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LOOP-ERASED RANDOM WALKS ON RANDOM FRACTALS

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Abstract

We introduce random pre-branched Koch curves and construct loop-erased random walks on these graphs. We prove the existence of the scaling limit and show that the sample path of the limit process is almost surely self-avoiding, while it has box-counting dimension strictly greater than 1.

1. Introduction

A loop-erased random walk (LERW) is a process wherein loops are erased from a simple random walk in chronological order. It is a non-Markov walk whose path has no self-intersection. Since its introduction on $\mathbb{Z}^d$ by Lawler ([11]), this process has been extensively studied. These studies have demonstrated the existence of the scaling limit on $\mathbb{Z}^d$ for all d. See, for example, [15] and [16] for $d = 2$, [10], [17], [18] and [22] for $d = 3$, and [12] and [13] for $d \geqq 4$. Also the growth exponents for LERW have been obtained. The growth exponent for a random walk is the exponent for the number of steps needed to travel distance N as N tends to infinity. See for example, [9], [19] and [14] for $d = 2$, [21] for $d = 3$, and [12] and [13] for $d \geqq 4$.

Studies such as [20], [8] and [7] have investigated LERWs on a fractal space, namely, the Sierpiński gasket. Cao ([2]) studied loop-erased random paths on more general graphs including some fractals.

A next step in investigating LERWs is to consider thier behavior within a random environment. Brownian motion, which is the scaling limit of a simple random walk, has been studied on random fractals, including random Sierpiński gaskets. See, for example, [5], [6], and [1]. Whilest these papers studied Markov processes on random fractals, in this paper, we deal with a non-Markov process on a random fractal. As far as the authors know, this paper will be the first attempt in this direction. We work on a random branched Koch curve so that we can make use of some results in [5] and to see how far we can go. In the process we found that the 3-dimensional version of the branched Koch curve is of interest in itself; it is not clear at first sight that the fractal satisfies the open set condition; in fact, it requires a somewhat lengthy proof.

We prove the following theorems:

Theorem 1. *For almost every environment $\overline{\nu}$, the loop-erased random walk on the random pre-branched Koch curve converges almost surely to a continuous process X as the edge length tends to* 0.

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Theorem 2. *For almost every environment $\overline{\nu}$, the sample path of the limit process is almost surely self-avoiding. Specifically, for any $0 \leqq t_1 < t_2 \leqq T$, $X(t_1) \neq X(t_2)$, where T is the time when the process reaches the end point of the random Koch curve.*

Theorem 3. *For almost every environment $\overline{\nu}$, the sample path regarded as a closed set almost surely has box-counting dimension*

$$d_B = \frac{\log \lambda_2^{1-p} \lambda_3^{p}}{\log 3},$$

where $0 < p < 1$ is a constant that determines our random environment, $\lambda_2 = 10/3$, and $\lambda_3 = 29/8$. In particular, d_B is strictly greater than 1.

Our main tool for proving the above results is the 'erasing-larger-loops-first' (ELLF) method, which was introduced to study LERW on the Sierpiński gasket [8]. In contrast to the 'standard' LERW obtained by erasing loops in chronological order, our LERW is constructed by erasing loops in descending order of the size of the loops; the resulting LERW is proved to have the same distribution as the 'standard' LERW.

The structure of this paper is as follows. In Section 2, we define the random branched Koch curve, which is the space we work on, and in Section 3 we construct random walks on the random pre-branched Koch curves. In Section 4, we recall the ELLF method of loop-erasing. Section 5 focuses on the generating functions of hitting times, which are crucial for all of the proofs concerning the existence of the scaling limit in Section 6. Section 7 is devoted to the proof for the self-avoiding property and the derivation of the box-counting dimension of the limit process.

2. Construction of the random branched Koch curve

2.1. Branched Koch curve and its 3-dimensional version. In this subsection, we define the (non-random) branched Koch curve and its three-dimensional version.

