 0 \quad (\forall \mathbf{x} \in C); \\ &\langle \mathbf{w}_2, \mathbf{v}_2 \rangle = 0 \text{ and } \langle \mathbf{w}_2, \mathbf{x} \rangle \geq 0 \quad (\forall \mathbf{x} \in C). \end{aligned}$$

In other words, $\mathbf{w}_1$ and $\mathbf{w}_2$ are chosen as they define the facets $\mathbb{R}_{\geq 0}\mathbf{v}_1$ and $\mathbb{R}_{\geq 0}\mathbf{v}_2$ of C, respectively. Since $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_s \in C^\circ$, we have $\langle \mathbf{w}_2, \mathbf{u}_i \rangle > 0$ for each i. Let, say, $\mathbf{u}_1$ attain $\langle \mathbf{w}_2, \mathbf{u}_1 \rangle \leq \langle \mathbf{w}_2, \mathbf{u}_i \rangle$ for each i. Let $L_1 \coloneqq \{\mathbf{u}_1 + m\mathbf{v}_2 \mid m \in \mathbb{Z}_{\geq 0}\}$ be the N-module generated by $\mathbf{u}_1$. Then L_1 is an infinite subset of M. In what follows, we prove that all elements of L_1 are irreducible on M. On the contrary, suppose that

$$\mathbf{u}_1 + m\mathbf{v}_2 = \mathbf{x}_1 + \mathbf{x}_2$$

for some $m \in \mathbb{Z}_{\geq 0}$ and $\mathbf{x}_1, \mathbf{x}_2 \in M \backslash \{\mathbf{0}\}$. Then $\mathbf{x}_1$ and $\mathbf{x}_2$ can be written as follows:

$$\mathbf{x}_1 = \sum a_s \mathbf{y}_s \text{ and } \mathbf{x}_2 = \sum b_t \mathbf{z}_t, \tag{1}$$

where $\mathbf{y}_s, \mathbf{z}_t \in \{\mathbf{v}_1\} \cup L$ and $a_s, b_t \in \mathbb{Z}_{\geq 0}$. Since generators of M are a subset of $C \backslash \mathbb{R}_{\geq 0}\mathbf{v}_2$, we have $\langle \mathbf{w}_2, \mathbf{x} \rangle > 0$ for all $\mathbf{x} \in \{\mathbf{v}_1\} \cup L$. If $\mathbf{u}_i + m'\mathbf{v}_2 \in L$ appears in the summand of $\mathbf{x}_1$ or $\mathbf{x}_2$ of (1), say, in $\mathbf{x}_1$, then we obtain that $\langle \mathbf{w}_2, \mathbf{x}_1 \rangle \geq \langle \mathbf{w}_2, \mathbf{u}_1 \rangle$. Since $\langle \mathbf{w}_2, \mathbf{x}_2 \rangle > \langle \mathbf{w}_2, \mathbf{v}_2 \rangle = 0$, we have $\langle \mathbf{w}_2, \mathbf{x}_1 + \mathbf{x}_2 \rangle > \langle \mathbf{w}_2, \mathbf{u}_1 + m\mathbf{v}_2 \rangle$, a contradiction to $\mathbf{u}_1 + m\mathbf{v}_2 = \mathbf{x}_1 + \mathbf{x}_2$. Thus, no elements in L appear in the summand of (1). Hence, $\mathbf{x}_1 + \mathbf{x}_2$ is a positive integer multiple of $\mathbf{v}_1$. We can rewrite it as like

$$\mathbf{u}_1 + m\mathbf{v}_2 = \mathbf{x}_1 + \mathbf{x}_2 = \ell\mathbf{v}_1$$

with some positive integer ℓ. However, by applying $\langle \mathbf{w}_1, - \rangle$ to both sides of these equations, we obtain that

$$0 < \langle \mathbf{w}_1, \mathbf{u}_1 + m\mathbf{v}_2 \rangle = \langle \mathbf{w}_1, \ell\mathbf{v}_1 \rangle = 0,$$

a contradiction.

Therefore, we conclude that all elements of L_1 are irreducible on M, implying the non-finite generation of M. $\square$

Now, we provide a family of examples of finitely generated $\Bbbk$-algebras whose initial algebras are equal to $\Bbbk[M]$ with M defined in Lemma 3.3. This is the first main theorem of this paper.

Theorem 3.4. *Work with the same notation as in Lemma 3.3. Let R be the $\Bbbk$-algebra generated by*

$$G \coloneqq \{x^{\mathbf{v}_1} + x^{\mathbf{v}_2}\} \cup \{x^{\mathbf{u}} \mid \mathbf{u} \in L\}.$$

Then R is finitely generated. Moreover, given a monomial order $\preceq$ with $\mathbf{v}_1 \succeq \mathbf{v}_2$, we have $\mathrm{in}_{\preceq} R = \Bbbk[M]$. In particular, G is a SAGBI basis of some R consisting of infinitely many polynomials (most of which are monomials).

