Zeng, L., Li, G. and Mao, J.
Osaka J. Math.
62 (2025), 1–17

SPECTRUM AND PARABOLIC HEAT KERNEL ON SMOOTH COMPACT METRIC MEASURE SPACES

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(Received September 21, 2021, revised October 31, 2023)

Abstract
In this paper, by utilizing some functional inequalities and combining with De Giorgi-Moser iteration technique, we obtain some estimates for the weighted L^∞-norm of eigenfunctions of the Witten-Laplacian on smooth metric measure spaces, which enables us to establish the upper bounds for multiplicity of corresponding eigenvalues. We remark that those estimates are established without any assumption on curvature. As an important application, we prove the uniform convergence of the heat kernel of weighted parabolic equation on smooth metric measure spaces. In addition, our result remains ture for some isomorphism classes of weighted Riemannian manifold with boundary satisfying certain conditions. Furthermore, by the standard theory of elliptic and parabolic equations, we prove the regularity and uniqueness of parabolic heat kernel on smooth metric measure spaces.

1. Introduction

Let M^n be an n-dimensional compact Riemannian manifold with nonempty boundary ∂M^n and f be a smooth function on M^n. We say that the triple $(M^n, g, e^{-f}dv)$ is a metric measure space, where $e^{-f}dv$ denotes the weighted Riemannian volume density on M^n. Metric measure spaces also arise in smooth collapsed Gromov-Hausdorff limits. The metric measure space has been studied extensively in geometric analysis and Kähler geometry in recent years. The most remarkable example is Perelman's work in [19], where he introduced a functional involving an integral of the scalar curvature with respect to a weighted measure such that the Ricci flow becomes a gradient flow of such a functional. During the last two decades, metric measure spaces has been studied extensively in geometric analysis. For instances, O. Munteanu and J. Wang [16–18] have obtained gradient estimates for positive weighted harmonic functions. G. F. Wei and W. Wylie in [22] have proved the weighted volume comparison theorems.

The so-called Witten-Laplacian (or f-Laplacian, drifting Laplacian), which is associated with the metric measure space, is defined by

$$\Delta_f u =: \Delta u - \langle \nabla f, \nabla u \rangle = e^f \operatorname{div}\left(e^{-f}\nabla u\right),$$

where Δ is the Laplacian on M^n. Clearly, if f is a constant, the Witten-Laplacian is exactly the Laplacian. Next, we introduce two inner products of Dirichlet type as follows: for any $u, w \in C_0^2(M)$,

2020 Mathematics Subject Classification. 35K08, 35P15, 53C21.

$$[u,w] := \int_{M^n} u\Delta_f w e^{-f} dv, \quad \text{and} \quad \mathfrak{D}[u,w] := \int_{M^n} \langle \nabla u, \nabla w \rangle e^{-f} dv.$$

Then, for the functions u, w given above, we have the following weighted Green formula

$$-\mathfrak{D}[u,w] = [u,w] = [w,u] = -\mathfrak{D}[w,u], \tag{1.1}$$

which means that the Witten-Laplacian is a self-adjoint operator with respect to the weighted measure $e^{-f}dv$ and Dirichlet boundary value. The Witten-Laplacian is an important elliptic operator and has been widely studied in the probability theory and geometric analysis (see, e.g., [4,7,8,15,16,22]). We consider the following Dirichlet problem of the Witten-Laplacian

$$\begin{cases} \Delta_f u + \Gamma u = 0, & \text{in} \quad M^n \\ \quad\quad u = 0, & \text{on} \quad \partial M^n \end{cases} \tag{1.2}$$

on a given smooth compact metric measure space $(M^n, g, e^{-f}dv)$ with the boundary ∂M^n. It is easy to know that there exists a discrete set of eigenvalues

$$0 < \Gamma_1 < \Gamma_2 \leq \Gamma_3 \leq \cdots \leq \Gamma_k \leq \cdots \to +\infty,$$

where each eigenvalue is repeated according to its multiplicity. Let Γ_i be the i-th eigenvalue of the eigenvalue problem (1.2) and u_i be the corresponding eigenfunction of Γ_i. It is easy to check that the set $\{u_i\}_{i=1}^{\infty}$ forms an orthonormal basis with respect to the weighted L^2_f-norm. In particular, when f is a constant, (1.2) is just the Dirichlet eigenvalue problem of the Laplacian on smooth Riemannian manifolds. For this special case, Li [12] investigated the spectrum of Laplacian, where he estimated for the upper bound of eigenfunctions and the multiplicities of eigenvalues of the Laplacian. As applications, Chavel and Feldman [2] studied the spectrum of manifolds with small handles, where they proved the convergence of eigenvalues of small handles. For the general case, estimates for eigenvalues of the Witten-Laplacian have been attracting more and more attention of many mathematicians in recent years (see, e.g., [3, 8, 11, 24]). In [8], Futaki, Li and Li gave a lower bound estimate for the first non-zero eigenvalue of the Witten-Laplacian on compact Riemannian manifolds. As an application, they derived a lower bound estimate for the diameter of compact gradient shrinking Ricci solitons. By using the upper bound estimate of the heat kernel and an argument of Li-Yau [13], Wu-Wu [23] derived eigenvalue estimates of the Witten-Laplacian on compact smooth metric measure spaces. Based on the functional inequalities established in Section 2, we obtain some estimates for the upper bounds of the eigenfunctions and the multiplicities of eigenvalues of the Witten-Laplacian. We refer the reader to Theorem 3.1 in Section 3 and Corollary 4.2 in Section 4, respectively. Also, we give some lower bounds for the eigenvalues of Dirichlet eigenvalue problem (1.2) of Witten-Laplacian without any curvature condition. See Theorem 4.9 in Section 4.