To construct our fractals we begin with the definition of similitudes $f_i : \mathbb{R}^3 \to \mathbb{R}^3$, $i = 1, 2, \ldots, 8$:

$$f_1((x, y, z)) = \frac{1}{3}(x, y, z),$$

$$f_2((x, y, z)) = \frac{1}{3}(x, y, z) + \left(\frac{2}{3}, 0, 0\right),$$

$$f_3((x, y, z)) = \frac{1}{3}(-x, -y, -z) + \left(\frac{2}{3}, 0, 0\right),$$

$$f_4((x, y, z)) = \left(-\frac{1}{6}x + \frac{\sqrt{3}}{6}y + \frac{2}{3},\ \frac{\sqrt{3}}{6}x + \frac{1}{6}y,\ -\frac{1}{3}z\right),$$

$$f_5((x, y, z)) = \left(-\frac{1}{6}x - \frac{\sqrt{3}}{6}y + \frac{1}{2},\ -\frac{\sqrt{3}}{6}x + \frac{1}{6}y + \frac{\sqrt{3}}{6}\ , -\frac{1}{3}z\right),$$

$$f_6((x,y,z)) = \left(\frac{1}{6}x - \frac{\sqrt{3}}{6}y + \frac{1}{3},\ \frac{\sqrt{3}}{18}x + \frac{1}{18}y - \frac{2\sqrt{2}}{9}z,\ \frac{\sqrt{6}}{9}x + \frac{\sqrt{2}}{9}y + \frac{1}{9}z\right),$$

$$f_7((x,y,z)) = \left(\frac{1}{6}x + \frac{\sqrt{3}}{6}y + \frac{1}{2},\ -\frac{\sqrt{3}}{18}x + \frac{1}{18}y - \frac{2\sqrt{2}}{9}z + \frac{\sqrt{3}}{18},\ -\frac{\sqrt{6}}{9}x + \frac{\sqrt{2}}{9}y + \frac{1}{9}z + \frac{\sqrt{6}}{9}\right),$$

$$f_8((x,y,z)) = \left(-\frac{\sqrt{3}}{9}y + \frac{\sqrt{6}}{9}z + \frac{1}{2},\ \frac{\sqrt{3}}{9}x + \frac{2}{9}y + \frac{\sqrt{2}}{9}z + \frac{\sqrt{3}}{18},\ -\frac{\sqrt{6}}{9}x + \frac{\sqrt{2}}{9}y + \frac{1}{9}z + \frac{\sqrt{6}}{9}\right).$$

We define two transformations $f^{(2)}$ and $f^{(3)}$ on the class of non-empty compact subsets of $\mathbb{R}^3$: for a non-empty compact set $A \in \mathbb{R}^3$, let

$$f^{(2)}(A) = \bigcup_{i=1}^{5} f_i(A),$$

$$f^{(3)}(A) = \bigcup_{i=1}^{8} f_i(A).$$

The branched Koch curve F^2 and the 3-dimensional branched Koch curve F^3 are fractals uniquely defined as the compact sets satisfying

$$F^2 = f^{(2)}(F^2),$$

and

$$F^3 = f^{(3)}(F^3),$$

respectively.

Since we study random walks, let us define pre-branched Koch curves, which are discrete versions of the fractals.

Let $F_0^2 = F_0$ be a unit line segment placed on the x-axis, namely, $F_0 = \{(s,0,0) \in \mathbb{R}^3 : 0 \leqq s \leqq 1\}$. We define recursively a sequence of closed sets $\{F_N^2\}$ starting from F_0^2: $F_{N+1}^2 = f^{(2)}(F_N^2)$ for $N \in \mathbb{Z}_+ = \{0,1,2,\ldots\}$. We call the sets F_N^2, $N \in \mathbb{Z}_+$, the **pre-branched Koch curves** (Fig.1 and Fig.2). We obtain the **branched Koch curve** F^2 by taking the closure of $\bigcup_{N=0}^{\infty} F_N^2$, which is a well-known fractal with Hausdorff dimension $\log 5/\log 3$ [3].