Proof. First, we prove that R is finitely generated.

Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent and $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_s \in C^\circ$, there exist positive integers ℓ_i, a_i, b_i such that

$$\ell_i \mathbf{u}_i = a_i \mathbf{v}_1 + b_i \mathbf{v}_2$$

holds for each $i = 1, \dots, s$. Fix i. Let

$$f_0 = x^{\mathbf{u}_i + a_i\mathbf{v}_1 + b_i\mathbf{v}_2} \ \ (= (x^{\mathbf{u}_i})^{\ell_i + 1})$$

and

$$\begin{aligned} f_k &= x^{\mathbf{u}_i + (b_i - 1 + k)\mathbf{v}_2}\,(x^{\mathbf{v}_1} + x^{\mathbf{v}_2})^{a_i + 1 - k} \\ &= \sum_{j=0}^{a_i+1-k} \binom{a_i + 1 - k}{j} x^{\mathbf{u}_i + (a_i + 1 - k - j)\mathbf{v}_1 + (b_i - 1 + k + j)\mathbf{v}_2} \end{aligned}$$

for $k = 1, \dots, a_i$. Let V be the $\Bbbk$-vector space with a basis

$$\{m_p \coloneqq x^{\mathbf{u}_i + (a_i - p)\mathbf{v}_1 + (b_i + p)\mathbf{v}_2} \mid p = 0, 1, \dots, a_i\}.$$

Then $f_0, f_1, \dots, f_{a_i}$ belong to V. Let

$$A = \begin{pmatrix} 1 & \binom{a_i}{0} & 0 & & 0 & 0 \\ 0 & \binom{a_i}{1} & \binom{a_i-1}{0} & & \vdots & \vdots \\ & & \binom{a_i-1}{1} & \cdots & 0 & \\ \vdots & \vdots & \vdots & & \binom{2}{0} & 0 \\ & \binom{a_i}{a_i-1} & \binom{a_i-1}{a_i-2} & & \binom{2}{1} & \binom{1}{0} \\ 0 & \binom{a_i}{a_i} & \binom{a_i-1}{a_i-1} & & \binom{2}{2} & \binom{1}{1} \end{pmatrix}$$

be the $(a_i + 1) \times (a_i + 1)$-matrix. Then

$$(m_0\ m_1\ \cdots\ m_{a_i})A = (f_0\ f_1\ \cdots\ f_{a_i}).$$

Now, we claim that $f_0, f_1, \dots, f_{a_i}$ also form a basis of V by seeing that A is invertible. We subtract 3-rd column from 2-nd column, 4-th column from 3rd column, ..., and $(a_i + 1)$-th column from a_i-th column. Then A is transformed into

$$\begin{pmatrix} 1 & \binom{a_i-1}{0} & 0 & & 0 & 0 \\ 0 & \binom{a_i-1}{1} & \binom{a_i-2}{0} & & \vdots & \vdots \\ & & \binom{a_i-2}{1} & \cdots & 0 & \\ \vdots & \vdots & \vdots & & \binom{1}{0} & 0 \\ & \binom{a_i-1}{a_i-1} & \binom{a_i-2}{a_i-2} & & \binom{1}{1} & \binom{1}{0} \\ 0 & 0 & 0 & & 0 & \binom{1}{1} \end{pmatrix}.$$

Hence, by induction on a_i, we can see that A is invertible.

Therefore, the monomial $m_{a_i} = x^{\mathbf{u}_i+(a_i+b_i)\mathbf{v}_2} \in V$ can be written as a (unique) $\Bbbk$-linear combination of $f_0, f_1, \dots, f_{a_i}$. This means that the monomial $x^{\mathbf{u}_i+(a_i+b_i)\mathbf{v}_2}$ belongs to

$$\Bbbk[x^{\mathbf{v}_1} + x^{\mathbf{v}_2}, x^{\mathbf{u}_i}, x^{\mathbf{u}_i+b_i\mathbf{v}_2}, x^{\mathbf{u}_i+(b_i+1)\mathbf{v}_2}, \dots, x^{\mathbf{u}_i+(a_i+b_i-1)\mathbf{v}_2}].$$

Similarly to the above discussion, we can show that the monomial $x^{\mathbf{u}_i+(a_i+b_i+1)\mathbf{v}_2}$ can be written as a $\Bbbk$-linear combination of the polynomials

$$(x^{\mathbf{u}_i})^{\ell_i}(x^{\mathbf{u}_i+\mathbf{v}_2}) \text{ and } x^{\mathbf{u}_i+(b_i+k)\mathbf{v}_2}(x^{\mathbf{v}_1} + x^{\mathbf{v}_2})^{a_i+1-k} \quad \text{for } k = 1, \dots, a_i.$$