We consider the heat kernel of the Witten-Laplacian on smooth metric measure spaces with nonempty boundary. The heat equation on $M^n \times (0, T)$, $0 < T \leq \infty$, of the Witten-Laplacian is given by

$$\left(\Delta_f - \frac{\partial}{\partial t}\right) \upsilon(x,t) = 0.$$

Typically, we consider solving it by prescribing an initial data υ_0 on M^n with $\upsilon_0 \in L^2_f(M^n)$, i.e.,

$$\lim_{t\to 0} \upsilon(x,t) = \upsilon_0(x).$$

Here, as for the notation $L^2_f(M^n)$, we refer the reader to Section 2. The fundamental solution for the heat equation is a kernel function $\Psi_f(x,y,t)$ defined on $M^n \times M^n \times (0,\infty)$ with the property that the function $\upsilon(x,t)$ defined by

$$\upsilon(x,t) = \int_{M^n} \Psi_f(x,y,t)\upsilon_0(y)e^{-f}dy$$

solves the heat equation

$$\left(\Delta_f - \frac{\partial}{\partial t}\right)\upsilon(x,t) = 0 \text{ on } M^n \times (0,\infty) \tag{1.3}$$

with the initial condition (IC for short)

$$\lim_{t\to 0} \upsilon(x,t) = \upsilon_0(x), \tag{1.4}$$

in the sense of $L^2_f(M^n)$. When M^n is a compact manifold with boundary, a natural boundary condition (BC for short) is the Dirichlet BC. In this situation, the solution is required to satisfy

$$\upsilon(x,t) = 0 \text{ on } \partial M^n \times (0,\infty) \tag{1.5}$$

in addition to (1.3) and (1.4). Charalambous-Lu [1] obtained f-heat kernel estimates and essential spectrum by analyzing a family of warped product manifolds. Wu-Wu [23] derived a local Gaussian upper bound for the f-heat kernel on complete smooth metric measure spaces with nonnegative Bakry-Émery Ricci curvature. By De Giorgi-Nash-Moser theory, Wu and Wu derived a Harnack inequality for positive solutions of the f-heat equation and Gaussian upper and lower bound estimates for the f-heat kernel on complete smooth metric measure spaces with Bakry-Émery Ricci curvature bounded from below. Both upper and lower bound estimates are sharp when the Bakry-Émery Ricci curvature is nonnegative (see [24]). In this paper, without any assumption on curvature, we prove the uniform convergence of the heat kernel of weighted parabolic equation on smooth compact metric measure spaces, and refer the readers to Theorem 5.1 in Section 5. We note that the uniform convergence still holds for the set of isomorphism classes of weighted Riemannian manifold with nonempty boundary satisfying certain conditions. Furthermore, we discuss some important properties of heat kernel with initial-boundary conditions (1.4) and (1.5). See Theorem 5.5 in Section 5.

2. Preliminaries

Denote by $L^p_f(M^n)$ the class of all measurable functions ϕ defined on M^n for which $\int_{M^n} |\phi|^p e^{-f} dv < \infty$. Then, we shall verify that

$$\|\phi\|_{L^p_f(M^n)} = \left\{\int_{M^n} |\phi|^p e^{-f} dV\right\}^{\frac{1}{p}}$$

is a norm on $L^p_f(M^n)$ provided that $1 \le p < \infty$. In particular, $L^p_f(M^n)$ will be denoted by $L^p(M^n)$ when f vanishes. For any positive integer k and $1 \leq p < \infty$, we consider the following vector space $W^{k,p}_f(M^n)$ consists of all locally summable functions: $\phi : M^n \to \mathbb{R}$ such that for each multiindex α with $\alpha \leq k$, $D^\alpha \phi$ exists in the weak sense and belongs to $L^p_f(M^n)$. The vector space $W^{k,p}_f(M^n)$ is called the weighted Sobolev space. In addition, when $p = 2, k = 1$, we denote $W^{1,2}_f(M^n)$ by $H^{1,2}_f(M^n)$. On the weighted Sobolev space, we define a functional $\|\cdot\|_{k,p,f}$, where k is a positive integer and $1 \leq p < \infty$, as follows:

$$\|\phi\|_{k,p,f} = \left(\sum_{0 \leq |\beta| \leq k} \|D^\beta \phi\|^p_{L^p_f(M^n)} \right)^{\frac{1}{p}}, \quad \text{if} \quad 1 \leq p < \infty.$$

Note that since any function is contained in $L^2_f(M^n)$, it is contained in $L^p_f(M^n)$ for all $p \in [1,2]$. Since any eigenfunction u is continuous on $\overline{M}^n = M^n \cup \partial M^n$, it is in $L^p_f(M^n)$ for all $p > 1$. Set

$$V_f(M^n) := \int_{M^n} e^{-f} dv = \|1\|_{L^1_f(M^n)}.$$

The L^∞-norm $\|u\|_{L^\infty_f(M^n)}$, given by

$$\|u\|_{L^\infty_f(M^n)} = \sup_{M^n} |u|,$$

is known to satisfy

$$\lim_{p\to\infty} \|u\|_{L^p_f(M^n)} V_f^{1/p}(M^n) = \|u\|_{L^\infty_f(M^n)}. \tag{2.1}$$

Denote by $W^{k,p}_{0,f}(M^n)$ the closure if $C^\infty_0(M^n)$ in $W^{k,p}_f(M^n)$. In particular, when $p = 2$, we denote $W^{k,2}_f(M^n)$ and $W^{k,2}_{0,f}(M^n)$ by $\mathcal{H}^k_f(M^n)$ and $\mathcal{H}^k_{0,f}(M^n)$, respectively. As we shall only consider the case $p > 1$, here, no confusion would happen for the various norms. Hölder's inequality states that for $p, q > 1$ satisfying $1/p + 1/q = 1$, we have

$$\|\varphi\psi\|_{L^1_f(M^n)} \leq \|\varphi\|_{L^p_f(M^n)} \|\psi\|_{L^q_f(M^n)},$$

for $\phi \in L^p_f(M^n)$, $\psi \in L^q_f(M^n)$.

Definition 2.1. Let M^n be an n-dimensional compact Riemannian manifold with nonempty boundary ∂M^n. Define the Dirichlet f-Sobolev constant of M^n as

$$\mathfrak{S}^f_D = \inf_{\substack{\varphi \in W^{1,1}_f(M^n) \\ \varphi|_{\partial M^n} = 0}} \frac{\|\nabla\varphi\|^n_{L^1_f(M^n)}}{\|\varphi\|^n_{L_f^{n/(n-1)}(M^n)}}$$

where infimum is taken over all functions φ in the Sobolev space with Dirichlet boundary condition. Furthermore, if f vanishes, we write $\mathfrak{S}^f_D$ as $\mathfrak{S}_D$.

Recall that the classical isoperimetric constant and Sobolev constant well reveal the relations among the curvature, the topology and the function theory, and thus they are very important concepts in geometric analysis. It is well-known that there are closed relationships between isoperimetric constant and Sobolev constants. The relationship between these constants were exploited in the study of eigenvalues of the Laplacian as early as the 1920's by Faber [6] and Krahn [10]. In [25], Zeng investigated the weighted isoperimetric constant and the weighted Sobolev constant in the sense of the weighted metric measure space. By Definition 2.1, we know that, for all $\varphi \in W_f^{1,1}(M^n)$ with $\varphi|_{\partial M^n} = 0$,

$$\|\nabla\varphi\|_{L_f^1(M^n)} \geq \left(\mathfrak{S}_D^f\right)^{1/n} \|\varphi\|_{L_f^{n/(n-1)}(M^n)}. \tag{2.2}$$

Theorem 2.2. *If $n = 2$, then $\mathcal{H}_{0,f}^1(M^n) \subset L_f^4(M^n)$, and*

$$\|\nabla\varphi\|_{L_f^2(M^n)}^2 \geq \frac{\mathfrak{S}_D^f}{4\left[V_f(M^n)\right]^{1/2}} \|\varphi\|_{L_f^4(M^n)}^2 \tag{2.3}$$

for all $\varphi \in \mathcal{H}_{0,f}^1(M^n)$.