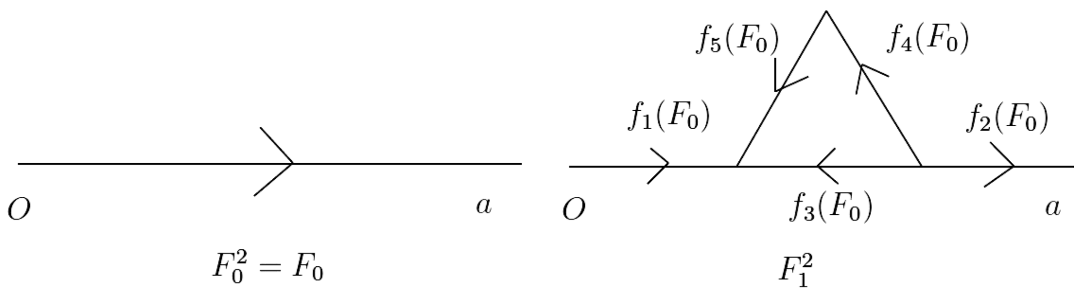

Fig. 1. Similitudes f_1–f_5 with their orientations.

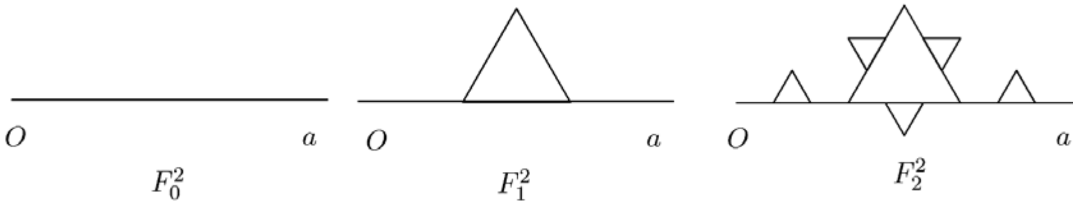

Fig. 2. First three steps of the construction of F^2.

To consider random walks on the pre-branched Koch curves, we regard F_N^2, $N \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, also as graphs consisting of the set of vertices G_N^2 and the set of edges E_N^2 defined as

$$O = (0,0,0), \quad a = (1,0,0),$$

$$G_0^2 = \{O, a\}, \quad G_{N+1}^2 = f^{(2)}(G_N^2), \quad N \in \mathbb{Z}_+,$$

and

$$E_0^2 = \{\{O, a\}\}, \quad E_{N+1}^2 = f^{(2)}(E_N^2), \quad N \in \mathbb{Z}_+.$$

The same notation F_N^2 will be used to represent both the closed set and the graph (G_N^2, E_N^2), for it will be clear in context what it means. In particular, F_0 can mean a unit line segment or a graph consisting of two vertices and an edge connecting them.

Next, we introduce the 3-dimensional pre-branched Koch curve.

Let $F_0^3 = F_0 = \{(s, 0, 0) \in \mathbb{R}^3 : 0 \leqq s \leqq 1\}$. We define recursively a sequence of closed sets $\{F_N^3\}$ using $F_{N+1}^3 = f^{(3)}(F_N^3)$ for $N \in \mathbb{Z}_+$. We call the sets F_N^3, $N \in \mathbb{Z}_+$, the **3-dimensional pre-branched Koch curves** (Fig.3). We obtain the **3-dimensional branched Koch curve** F^3 by taking the closure of $\bigcup_{N=0}^{\infty} F_N^3$.