By applying these repeatedly, we can show that all monomials $x^{\mathbf{u}_i+j\mathbf{v}_2}$ for $j \in \mathbb{Z}_{\geq 0}$ belong to $\Bbbk[x^{\mathbf{v}_1} + x^{\mathbf{v}_2}, x^{\mathbf{u}_i}, x^{\mathbf{u}_i+\mathbf{v}_2}, \dots, x^{\mathbf{u}_i+(a_i+b_i-1)\mathbf{v}_2}]$. Thus, the $\Bbbk$-algebra generated by

$$\{x^{\mathbf{v}_1} + x^{\mathbf{v}_2}\} \cup \bigcup_{i=1}^{s} \{x^{\mathbf{u}_i}, x^{\mathbf{u}_i+\mathbf{v}_2}, \dots, x^{\mathbf{u}_i+(a_i+b_i-1)\mathbf{v}_2}\}$$

coincides with R, which is finitely generated.

Next, we prove that $\mathrm{in}_\preceq R = \Bbbk[M]$. It is sufficient to prove that G is a SAGBI basis of R. Namely, we prove that for any polynomial $f \in R$, $\mathrm{in}_\preceq f$ can be written as a product of finitely many monomials in $\{\mathrm{in}_\preceq g \mid g \in G\}$. Since f can be written as

$$f = \sum_i c_i (x^{\mathbf{v}_1} + x^{\mathbf{v}_2})^{\alpha^{(i)}} (x^{\mathbf{w}_1})^{\beta_1^{(i)}} (x^{\mathbf{w}_2})^{\beta_2^{(i)}} \cdots (x^{\mathbf{w}_t})^{\beta_t^{(i)}}, \tag{2}$$

where $c_i \in \Bbbk, \alpha^{(i)}, \beta_1^{(i)}, \beta_2^{(i)}, \dots, \beta_t^{(i)} \in \mathbb{Z}_{\geq 0}$ and $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_t \in L$, the initial monomial $\mathrm{in}_\preceq f$ belongs to

$$\mathcal{S} := \mathrm{supp}((x^{\mathbf{v}_1} + x^{\mathbf{v}_2})^{\alpha^{(i)}} (x^{\mathbf{w}_1})^{\beta_1^{(i)}} (x^{\mathbf{w}_2})^{\beta_2^{(i)}} \cdots (x^{\mathbf{w}_t})^{\beta_t^{(i)}})$$

for some i.

- If one of $\beta_1^{(i)}, \beta_2^{(i)}, \dots, \beta_t^{(i)}$ is not 0, then all monomials in $\mathcal{S}$ can be written as $x^{m_1\mathbf{u}_1+m_2\mathbf{u}_2+\ldots+m_s\mathbf{u}_s+a\mathbf{v}_1+b\mathbf{v}_2}$ with $m_1, m_2, \dots, m_s, a, b \in \mathbb{Z}_{\geq 0}$ and one of $m_1, m_2, \dots, m_s$ is positive, say, $m_1 > 0$. Then we have

$$x^{m_1\mathbf{u}_1+m_2\mathbf{u}_2+\cdots+m_s\mathbf{u}_s+a\mathbf{v}_1+b\mathbf{v}_2} = (x^{\mathbf{v}_1})^a (x^{\mathbf{u}_1})^{m_1-1} (x^{\mathbf{u}_2})^{m_2} \cdots (x^{\mathbf{u}_s})^{m_s} (x^{\mathbf{u}_1+b\mathbf{v}_2}).$$

- If $\beta_1^{(i)} = \beta_2^{(i)} = \ldots = \beta_t^{(i)} = 0$, i.e., $\mathrm{in}_\preceq f \in \mathrm{supp}\,(x^{\mathbf{v}_1} + x^{\mathbf{v}_2})^{\alpha^{(i)}}$, since $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_s \in C^\circ$, the monomial $x^{\alpha^{(i)}\mathbf{v}_1}$ cannot be written like

$$x^{m_1\mathbf{u}_1+m_2\mathbf{u}_2+\cdots+m_s\mathbf{u}_s+a\mathbf{v}_1+b\mathbf{v}_2}$$

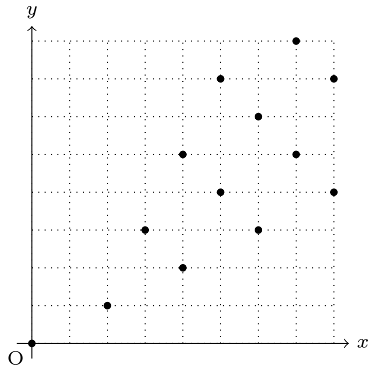

Fig. 1. The monoid M of Example 3.5 in the case with $\mathbf{v}_1 = (2,1), \mathbf{v}_2 = (1,2)$.