Proof. Given $\varphi \in C_0^\infty(M^n)$, applying (2.2) to the function $\xi = \varphi^2$, and then

$$\begin{aligned}\mathfrak{S}_D^f \|\varphi\|_{L_f^4(M^n)}^4 &\leq \||\nabla\varphi^2|\|_{L_f^1(M^n)}^2 = 4\|\varphi|\nabla\varphi|\|_{L_f^1(M^n)}^2 \leq 4\|\varphi\|_{L_f^2(M^n)}^2 \||\nabla\varphi|\|_{L_f^2(M^n)}^2 \\ &\leq 4\left[V_f(M^n)\right]^{1/2} \|\varphi\|_{L_f^4(M^n)}^2 \||\nabla\varphi|\|_{L_f^2(M^n)}^2,\end{aligned}$$

which implies the conclusion. □

However, for the case that $n > 2$, we need the following theorem.

Theorem 2.3. *If $n > 2$, then $\mathcal{H}_{0,f}^1(M^n) \subset L_f^{2n/(n-2)}(M^n)$, and*

$$\||\nabla\varphi|\|_{L_f^2(M^n)}^2 \geq \mathfrak{c}_1(n)\|\varphi\|_{L_f^{2n/(n-2)}(M^n)}^2 \tag{2.4}$$

for all $\varphi \in \mathcal{H}_{0,f}^1(M^n)$, where

$$\mathfrak{c}_1(n) = \left[\mathfrak{S}_D^f\right]^{2/n} \{(n-2)/2(n-1)\}^2. \tag{2.5}$$

Proof. Let $h \in \mathcal{H}_{0,f}^1(M^n)$, by Definition 2.1, we have

$$\|\nabla h\|_{L_f^1(M^n)}^n \geq \mathfrak{S}_D^f \|h\|_{L_f^{n/(n-1)}(M^n)}^n, \tag{2.6}$$

If we apply (2.6) to $h = |\varphi|^{2(n-1)/(n-2)}$, then we have

$$\|\nabla|\varphi|^{2(n-1)/(n-2)}\|_{L_f^1(M^n)}^n \geq \mathfrak{S}_D^f \||\varphi|^{2(n-1)/(n-2)}\|_{L_f^{n/(n-1)}(M^n)}^n. \tag{2.7}$$

Since

$$\|\nabla|\varphi|^{2(n-1)/(n-2)}\|_{L_f^1(M^n)}^n = \left\{\frac{2(n-1)}{n-2}\||\varphi|^{n/(n-2)}|\nabla\varphi|\|_{L_f^1(M^n)}\right\}^n,$$

it follows from (2.7) that

$$\left\{\frac{2(n-1)}{n-2}\||\varphi|^{n/(n-2)}|\nabla\varphi|\|_{L_f^1(M^n)}\right\}^n \geq \mathfrak{S}_D^f \||\varphi|^{2(n-1)/(n-2)}\|_{L_f^{n/(n-1)}(M^n)}^n = \mathfrak{S}_D^f \|\varphi\|_{L_f^{2n/(n-2)}(M^n)}^{\frac{2n(n-1)}{n-2}}. \tag{2.8}$$

The Cauchy-Schwartz inequality implies that

$$\left\{\frac{2(n-1)}{n-2}\||\varphi|^{n/(n-2)}|\nabla\varphi|\|_{L_f^1(M^n)}\right\}^n \leq \left\{\frac{2(n-1)}{n-2}\right\}^n \||\varphi|\|_{L_f^{2n/(n-2)}(M^n)}^{\frac{n^2}{n-2}} \||\nabla\varphi|\|_{L_f^2(M^n)}^n. \tag{2.9}$$

Therefore, uniting (2.8) and (2.9), we have

$$\left\{\frac{2(n-1)}{n-2}\right\}^n |||\varphi|||^{\frac{n^2}{n-2}}_{L_f^{2n/(n-2)}(M^n)} |||\nabla\varphi|||^n_{L_f^2(M^n)} \geq \mathfrak{S}_D^f ||\varphi||^{\frac{2n(n-1)}{n-2}}_{L_f^{2n/(n-2)}(M^n)}$$

from which one can get (2.4). □

In addition, we need the following lemma.

Lemma 2.4. *If $\varphi_1, \cdots, \varphi_k$ are linearly independent on M^n, $q > 2$, and $0 < \Gamma_1 \leq \Gamma_2 \leq \cdots \leq \Gamma_k$, then there exists a subset $\mathcal{T} \subset \{1, \cdots, k\}$ such that*

$$\int_{M^n} \left|\sum_{i=1}^k \Gamma_i \varphi_i\right|^q e^{-f} dv \leq \int_{M^n} \Gamma_k^q \left|\sum_{t\in\mathcal{T}} \varphi_t\right|^q e^{-f} dv. \tag{2.10}$$

Proof. Let

$$F(\Gamma_1, \cdots, \Gamma_k) = \int_{M^n} \left|\sum_{i=1}^k \Gamma_i \varphi_i\right|^q e^{-f} dv$$

and note that

$$\frac{\partial^2 F}{\partial \Gamma_i^2} = q(q-1) \int_{M^n} \varphi_i^2 \left|\sum_{i=1}^k \Gamma_i \varphi_i\right|^{q-2} e^{-f} dv \geq 0.$$

Thus F is convex in each variable. Therefore F may be majorized by, appropriately, replacing each Γ_i by 0 or Γ_k. But that is the claim. □

REMARK 2.5. The assumption of linearly independent in Lemma 2.4 can be removed. However, we will find that this condition automatically holds in the proof of Theorem 4.4.

3. Estimates for the weighted L^∞-norm of eigenfunctions

In this section, we investigate eigenfunctions of the Witten-Laplacian on smooth compact Riemannian manifolds with non-empty boundary. By making use of those functional inequalities and combining with De Giorgi-Moser iteration technique, we obtain some estimates for the L_f^∞-norm of eigenfunctions of the Witten-Laplacian. In other words, we prove the following conclusion.