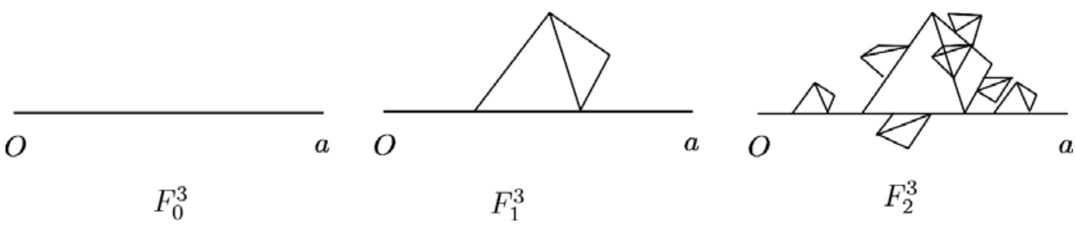

Fig. 3. First three steps of the construction of F^3.

We regard 3-dimensional pre-branched Koch curves also as graphs with the set of vertices G_N^3 and the set of edges E_N^3, defined as

$$G_0^3 = \{O, a\}, \quad G_{N+1}^3 = f^{(3)}(G_N^3) \quad N \in \mathbb{Z}_+,$$

and

$$E_0^3 = \{\{O, a\}\}, \quad E_{N+1}^3 = f^{(3)}(E_N^3) \quad N \in \mathbb{Z}_+.$$

Here again the same notation F_N^3 will be used to represent both the closed set and the graph (G^3_N, E^3_N).

Proposition 4. *There exists a nonempty open set V such that*

(i) $F^3 \subseteq \overline{V}$;

(ii) $\bigcup_{i=1}^{8} f_i(V) \subseteq V$;

(iii) $f_i(V) \cap f_j(V) = \emptyset, \quad if \ i \neq j$;

(iv) $f_i(\overline{V}) \cap f_j(\overline{V}) \subset G^3_1. \quad if \ i \neq j.$

Here $\overline{V}$ *denotes the closure of* V. *Conditions* (ii) *and* (iii) *constitute the open set condition* ([3])*, which leads to*

$$d_H(F^3) = \frac{\log 8}{\log 3}.$$

The explicit form of V and the proof of Proposition 4 are given in Appendix.

The 3-dimensional branched Koch curve may not be so well-known as its 2-dimensional couterpart, so let us give some geometrical explanations.

Let $O = (0,0,0)$, $a = (1,0,0)$, $b = \left(\frac{1}{3},0,0\right)$, $c = \left(\frac{2}{3},0,0\right)$, $d = \left(\frac{1}{2},\frac{\sqrt{3}}{6},0\right)$, $e = \left(\frac{1}{2},\frac{\sqrt{3}}{18},\frac{\sqrt{6}}{9}\right)$, and $f = \left(\frac{1}{2},-\frac{\sqrt{3}}{2},0\right)$.

Note that Oaf forms an equilateral triangle $\triangle Oaf$ with side length 1, and $bcde$ forms a tetrahedron with side length 1/3 (Fig.4).

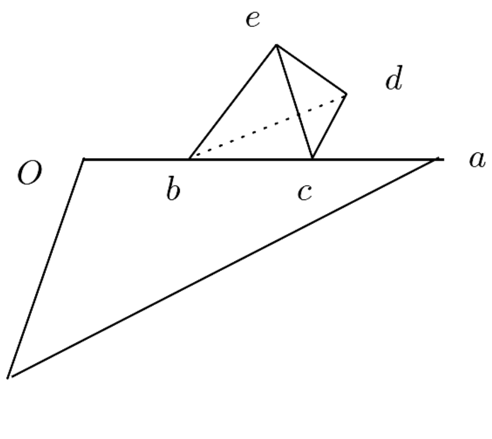

Fig.4. $\triangle Oaf$ and the tetrahedron $bcde$.