with $m_1, m_2, \dots, m_s, a, b \in \mathbb{Z}_{\geq 0}$ and one of $m_1, m_2, \dots, m_s$ is not 0. Therefore, the cancellation of the monomials of the form $x^{\alpha^{(i)}\mathbf{v}_1}$ in (2) never happens, i.e., $x^{\alpha^{(i)}\mathbf{v}_1}$ definitely appears in f. Since $x^{\alpha^{(i)}\mathbf{v}_1}$ is the strongest monomial with respect to $\preceq$ in $\operatorname{supp}(x^{\mathbf{v}_1} + x^{\mathbf{v}_2})^{\alpha^{(i)}}$, we have $\operatorname{in}_{\preceq} f = (x^{\mathbf{v}_1})^{\alpha^{(i)}}$.

By these discussions, we see that for any $f \in R$, the initial monomial $\operatorname{in}_{\preceq} f$ can be written as a product of finitely many monomials in $\{\operatorname{in}_{\preceq} g \mid g \in G\}$. Therefore, G is a SAGBI basis of R. $\square$

We provide three families of examples of Theorem 3.4. Each of Examples 3.5, 3.7 and 3.6 generalizes the initiated example [5, 1.20] of finitely generated $\mathbb{k}$-algebra whose initial algebra is not finitely generated. In what follows, let $\preceq$ be a monomial order with $x \succeq y$.

Example 3.5. Given $\mathbf{v}_1, \mathbf{v}_2 \in (\mathbb{Z}_{\geq 0})^2$ which are linearly independent and satisfy $x^{\mathbf{v}_1} \succeq x^{\mathbf{v}_2}$, let

$$R_1 = \mathbb{k}[x^{\mathbf{v}_1} + x^{\mathbf{v}_2}, x^{\mathbf{v}_1+\mathbf{v}_2}, x^{\mathbf{v}_1+2\mathbf{v}_2}].$$

Then

$$\{x^{\mathbf{v}_1} + x^{\mathbf{v}_2}\} \cup \{x^{\mathbf{v}_1+m\mathbf{v}_2} \mid m \geq 1\}$$

forms a SAGBI basis of R_1 with respect to $\preceq$. When $\mathbf{v}_1 = (2,1), \mathbf{v}_2 = (1,2)$, the initial algebra of R_1 corresponds to the monoid plotted in Fig. 1. In particular, the case with $\mathbf{v}_1 = (1,0), \mathbf{v}_2 = (0,1)$ is the same as that of [5, 1.20].

Example 3.6. Given a positive integer s, let

$$R_2 = \mathbb{k}[x^s + y^s, x^s y^s, x^s y^{2s}, x^{s+1}y^{s-1}, x^{s+1}y^{2s-1}, \dots, x^{2s-1}y, x^{2s-1}y^{s+1}].$$

Then

$$\{x^s + y^s\} \cup \bigcup_{i=0}^{s-1} \{x^{s+i}y^{s-i+sm} \mid m \in \mathbb{Z}_{\geq 0}\}$$

forms a SAGBI basis of R_2. When $s = 3$, the initial algebra of R_2 corresponds to the monoid plotted in Fig. 2. In particular, the case $s = 1$ is the same as [5, 1.20].

Example 3.7. Given positive integers a, b, let

$$R_3 = \mathbb{k}[x + y, x^a y^b, x^a y^{b+1}, \dots, x^a y^{a+2b-1}].$$

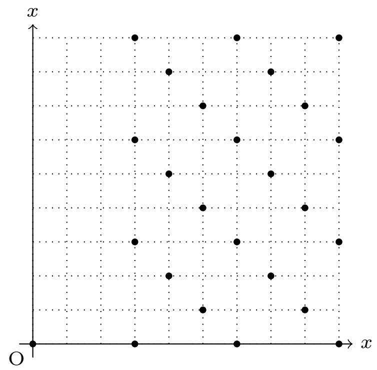

Fig. 2. The monoid M of Example 3.6 in the case with $s = 3$.

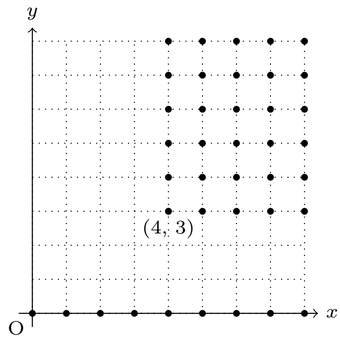

Fig. 3. The monoid M of Example 3.7 in the case with $a = 4, b = 3$.

Then

$$\{x + y\} \cup \{x^a y^m \mid m \geq b\}$$

forms a SAGBI basis of R_3. When $a = 4, b = 3$, the initial algebra of R_3 corresponds to the monoid plotted in Fig. 3. In particular, the case $a = b = 1$ is the same as [5, 1.20].