Theorem 3.1. *Let M^n be an n-dimensional compact Riemannian manifold with nonempty boundary. Denote $\mathfrak{c}_2(n)$ by*

$$\mathfrak{c}_2(n) = \prod_{j=0}^{\infty} \left(\frac{\beta^{2j}}{2\beta^j - 1}\right)^{1/\beta^j}$$

where

$$\beta = \begin{cases} \frac{n}{n-2}, & n > 2, \\ 2, & n = 2, \end{cases} \tag{3.1}$$

If u is an eigenfunction of the eigenvalue Γ with the Dirichlet problem (1.2)*, then*

$$\|u\|_{L_f^\infty(M)} \leq \mathfrak{c}_2(n)^{1/2}\left(\frac{\Gamma}{\mathfrak{c}_1(n)}\right)^{\frac{\beta}{2(\beta-1)}}\|u\|_{L_f^2(M)},$$

where $\mathfrak{c}_1(n)$ *is given by*

$$\mathfrak{c}_1(n) = \begin{cases} \left[\mathfrak{S}_D^f\right]^{2/n}\{(n-2)/2(n-1)\}^2, & n > 2, \\ \mathfrak{S}_D^f/\left(4\left(V_f(M)\right)^{1/2}\right), & n = 2. \end{cases}$$

Proof. Notice that for any $\psi \in \mathcal{H}^1_{0,f}(M^n) \cap C^2(M^n \backslash \partial M^n)$, and $\alpha \geq 1$, it follows from the weighted Green formula (1.1) that

$$\begin{aligned} -\left[|\psi|^{2\alpha-2}\psi, \psi\right] &= \mathfrak{D}\left[|\psi|^{2\alpha-2}\psi, \psi\right] = (2\alpha-1)\int_{M^n}|\psi|^{2\alpha-2}|\nabla\psi|^2 e^{-f}dv \\ &= \left[(2\alpha-1)/\alpha^2\right]\||\nabla|\psi|^\alpha|\|^2_{L_f^2(M^n)}. \end{aligned} \tag{3.2}$$

So for the eigenfunction u of the eigenvalue Γ, we have

$$\Gamma\|u\|^{2\alpha}_{L_f^{2\alpha}(M^n)} = \left[(2\alpha-1)/\alpha^2\right]\||\nabla|u|^\alpha|\|^2_{L_f^2(M^n)}. \tag{3.3}$$

Then (2.3), (2.4) and (3.3) imply

$$\Gamma\|u\|^{2\alpha}_{L_f^{2\alpha}(M^n)} \geq \mathfrak{c}_1(n)\left[(2\alpha-1)/\alpha^2\right]\||u|^\alpha\|^2_{L_f^{2\beta}(M^n)}. \tag{3.4}$$

Let α range over the numbers β^k, where $k = 0, 1, 2, \cdots$, then we obtain

$$\|u\|_{L_f^{2\beta^{k+1}}(M^n)} \leq \|u\|_{L_f^2(M^n)}\prod_{j=0}^{k}\left\{\frac{\Gamma}{\mathfrak{c}_1(n)}\frac{\beta^{2j}}{2\beta^j-1}\right\}^{1/2\beta^j},$$

for all $k = 0, 1, 2, \cdots$. By (2.1), we have

$$\begin{aligned} \|u\|_{L_f^\infty(M^n)} &\leq \|u\|_{L_f^2(M^n)}\prod_{j=0}^{\infty}\left\{\frac{\Gamma}{\mathfrak{c}_1(n)}\frac{\beta^{2j}}{2\beta^j-1}\right\}^{1/2\beta^j} \\ &= \|u\|_{L_f^2(M^n)}\left\{\frac{\Gamma}{\mathfrak{c}_1(n)}\right\}^{\frac{\beta}{2(\beta-1)}}\prod_{j=0}^{\infty}\left\{\frac{\beta^{2j}}{2\beta^j-1}\right\}^{1/2\beta^j} \\ &= \mathfrak{c}_2(n)^{1/2}\|u\|_{L_f^2(M^n)}\left\{\frac{\Gamma}{\mathfrak{c}_1(n)}\right\}^{\frac{\beta}{2(\beta-1)}}, \end{aligned}$$

which is the claim. □

Remark 3.2. By observing the proof of Theorem 3.1, we know that the De Giorgi-Moser iteration technique is used to give the estimates for the weighted L_f^∞ norm of the eigenfunctions. In fact, the similar iteration method will be used again to establish the upper bounds for the multiplicity of eigenvalues in next section.

4. Bounds for the multiplicity of eigenvalues

In this section, we investigate the multiplicity of eigenvalues. For this purpose, we firstly need to prove the following proposition.

Proposition 4.1. *Let E be a finite-dimensional subspace of $L^2_f(M^n)\cap C^0(M^n)$. Then there exists $\psi\in E\backslash\{0\}$ such that*

$$(\dim E)\|\psi\|^2_{L^2_f(M^n)}\leq\|\psi\|^2_{L^\infty_f(\Omega)}V_f(M^n).$$

Proof. Let $\{\psi_i\}_{i=1}^r$ be an orthonormal basis of E with respect to the inner product of $L^2_f(M^n)$, and set

$$F(x)=\sum_i\psi_i^2>0. \tag{4.1}$$

One easily sees that F is independent of the choice of the orthonormal basis. Also we may assume that F is not identically zero. Let $\|F\|_{L^\infty_f(\Omega)}=F(x_0)>0$ for some $x_0\in M^n$. Define the subspace E_0 of E, which consists of those functions in E vanishing at x_0. Then, we have $\dim E_0=\dim E-1$ and $E=E_0\oplus E_0^\perp$, where $E_0^\perp$ denotes the orthocomplement space of E_0 in E. We can assume that $\psi_1\in E_0^\perp$, which implies that $\psi_1(x_0)\neq 0$ and $\psi_i(x_0)=0$ for any $i=2,\ldots,r$. Therefore, from (4.1), we have

$$\begin{aligned}\dim E&=\sum_i\|\psi_i\|^2_{L^2_f(M^n)}=\|F\|_{L^1_f(M^n)}\leq\|F\|_{L^\infty_f(\Omega)}V_f(M^n)\\&=\psi_1^2(x_0)V_f(M^n)\leq\|\psi_1\|^2_{L^\infty_f(\Omega)}V_f(M^n),\end{aligned}$$

which implies that $\psi=\psi_1$ is the desired function. □

Clearly, according to Theorem 3.1 and Proposition 4.1, we have the following.