Remark 3.8. Readers may have a doubt about the similarity between our examples and the ones developed in [4]. On the one hand, Examples 3.5 and 3.6 can be obtained via the method developed in [4]. In fact, for Example 3.5, we may assign $x^{\mathbf{v}_1}$ and $x^{\mathbf{v}_2}$ instead of x and y in [4, Example 2.7], and for Example 3.6, we may assign x^s and y^s and $U = \{x^{s+1}y^{s-1}, x^{s+2}y^{s-2}, \dots, x^{2s-1}y\}$, where U appears in (A3) of the construction developed in [4].

On the other hand, most cases of Example 3.7 cannot be obtained in that way. In fact, we can observe that the subalgebra R constructed in the way of [4] always contains xy whenever R contains $x + y$. This implies that the algebra R_3 cannot be equal to R constructed in [4].

4. Nonexistence of the monoid algebra as an initial algebra

This section is devoted to proving the second main theorem, which provides an example of a monoid whose algebra cannot be realized as any initial algebra of a finitely generated homogeneous $\Bbbk$-algebra.

Theorem 4.1. *Let M be a submonoid of $(\mathbb{Z}_{\geq 0})^2$ generated by infinitely many irreducible elements $\{(1, n^2) \mid n \in \mathbb{Z}_{\geq 0}\}$. For any subalgebra R generated by finitely many homogeneous polynomials in $\Bbbk[x, y]$ and any monomial order $\preceq$ in $(\mathbb{Z}_{\geq 0})^2$, the initial algebra $\mathrm{in}_{\preceq} R$ cannot be equal to $\Bbbk[M]$.*

Proof. Suppose the existence of a subalgebra R and a monomial order $\preceq$ such that $\mathrm{in}_\preceq R = \Bbbk[M]$. Let g_n be a polynomial with $\mathrm{in}_\preceq g_n = xy^{n^2}$ for each n and let $G = \{g_0, g_1, \ldots\}$ be the reduced SAGBI basis of R. Since we assume that R is generated by homogeneous polynomials of $\Bbbk[x, y]$, the reduced SAGBI basis G can be chosen as a set of homogeneous polynomials. Since R is finitely generated and $R = \Bbbk[g_0, g_1, \ldots]$, there exists $m \in \mathbb{Z}_{\geq 0}$ such that $R = k[g_0, g_1, \ldots, g_m]$. Then $\deg_\preceq R$ is a submonoid of the monoid generated by $\bigcup_{i=0}^m \operatorname{supp} g_i$ because all monomials appearing in polynomials of R can be written as a product of finitely many monomials appearing in $g_0, g_1, \ldots, g_m$.

(The first step): First, we prove that there exists a positive integer a with $(0, a) \in \bigcup_{i=0}^m \operatorname{supp} g_i$. On the contrary, suppose that $(0, a) \notin \bigcup_{i=0}^m \operatorname{supp} g_i$ for any $a \geq 1$. Let C be the cone generated by $\bigcup_{i=0}^m \operatorname{supp} g_i$. Since

$$\bigcup_{i=0}^{m} \operatorname{supp} g_i \subset \{(x, y) \in (\mathbb{Z}_{\geq 0})^2 \mid x + y \leq m^2 + 1\} \setminus \{(0, y) \mid y \geq 1\}$$

by our assumption, the ray generated by $(1, m^2)$ is a face of C, and $y \leq m^2 x$ for any $(x, y) \in C$. However, for all integer $\ell > m$, $(1, \ell^2)$ is out of C, i.e., out of $\deg_\preceq R$, a contradiction. Hence, we have $(0, a) \in \bigcup_{i=0}^m \operatorname{supp} g_i$ for some $a \geq 1$. In particular, if $(0, a) \in \bigcup_{i=0}^m \operatorname{supp} g_i$, then $a = i^2 + 1$ for some $i \in \{0, 1, \ldots, m\}$ because g_i's are homogeneous.

(The second step): Next, we prove that each g_i can be written like $g_i = y^{i^2}(x + ay)$ for some $a \in \Bbbk \setminus \{0\}$. Note that a is independent of i.