Corollary 4.2. *If Γ is an eigenvalue of Dirichlet eigenvalue problem* (1.2)*, with multiplicity m_Γ, then*

$$m_\Gamma\leq\mathfrak{c}_2(n)\begin{cases}(\Gamma/\mathfrak{c}_1(n))^{n/2}V_f(M^n), & n>2,\\ \left(4\Gamma V_f(M^n)/\Im^f_D\right)^2, & n=2,\end{cases} \tag{4.2}$$

where $\mathfrak{c}_1(n)$ is given by (2.5)*.*

Remark 4.3. One might describe (4.2) as providing a lower bound of Γ in terms of its multiplicity m_Γ.

We now give an adjusted version of the above argument which will produce a lower bound for Γ_k, in terms of k. To this aim, we let $0<\Gamma_1\leq\Gamma_2\leq\cdots\leq\Gamma_k$ be the first k eigenvalues of M^n, $\{u_1,u_2,\cdots,u_k\}$ orthonormal eigenfunctions such that u_i is an eigenfunction of Γ_i for each $i=1,\cdots,k$. Set $E=span\{u_1,\cdots,u_k\}$. We assume that $n>2$, and let $\mathfrak{c}_1(n)$, β, be defined as in (2.5), (3.1), respectively. By Theorems 2.3 and 3.1, we have for any $\psi\in E$, $\alpha\geq 1$,

$$-\left[|\psi|^{2\alpha-2}\psi,\psi\right]\geq\mathfrak{c}_1(n)[(2\alpha-1)/\alpha^2]\||\psi|^{2\alpha}\|_{L_f^{n/(n-2)}(M^n)}. \tag{4.3}$$

Let $\alpha=1$, then (4.3) implies

$$\|\psi\|^2_{L_f^{2n(n-2)}(M^n)}\leq-\mathfrak{c}_1(n)^{-1}\left[\psi,\psi\right]\leq(\Gamma_k/\mathfrak{c}_1(n))\,\|\psi\|^2_{L^2_f(M^n)},$$

that is,

$$\|\psi\|_{L_f^{2n(n-2)}(M^n)} \le (\Gamma_k/\mathfrak{c}_1(n))^{1/2}\,\|\psi\|_{L_f^2(M^n)}. \tag{4.4}$$

For any fixed j in $\{0, 1, 2, \cdots\}$, let $h_j \in E$ have the property that

$$\frac{\|\varphi\|_{L_f^{2\beta^{j+1}}(M^n)}}{\|\varphi\|_{L_f^2(M^n)}} \le \frac{\|h_j\|_{L_f^{2\beta^{j+1}}(M^n)}}{\|h_j\|_{L_f^2(M^n)}} \tag{4.5}$$

for all $\varphi \in E$. The function h_j, can be written as

$$h_j = \sum_{i=1}^{k} \eta_i u_i.$$

By setting $\alpha = \beta^j$ in (4.3), and using Lemma 2.4, we obtain

$$\left\||h_j|^{2\beta^j}\right\|_{L_f^{\beta}(M^n)} \le -\frac{\beta^{2j}}{2\beta^j - 1}\frac{1}{\mathfrak{c}_1(n)}\left[|h_j|^{2\beta^j-2}h_j, h_j\right] \le \frac{\beta^{2j}}{2\beta^j - 1}\frac{1}{\mathfrak{c}_1(n)}\left\|h_j\right\|^{2\beta^j-1}_{L_f^{2\beta^j}(M^n)}\left\|\Delta_f h_j\right\|_{L_f^{2\beta^j}(M^n)},$$

and, by noticing (4.5),

$$\begin{aligned}
\left\|\Delta_f h_j\right\|_{L_f^{2\beta^j}(M^n)} &\le \left\|\Delta_f h_j\right\|_{L_f^{2\beta^{j+1}}(M^n)}\left[V_f(M^n)\right]^{(\beta-1)/2\beta^{j+1}}\\
&= \Big\|\sum_i \Gamma_i\eta_i u_i\Big\|_{L_f^{2\beta^{j+1}}(M^n)}\left[V_f(M^n)\right]^{(\beta-1)/2\beta^{j+1}}\\
&\le \Gamma_k\Big\|\sum_{t\in\mathcal{T}} \eta_t u_t\Big\|_{L_f^{2\beta^{j+1}}(M^n)}\left[V_f(M^n)\right]^{(\beta-1)/2\beta^{j+1}}\\
&\le \Gamma_k\Big\{\|h_j\|_{L_f^{2\beta^{j+1}}(M^n)}/\|h_j\|_{L_f^2(M^n)}\Big\}\Big\|\sum_{t\in\mathcal{T}} \eta_t u_t\Big\|_{L_f^2(M^n)}\left[V_f(M^n)\right]^{(\beta-1)/2\beta^{j+1}}\\
&\le \Gamma_k\|h_j\|_{L_f^{2\beta^{j+1}}(M^n)}\left[V_f(M^n)\right]^{(\beta-1)/2\beta^{j+1}}.
\end{aligned}$$

Therefore,

$$\left\{\|h_j\|_{L_f^{2\beta^{j+1}}(M^n)}\right\}^{2\beta^j-1} \le \frac{\Gamma_k}{\mathfrak{c}_1(n)}\frac{\beta^{2j}}{2\beta^j-1}\left[V_f(M^n)\right]^{(\beta-1)/2\beta^{j+1}}\left\{\|h_j\|_{L_f^{2\beta^j}(M^n)}\right\}^{2\beta^j-1},$$

which implies

$$\frac{\|h_j\|_{L_f^{2\beta^{j+1}}(M^n)}}{\left[V_f(M^n)\right]^{1/2\beta^{j+1}}} \le \left\{\frac{\Gamma_k\left[V_f(M^n)\right]^{2/n}}{\mathfrak{c}_1(n)}\frac{\beta^{2j}}{2\beta^j-1}\right\}^{1/(2\beta^j-1)}\frac{\|h_j\|_{L_f^{2\beta^j}(M^n)}}{\left[V_f(M^n)\right]^{1/2\beta^j}}. \tag{4.6}$$

Furthermore, one can obtain via iteration

$$\frac{\|\psi\|_{L_f^{2\beta^{j+1}}(M^n)}}{\|\psi\|_{L_f^2(M^n)}\left[V_f(M^n)\right]^{1/2\beta^{j+1}}} \le \frac{\|h_0\|_{L_f^{2\beta}(M^n)}}{\|h_0\|_{L_f^2(M^n)}}\left[V_f(M^n)\right]^{-1/2\beta}\prod_{i=1}^{j}\left\{\frac{\Gamma_k\left[V_f(M^n)\right]^{2/n}}{\mathfrak{c}_1(n)}\frac{\beta^{2i}}{2\beta^i-1}\right\}^{1/(2\beta^i-1)}$$

for all $\psi \in E$, $j = 0, 1, 2, \cdots$, which implies

$$\frac{\|\psi\|_{L_f^{\infty}(M^n)}}{\|\psi\|_{L_f^2(M^n)}} \le \frac{\|h_0\|_{L_f^{2\beta}(M^n)}}{\|h_0\|_{L_f^2(M^n)}}\left[V_f(M^n)\right]^{-1/2\beta}\prod_{i=1}^{\infty}\left\{\frac{\Gamma_k\left[V_f(M^n)\right]^{2/n}}{\mathfrak{c}_1(n)}\frac{\beta^{2i}}{2\beta^i-1}\right\}^{1/(2\beta^i-1)}, \tag{4.7}$$

for all $\psi \in E$. Now $i \geq 1, \beta > 1$ imply

$$1/2\beta^i \leq 1/(2\beta^i - 1) \leq 1/\beta^i,$$

from which one obtains

$$1/2(\beta - 1) \leq \sum_{i=1}^{\infty} 1/(2\beta^i - 1) \leq 1/(\beta - 1).$$

Set

$$C = \prod_{i=1}^{\infty} \left\{ \frac{\beta^{2i}}{2\beta^i - 1} \right\}^{1/(2\beta^i - 1)}.$$

If

$$\frac{\Gamma_k \left[V_f(M^n) \right]^{2/n}}{\mathfrak{c}_1(n)} \geq 1, \tag{4.8}$$

then (4.4) and (4.7) imply

$$\frac{\|\psi\|_{L_f^\infty(M^n)}}{\|\psi\|_{L_f^2(M^n)}} \leq C \left\{ \frac{\Gamma_k}{\mathfrak{c}_1(n)} \right\}^{(n-1)/2} \left[V_f(M^n) \right]^{(n-2)/2n}$$

for all $\psi \in E$, and Proposition 4.1 implies

$$k \leq C^2 \left\{ \frac{\Gamma_k \left[V_f(M^n) \right]^{2/n}}{\mathfrak{c}_1(n)} \right\}^{n-1}$$

for all k for which (4.8) is valid. If, on the other hand,

$$\frac{\Gamma_k \left[V_f(M^n) \right]^{2/n}}{\mathfrak{c}_1(n)} \leq 1,$$

then (4.6), for $j = 0$, and (4.7) imply

$$\frac{\|\psi\|_{L_f^\infty(M^n)}}{\|\psi\|_{L_f^2(M^n)}} \leq \frac{C}{\left[V_f(M^n) \right]^{1/2}} \left\{ \frac{\Gamma_k V_f(M^n)}{\mathfrak{c}_1(n)} \right\}^{(n+2)/4},$$

and, by Proposition 4.1, we have

$$k \leq C^2 \left\{ \frac{\Gamma_k V_f(M^n)}{\mathfrak{c}_1(n)} \right\}^{n/2+1}.$$

When $n = 2$, using the same argument as before, one can concludes that

$$k \leq C^2 \left\{ \frac{4\Gamma_k V_f(M^n)}{\mathfrak{S}_D^f} \right\}^3$$

for all $k = 1, 2, 3, \cdots$, where

$$C = \prod_{j=1}^{\infty} \left\{ \frac{4^j}{2^{j+1} - 1} \right\}^{1/(2^{j+1} - 1)}.$$

We summarize the above discussion as follows.

Theorem 4.4. *There exists a positive constant $C(n)$, depending only on $n = \dim M^n$, such that*

$$k \leq C(n)\begin{cases} \left\{\frac{\Gamma_k\left[V_f(M^n)\right]^{n/2}}{\mathfrak{c}_1(n)}\right\}^{n-1}, & n > 2, \\ \left[\frac{4\Gamma_k V_f(M^n)}{\mathfrak{S}_D^f}\right]^3, & n = 2, \end{cases} \tag{4.9}$$

for all $k \geq C(n)$. If $k \leq C(n)$ then (4.9) remains valid if $\Gamma_k\left[V_f(M^n)\right]^{2/n}/\mathfrak{c}_1(n) > 1$ (resp., $4\Gamma_k V_f(M^n)/\mathfrak{S} > 1$) and $n > 2$ (resp., $n = 2$). Otherwise, we have

$$k \leq C(n)\begin{cases} \left\{\frac{\Gamma_k\left[V_f(M^n)\right]^{n/2}}{\mathfrak{c}_1(n)}\right\}^{n/2+1}, & n > 2, \\ \left[\frac{4\Gamma_k V_f(M^n)}{\mathfrak{S}_D^f}\right]^3, & n = 2, \end{cases}$$

when

$$\left\{\frac{\Gamma_k\left[V_f(M^n)\right]^{n/2}}{\mathfrak{c}_1(n)}\right\}^{n-1} < 1 \quad \left(resp., \frac{4\Gamma_k V_f(M^n)}{\mathfrak{S}_D^f} < 1\right)$$

and $n > 2$ (resp., $n = 2$).

Remark 4.5. In Theorem 4.4, we also give a lower bound of eigenvalues of the eigenvalue problem (1.2) without any curvature assumption.

5. Uniform convergence and uniqueness for the heat kernel

In this section, let us first consider the heat kernel of parabolic equation with IC (1.4) and Dirichlet BC (1.5) of the Witten-Laplacian on smooth metric measure spaces. In particular, any function $\upsilon_0 \in L_f^2(M^n)$ can be written in the form

$$\upsilon_0(x) = \sum_{i=1}^{\infty} a_i\phi_i(x)$$

with

$$a_i = \int_{M^n} \upsilon_0\phi_i e^{-f}dx.$$

Formally, the function given by

$$\upsilon(x,t) = \sum_{i=1}^{\infty} e^{-\Gamma_i t}a_i\phi_i(x)$$

will satisfy (1.3) with Dirichlet BC (1.5) and IC (1.4). This is equivalent to saying that

$$\upsilon(x,t) = \int_{M^n} \Psi_f(x,y,t)\upsilon_0(y)e^{-f}dy, \tag{5.1}$$

if we define the kernel by

$$\Psi_f(x,y,t) = \sum_{i=1}^{\infty} e^{-\Gamma_i t}\phi_i(x)\phi_i(y). \tag{5.2}$$

Let us point out that the heat kernel defined by (5.2) is a formal sum, which means that $\upsilon(x,t)$ defined by (5.1) is just a formal solution to (1.3) with initial-boundary value conditions (1.4) and (1.5). Therefore, it is necessary for us to discuss the uniform convergence of the heat kernel (5.2) to guarantee that the formal solution is a suitable candidate. To this end, by applying Theorems 3.1 and 4.4, we give the uniform upper bound of the heat kernel of parabolic equation of Witten-Laplacian on smooth metric measure spaces. At this point, we extend a result given by Chavel and Feldman in [2] to the metric measure spaces.