Since each g_i is homogeneous, we can write

$$\begin{aligned} g_0 &= x + a_0 y \\ g_1 &= xy + \cdots + a_1 y^2 + \cdots \\ &\vdots \\ g_m &= xy^{m^2} + \cdots + a_m y^{m^2+1} + \cdots \\ &\vdots \end{aligned}$$

with $a_0, a_1, \ldots, a_m, \ldots \in \Bbbk$ and one of $a_0, a_1, \ldots, a_m$ is not 0. Since $\mathrm{in}_\preceq g_i = xy^{i^2}$, we get $xy^{i^2} \succeq y^{i^2+1}$. Therefore, we have $x \succeq y$. For $b \geq 2$, monomials $x^b y^{i^2+1-b}$ do not appear in g_i because $x^b y^{i^2+1-b} \succeq xy^{i^2}$. Thus we can rewrite g_i's by

$$g_i = xy^{i^2} + a_i y^{i^2+1}.$$

We prove $a_0 = a_1 = \ldots = a_m = \cdots$ by induction on m. In the case of $m = 2$, we consider

$$\begin{aligned} g &\coloneqq g_0^3 g_2 - g_1^4 - (3a_0 + a_2 - 4a_1) g_0 g_1 g_2 \\ &= (a_0 a_1 - a_0 a_2 - 2a_1^2 + 3a_1 a_2 - a_2^2) x^2 y^6 \\ &\quad + (a_0^3 - 3a_0^2 a_1 + 4a_0 a_1^2 - a_0 a_2^2 - 4a_1^3 + 4a_1^2 a_2 - a_1 a_2^2) xy^7 \\ &\quad + (a_0^3 a_2 - 3a_0^2 a_1 a_2 + 4a_0 a_1^2 a_2 - a_0 a_1 a_2^2 - a_1^4) y^8. \end{aligned}$$

Though $\mathrm{in}_\preceq g \in \mathrm{in}_\preceq R$, any monomials in $\{x^2y^6, xy^7, y^8\}$ cannot be written as a product of $x, xy, xy^4, xy^9, \ldots$. Thus, all coefficients of g should be 0. Let h_1, h_2, h_3 be the following polynomials in $\Bbbk[X_0, X_1, X_2]$:

$$h_1 = X_0X_1 - X_0X_2 - 2X_1^2 + 3X_1X_2 - X_2^2;$$
$$h_2 = X_0^3 - 3X_0^2X_1 + 4X_0X_1^2 - X_0X_2^2 - 4X_1^3 + 4X_1^2X_2 - X_1X_2^2;$$
$$h_3 = X_0^3X_2 - 3X_0^2X_1X_2 + 4X_0X_1^2X_2 - X_0X_1X_2^2 - X_1^4.$$

Then a_0, a_1, a_2 satisfy $h_1(a_0, a_1, a_2) = h_2(a_0, a_1, a_2) = h_3(a_0, a_1, a_2) = 0$. The reduced Gröbner basis for the ideal generated by h_1, h_2, h_3 with respect to a lexicographic ordering is given by the following polynomials:

$$\tilde{h}_1 = X_1^4 - 4X_1^3X_2 + 6X_1^2X_2^2 - 4X_1X_2^3 + X_2^4 = (X_1 - X_2)^4;$$
$$\tilde{h}_2 = X_0X_1 - X_0X_2 - 2X_1^2 + 3X_1X_2 - X_2^2;$$
$$\tilde{h}_3 = X_0^3 - 3X_0^2X_2 + 3X_0X_2^2 - 8X_1^3 + 24X_1^2X_2 - 24X_1X_2^2 + 7X_2^3.$$

Hence, a_0, a_1, a_2 also satisfy $\tilde{h}_1(a_0, a_1, a_2) = \tilde{h}_2(a_0, a_1, a_2) = \tilde{h}_3(a_0, a_1, a_2) = 0$. We get $a_1 = a_2$ from $\tilde{h}_1(X_0, X_1, X_2)$, and $a_0 = a_1$ from $a_1 = a_2$ and $\tilde{h}_3(a_0, a_1, a_2) = 0$. Thus, we conclude $a_0 = a_1 = a_2$.

Now, assume $a_0 = a_1 = \ldots = a_{i-1} = a$ for some $i \geq 3$. Let $b = a_i - a$ and suppose $b \neq 0$. If $i = 2k + 1$ for some $k \geq 1$, then a direct computation implies that

$$b^2 g_1 g_2 g_{2k+1} - b g_0^2 g_2 g_{2k+1} + g_0 g_1^3 g_{2k+1} - g_0 g_{k-1} g_{k+1}^3 = b^3 y^{4k^2+4k+7}(x + ay)^2$$

by using $g_j = y^{j^2}(x + ay), (0 \leq j \leq i - 1)$ and $g_i = y^{i^2+1}(x + a_i y)$. The initial monomial of this polynomial is $x^2 y^{4k^2+4k+7}$, but $x^2 y^{4k^2+4k+7}$ cannot be written as a product of two monomials of $x, xy, xy^4, \ldots$ because any squares of integers are 0 or 1 modulo 4, a contradiction. Similarly, if $i = 2k$ for some $k \geq 2$, then we get a polynomial in R

$$b^2 g_1^2 g_{2k} - b g_0^2 g_1 g_{2k} + g_0^4 g_{2k} - g_0 g_k^4 = b^3 y^{4k^2+3}(x + ay)^2,$$

a contradiction. Thus, we have $b = 0$, i.e., $a_i = a$ for each $i \geq 0$.