Theorem 5.1. *Given positive constants τ,μ,γ, then the formal heat kernel, $\Psi_f:\overline{M}^n\times\overline{M}^n\times(0,\infty)\to\mathbb{R}$, associated to the Dirichlet eigenvalue problem* (1.2)*, and given by*

$$\Psi_f(x,y,t) = \sum_{i=1}^{\infty} e^{-\Gamma_i t}u_i(x)u_i(y) \tag{5.3}$$

(where $u_1,u_2,\cdots$ is a complete orthonormal basis of $L^2_f(M^n)$, with each u_i an eigenfunction of Γ_i, $i=1,2,\cdots$) converges uniformly on $\overline{M}^n\times\overline{M}^n\times[\tau,\infty)$, and for $V_f(M^n)\in(0,\mu)$, $\mathfrak{S}^f_D\in[\gamma,\infty)$. Furthermore, an upper bound on $V_f(M^n)$, and a lower bound on $\mathfrak{S}^f_D$ provide a uniform upper bound on the heat kernel, for times bounded away from zero.

Proof. We give the proof for $n>2$. For $n=2$ the argument is similar. For all $t\geq\tau>0$, by Theorem 3.1, we have

$$|\Psi_f(x,y,t)| \leq c_1\sum_{k=1}^{\infty}\Gamma_k^{n/2}e^{-\Gamma_k\tau},$$

where c_1 only depends on γ and the dimension n. Next, one determines $\alpha_0 := \alpha_0(n,\tau)>0$ for which

$$\alpha^{5n/2-2}e^{-\alpha\tau}\leq 1$$

for all $\alpha^2\geq\alpha_0$. From Theorem 4.4 we have

$$\Gamma_k\geq c_2(n,\mu,\gamma)k^{1/(n-1)}$$

for all $k\geq C(n)$, where c_2 is a constant depending on n,μ,γ. Taking

$$N=N(n,\mu,\gamma)=\left[\left(\frac{\alpha_0}{c_2(n,\mu,\gamma)}\right)^{n-1}\right]+1,$$

where $[x]$ denotes a positive integer which is larger than or equals to x, we have

$$\Gamma_k\geq\alpha_0$$

for all $k>N$, which implies

$$\sum_{k=N+1}^{\infty}\Gamma_k^{n/2}e^{-\Gamma_k\tau}\leq c_3(n,\mu,\gamma)\sum_{k=N+1}^{\infty}\Gamma_k^{-2(n-1)}\leq c_4(n,\mu,\gamma)\sum_{k=N+1}^{\infty}k^{-2},$$

where both c_3 and c_4 are two constants depending on n,μ,γ. Since the series

$$\sum_{k=N+1}^{\infty} k^{-2}$$

is convergent, the fundamental solution $\Psi_f(x, y, t)$ is uniformly convergent. Since

$$\lim_{\alpha\to 0} \alpha^{n/2} e^{-\alpha\tau} = \lim_{\alpha\to+\infty} \alpha^{n/2} e^{-\alpha\tau},$$

there is a critical point $\alpha_1 = \alpha_1(n, \tau) > 0$ of the function $\vartheta(\alpha) = \alpha^{n/2} e^{-\alpha\tau}$ with respect to the variable α, such that

$$c_5 := c_5(n, \tau) = \alpha_1^{n/2} e^{-\alpha_1\tau} = \sup_{\alpha>0} \alpha^{n/2} e^{-\alpha\tau},$$

which implies that

$$\sum_{k=1}^{N} \Gamma_k^{n/2} e^{-\Gamma_k\tau} \leq c_5 N.$$

Therefor, we finish the proof of this theorem. □

Remark 5.2. Note that any compact Riemannian manifold with boundary has a geodesically complete Riemannian extension (see Theorem A and Corollary B in [20]), so it can be treated as a domain of a complete Riemannian manifold. However, the convergence of (5.3) is uniform not only for fixed manifolds, but also for the set of isomorphism classes of n-dimensional weighted Riemannian manifold with boundary satisfying $V_f \leq \mu$ and $\mathfrak{S}_D^f \geq \gamma$.

Remark 5.3. According to Theorem 5.1, we know that kernel function given by (5.3) is well defined on $M^n \times M^n \times (0, \infty)$ and satisfies the Dirichlet BC (1.5) in both x and y for $t > 0$.

Remark 5.4. By replacing the compact smooth metric measure spaces by the closed bounded domain on the complete smooth metric measure spaces, the arguments will be same through out this paper.

Next, we will prove the uniqueness for the parabolic heat kernel.

Theorem 5.5. *Let M^n be a compact manifold with boundary. The kernel function given by* (5.3) *is the unique kernel, such that for any $\upsilon_0 \in L_f^2(M^n)$, the function given by*

$$\upsilon(x, t) = \int_{M^n} \Psi_f(x, y, t)\upsilon_0(y)e^{-f} dy$$

solves the heat equation (1.3) *with Dirichlet BC* (1.5) *and initial condition* (1.4). *Moreover, $\Psi_f(x, y, t)$ is a smooth function on $(M^n\backslash\partial M^n) \times (M^n\backslash\partial M^n) \times (0, \infty)$, which satisfies*

$$\int_{M^n} \Psi_f(x, y, t)e^{-f(y)} dy \leq 1,$$

and, it is nonnegative for all x, y, t. In particular, we have $\upsilon(x, t) \geq 0$ on $M^n \times [0, +\infty)$ when $\upsilon_0 \geq 0$.

Proof. Noticing that

$$v_0(x) = \sum_{i=1}^{\infty} a_i u_i(x) \ \text{ in } \ L^2_f(M^n),$$

it is easy to see that

$$v(x,t) = \int_{M^n} \Psi_f(x,y,t) v_0(y) e^{-f} dy = \sum_{i=1}^{\infty} e^{-\Gamma_i t} a_i u_i(x). \tag{5.4}$$