(The third step): Finally, we prove that $\{g_0, g_1, \ldots, g_m\}$ forms a reduced SAGBI basis of R. This claim contradicts the uniqueness of the reduced SAGBI basis of R, so we can prove the nonexistence of homogeneous finitely generated subalgebra R with $\mathrm{in}_{\preceq} R = \Bbbk[M]$.

Let σ be an automorphism of $\Bbbk[x, y]$ defined by

$$\sigma(x) = x - ay \text{ and } \sigma(y) = y.$$

Then $\sigma(g_i) = xy^{i^2}$ for each i. Let $I_{\mathcal{A}}$ be the toric ideal of $2 \times (m + 1)$-matrix

$$\mathcal{A} = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 0 & 1 & \ldots & m^2 \end{pmatrix}$$

For all binomials $t^{\alpha_0} \cdots t^{\alpha_m} - t^{\beta_0} \cdots t^{\beta_m}$ in generators of $I_{\mathcal{A}}$, the image of $g_0^{\alpha_0} \cdots g_m^{\alpha_m} - g_0^{\beta_0} \cdots g_m^{\beta_m}$ by σ is

$$\sigma(g_0^{\alpha_0} \cdots g_m^{\alpha_m} - g_0^{\beta_0} \cdots g_m^{\beta_m}) = (x)^{\alpha_0} \cdots (xy^{m^2})^{\alpha_m} - (x)^{\beta_0} \cdots (xy^{m^2})^{\beta_m} = 0.$$

This implies that $g_0^{\alpha_0} \cdots g_m^{\alpha_m} - g_0^{\beta_0} \cdots g_m^{\beta_m} = 0$. Therefore, all $g_0^{\alpha_0} \cdots g_m^{\alpha_m} - g_0^{\beta_0} \cdots g_m^{\beta_m}$ subduce to an element of $\Bbbk$. From Proposition 2.1, we conclude that $\{g_0, g_1, \ldots, g_m\}$ is a finite SAGBI basis, and clearly reduced. □

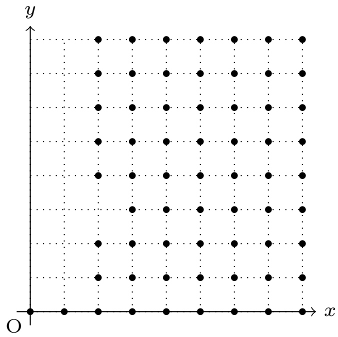

Fig. 4. Monoid of Example 5.2.

5. Other examples

In this section, we provide examples that do not suit Section 3. Since these examples were found by computer experiments, we omit proofs of non-finitely generation.

Our first interest is whether the converse of Lemma 3.3 and Theorem 3.4 is true. Our question can be rewritten more precisely into the following way.

Question 5.1. *Let C be the cone generated by $\mathbf{v}_1, \mathbf{v}_2 \in (\mathbb{Z}_{\geq 0})^2$, and let*

$$\{x^{\mathbf{v}_1} + x^{\mathbf{v}_2}\} \cup \{x^{\mathbf{u}_i}\}_{i=1}^{\infty}$$

be a reduced SAGBI basis of some finitely generated $\Bbbk$-subalgebra $R \subset \Bbbk[x, y]$, where each $\mathbf{u}_i$ belongs to C°. Then, does there exist any monoid M constructed in the way as in Lemma 3.3 together with a monomial order $\preceq$ such that $\mathrm{in}_{\preceq} R = \Bbbk[M]$?

This is not true in general as Examples 5.2 and 5.3 indicate. Those examples are counterexamples of Question 5.1.

Example 5.2. Let $R = \Bbbk[x + y, x^2y, x^2y^2, x^3y^3]$ and $\preceq$ a monomial order with $x \succeq y$. Then a SAGBI basis of R with respect to $\preceq$ seems to be

$$\{x + y, x^2y, x^2y^2, x^3y^3, x^2y^4, x^2y^5, \dots\}.$$

The expected initial ideal of R corresponds to the monoid plotted in Fig. 4. This example is similar to Example 3.7 with $a = 2, b = 1$, but x^2y^3 is not contained in R in this example. Therefore, generators of the monoid cannot be written as a union of $\{(1, 0)\}$ and any N-modules with $N = \mathbb{Z}_{\geq 0}(0, 1)$.

Example 5.3. Let $R = k[x^2 + y^2, x^2y, x^2y^2]$ and let $\preceq$ be a monomial order with $x \succeq y$. Then a SAGBI basis of R with respect to $\preceq$ seems to be

$$\{x^2 + y^2, x^2y, x^2y^2, x^2y^4, x^2y^6, \dots\}.$$

The expected initial ideal of R corresponds to the monoid plotted in Fig. 5.