Next, let us show that the series (5.3) converges in $C^\infty(N^o \times \mathbb{R}_+)$, where N^o denotes $(M^n \backslash \partial M^n) \times (M^n \backslash \partial M^n)$. For this purpose, we assume that N is a product manifold $M^n \times M^n$ with the smooth product Riemannian metric and the product volume measure $d\nu = d\mu d\mu$. Since from [9, Theorem 7.4], one has

$$\Delta_{f,x} \Psi_f(x,y,t) = \frac{\partial}{\partial t} \Psi_f(x,y,t) = \Delta_{f,y} \Psi_f(x,y,t),$$

the function $\Psi_f(x,y,t)$ satisfies the heat equation

$$\frac{\partial}{\partial t} w(x,y,t) = \frac{1}{2} \Delta_{f,N} w(x,y,t)$$

with respect to the drifting Laplace operator $\Delta_{f,N} := \Delta_{f,x} + \Delta_{f,y}$ on the product manifold N, where $\Delta_{f,x}, \Delta_{f,y}$ denote the drifting Laplacian on M^n with respect to the variables x and y, respectively, and it is straightforward to check that each function

$$\chi_k(x,y,t) = e^{-t\Gamma_k} u_k(x) u_k(y)$$

also satisfies the same equation. One then knows that the series $\sum_k \chi_k(x,y,t)$ locally converges to $\Psi_f(x,y,t)$ in $L^2_f(N^o \times \mathbb{R}_+)$, which means that $\Psi_f(x,y,t)$ and its derivatives are continuous functions on $N^o \times \mathbb{R}_+$, i.e., all these derivatives are the weak derivatives of $\Psi_f(x,y,t)$ on $N^o \times \mathbb{R}_+$. Hence, the following equation

$$\frac{\partial}{\partial t} \Psi_f(x,y,t) = \frac{1}{2} \Delta_{f,N} \Psi_f(x,y,t)$$

is satisfied in the weak sense in $N^o \times \mathbb{R}_+$. Furthermore, by Theorem 7.4 in [9], we know that it is C^∞ topologically convergent on $N^o \times \mathbb{R}_+$.

On the meanwhile, elliptic regularity theory (see Theorem 5 and Theorem 6 of Chapter 6 in [5]) guarantees that $v(x,t) \in C^\infty(M^n)$ for any fixed $t > 0$. For any non-negative integer m, since the sequence $\sum_{i=1}^{\infty} \Gamma_i^m e^{-(t+\varepsilon)\Gamma_i} a_i u_i(x)$ is uniformly convergent with respect to the variable t as $\varepsilon \to 0$, we have

$$\Delta_f^m \left(\sum_{i=1}^{\infty} e^{-\Gamma_i (t+\varepsilon)} a_i u_i(x) \right) = \sum_{i=1}^{\infty} \Gamma_i^m e^{-(t+\varepsilon)\Gamma_i} a_i u_i(x).$$

Furthermore, the function $\Gamma_i^m e^{-(t+\varepsilon)\Gamma_i} u_i(x)$ remains uniformly bounded in Γ_i as $\varepsilon \to 0$, and thus, one can switch the order of the limit under and the sum sign to yield

$$\Delta_f^m \left(\sum_{i=1}^{\infty} e^{-\Gamma_i (t+\varepsilon)} a_i u_i(x) \right) \to \Delta_f^m \left(\sum_{i=1}^{\infty} e^{-\Gamma_i t} a_i u_i(x) \right),$$

as $\varepsilon \to 0$ in the sense of L^2_f-norm. Therefore, for any non-negative integer k, we have

$$v(t+\varepsilon,\cdot)\xrightarrow{\mathcal{W}_f^{2k}} v(t,\cdot) \text{ as } \varepsilon\to 0,$$

i.e., $v(t+\varepsilon,\cdot)$ convergents to $v(t,\cdot)$, as $\varepsilon\to 0$ in the sense of the norm $\|\cdot\|_{\mathcal{W}_f^{2k}} := \|\cdot\|_{2k,2,f}$, which means that, for any $t>0$,

$$v(t+\varepsilon,\cdot)\xrightarrow{C^\infty} v(t,\cdot) \text{ as } \varepsilon\to 0,$$

which is equivalent to say that $v(t+\varepsilon,\cdot)$ is C^∞-topologically convergent to $v(t,\cdot)$ as $\varepsilon\to 0$. Thus, it is clear that $v(t,x)$ is continuous in t locally uniformly in x, which implies that $v(t,x)$ is continuous jointly in t,x. Noticing that $v_0\in L^2_f(M^n)$, applying the same argument as the proof of Theorem 7.10 in [9], one can show that $v(x,t)$ solves the Dirichlet heat equation (1.3) with initial-boundary conditions (1.4) and (1.5) in the weak sense. Furthermore, using Theorem 7.4 in [9] again, one can see that $v(x,t)$ is a classical solution of the Dirichlet heat equation (1.3) with initial-boundary conditions (1.4) and (1.5).

The strong maximum principle (see Theorem 2.9 in [14]) implies that

$$v(x,t)=\int_{M^n}\Psi_f(x,y,t)v_0(y)e^{-f}dy\geq 0$$

on $(M^n\backslash\partial M^n)\times(0,\infty)$ whenever $v_0\geq 0$ on M^n. Hence, it is not difficult to see that $\Psi_f(x,y,t)$ is also nonnegative on $N^o\times\mathbb{R}_+$. Next, we consider that $\overline{v}(x,t)$ is another solution of the heat equation satisfies IC (1.4). According to the maximum principle (see Theorem 2.9 in [14]), we know that the solution of the heat equation satisfies $v(x,t)-\overline{v}(x,t)\equiv 0$ since it vanishes on $(M^n\times\{0\})\cup(\partial M^n\times(0,\infty))$. Thus, we conclude that the fundamental solution $\Psi_f(x,y,t)$ is unique. Also, Theorem 2.9 in [14] implies that the property that

$$\int_{M^n}\Psi_f(x,y,t)e^{-f}dy\leq 1$$

follows from the fact that it solves the heat equation with initial condition $v_0=1$. □

REMARK 5.6. In particular, assume the boundary ∂M^n is smooth, then one can conclude that $\Psi_f(x,y,t)\in C^\infty(\overline{M}^n\times\overline{M}^n\times(0,\infty))$.

REMARK 5.7. Based on some properties of heat semigroup, Grigor'yan discussed some important properties of abstract kernel functions without initial-boundary conditions, see Theorem 7.13 and its remark in his celebrated book [9], where the existence of the heat kernel is established by the Riesz representation theorem. Therefore, Grigor'yan's argument is so abstruse and non-trivial. However, given initial-boundary conditions, we can concretely construct this heat kernel via the eigenfunction expansion formula, which enables us to reprove some properties of kernel functions by a simple and elementary strategy.

ACKNOWLEDGEMENTS. The first author is supported by the Natural Science Foundation of Jiangxi Province (Grant No. 20171ACB21023) and the National Natural Science Foundation of China (Grant No. 11861036, 11826213). The second author is supported by the National Natural Science Foundation of China (Grant No. 11761080). The corresponding author, Prof. J. Mao, was partially supported by the National Natural Science Foundation of China (Grant No. 11801496), the Fok Ying-Tung Education Foundation (China), and Hubei

Key Laboratory of Applied Mathematics (Hubei University). The authors will express their sincere thanks to the anonymous referee for helpful comments and suggestions.

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