Throughout this paper, we constructed finitely generated $\Bbbk$-subalgebras (having an infinite SAGBI basis) generated by exactly one binomial $x^{\mathbf{v}_1} + x^{\mathbf{v}_2}$ and finitely many monomials $x^{\mathbf{u}_1}, x^{\mathbf{u}_2}, \dots, x^{\mathbf{u}_t}$ with $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_t \in C^\circ$, where $C = \mathbb{R}_{\geq 0}\mathbf{v}_1 + \mathbb{R}_{\geq 0}\mathbf{v}_2$. However, the condition "$\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_t \in C^\circ$" is not necessary for the infiniteness of SAGBI basis.

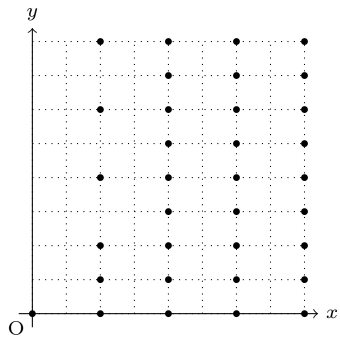

Fig. 5. Monoid of Example 5.3.

Example 5.4. Let $R = \Bbbk[xy + y^2, x, xy^2]$ and $\preceq$ a monomial order with $x \succeq y$. Then a SAGBI basis of R with respect to $\preceq$ seems to be

$$\{x, xy + y^2, xy^2, 2xy^3 + y^4, xy^4, 3xy^5 + y^6, \ldots\}$$

Computing SAGBI basis of R, monomials and binomials appear in this SAGBI basis alternately. This example is almost the same as [5, 4.11] up to sign.

Finally, we introduce the most complicated example through our experiments.

Example 5.5. Let $R = \Bbbk[x^2 - y^2, x^3 - y^3, x^4 - y^4]$ and $\preceq$ a monomial order with $x \succeq y$. Computing a SAGBI basis of R, we observe that the following monomials appear as initial terms:

$$x^2, x^3, x^2y^2, x^3y^3, x^5y^7, x^6y^8, x^6y^{10}, x^7y^{11}, x^7y^{13}, x^8y^{14}, x^8y^{16}, x^9y^{17}, x^9y^{19}, \ldots.$$

Thus, generators of M which satisfy $\mathrm{in}_\preceq R = \Bbbk[M]$ are

$$(2,0), (3,0), (2,2), (3,3), (5,7), (6,8), (6,10), (7,11), (7,13), (8,14), (8,16), (9,17), (9,19), \ldots.$$

The monoid M is plotted in Fig. 6. All points following $(3,3)$ are contained in

$$\{(5,7) + m(1,3) \mid m \in \mathbb{Z}_{\geq 0}\} \cup \{(6,8) + m(1,3) \mid m \in \mathbb{Z}_{\geq 0}\}.$$

However, first four points $(2,0), (3,0), (2,2), (3,3)$ are not contained there, and $(2,2) + (1,3) = (3,5)$ and $(3,3) + (1,3) = (4,6)$ are not contained in M. Moreover, the monomial xy^3 does not appear in generators of R.

In contrast, let us change the signs of generators of R, i.e. let $R = \Bbbk[x^2 + y^2, x^3 + y^3, x^4 + y^4]$. Then R has a finite SAGBI basis with respect to a monomial order with $x \succeq y$, which is

$$\{x^2 + y^2, x^3 + y^3, x^2y^2, x^3y^3\}.$$

CRediT authorship contribution statement

Akihiro Higashitani: Supervision, Project administration, Investigation, Formal analysis, Conceptualization. **Koichiro Tani:** Investigation, Formal analysis, Conceptualization.

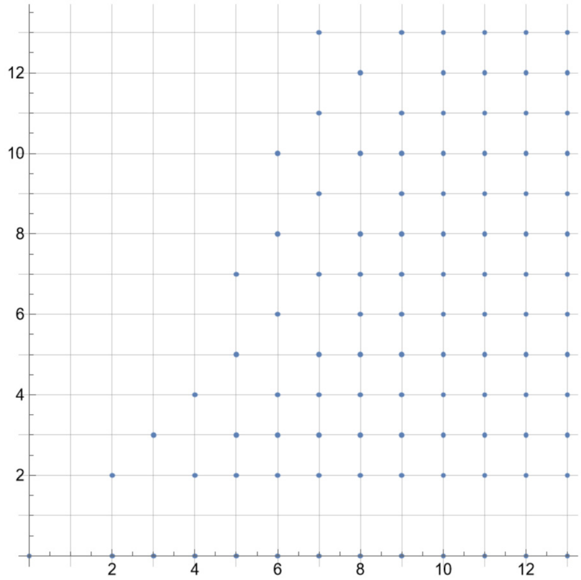

Fig. 6. Non-finitely generated monoid of Example 5.5 with `Mathematica`.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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