The similitude f_3 maps $\triangle Oaf$ onto the triangle $\triangle cbd$, preserving the order of the vertices, which means that $f_3(O) = c$, $f_3(a) = b$, and $f_3(f) = d$. f_4, f_5, f_6, f_7 and f_8 map $\triangle Oaf$ onto $\triangle cdb$, $\triangle dbc$, $\triangle bec$, $\triangle ecb$, and $\triangle edc$, respectively, each preserving the order of the vertices. If one maps the whole figure shown in Fig.4, then the image of e comes in the direction of $f_i(\overrightarrow{Of} \times \overrightarrow{Oa})$ relative to $f_i(\triangle Oaf)$.

2.2. Random branched Koch curve. We define a random fractal, using the definition of branched Koch curves given above. Let $\overline{\nu} = (\nu_1, \nu_2, \ldots)$ be an environment, where $\{\nu_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. random variables that take the value 2 with probability $1 - p$ and the value

3 with probability p, respectively, where $0 \leqq p \leqq 1$.

Let σ be the shift operator, $\sigma(\nu_1, \nu_2, \nu_3, \ldots) = (\nu_2, \nu_3, \ldots)$.

Recall that

$$f^{(2)}(A) = \bigcup_{i=1}^{5} f_i(A),$$

and

$$f^{(3)}(A) = \bigcup_{i=1}^{8} f_i(A).$$

Starting from $F_0 = \{(s, 0, 0) \in \mathbb{R}^3 : 0 \leqq s \leqq 1\}$, define

$$F_N(\overline{\nu}) = f^{(\nu_1)} \circ f^{(\nu_2)} \circ \cdots \circ f^{(\nu_N)}(F_0).$$

We call $F_N(\overline{\nu})$, $N \in \mathbb{N}$, the **random pre-branched Koch curves** and the closure of $\bigcup_{N=0}^{\infty} F_N(\overline{\nu})$, denoted by $F(\overline{\nu})$, the **random branched Koch curve**. In this paper, we will work on these random pre-fractals and the limiting random fractal. Some examples of $F_2(\overline{\nu})$ are shown in Fig.5.

We regard $F_N(\overline{\nu})$, $N \in \mathbb{Z}_+$ also as graphs; starting from $G_0(\overline{\nu}) = \{O, a\}$ and $E_0(\overline{\nu}) = \{\{O, a\}\}$, we define

$$G_N(\overline{\nu}) = f^{(\nu_1)} \circ f^{(\nu_2)} \circ \cdots \circ f^{(\nu_N)}(G_0(\overline{\nu})),$$

and

$$E_N(\overline{\nu}) = f^{(\nu_1)} \circ f^{(\nu_2)} \circ \cdots \circ f^{(\nu_N)}(E_0(\overline{\nu})).$$

Proposition 5. *The random fractal $F(\overline{\nu})$ has Hausdorff dimension d_H and Box dimension d_B:*

$$d_H = d_B = \frac{(1-p)\log 5 + p \log 8}{\log 3},$$

for almost every $\overline{\nu}$.

Since the proof is the same as that for the random Sierpiński gasket in [5] with equilateral triangles replaced by similitudes of the closure of V, we omit the proof here.

3. Random walk on the pre-RBK

3.1. Paths on the pre-RBK. In the following, we fix $\overline{\nu}$ and write F_N, G_N, and E_N for $F_N(\overline{\nu}), G_N(\overline{\nu})$ and $E_N(\overline{\nu})$, respectively, whenever no confusion occurs.

For each $N \in \mathbb{Z}_+$, define a set of finite paths $W'_N = W'_N(\overline{\nu})$ on the graph $F_N = (G_N, E_N)$ by

$$W'_N = \{w = (w(0), w(1), \cdots, w(n)) : \ w(i) \in G_N, \ \{w(i-1), w(i)\} \in E_N, \ 1 \leqq i \leqq n, \ n \in \mathbb{N}\}.$$

This gives the natural definition for the length (total number of steps) ℓ of a path $w = (w(0), w(1), \cdots, w(n)) \in W'_N$; namely, $\ell(w) = n$.

For a path $w \in W'_N$ and $A \subseteq G_M$ with $M \leqq N$, we define the hitting time of A by

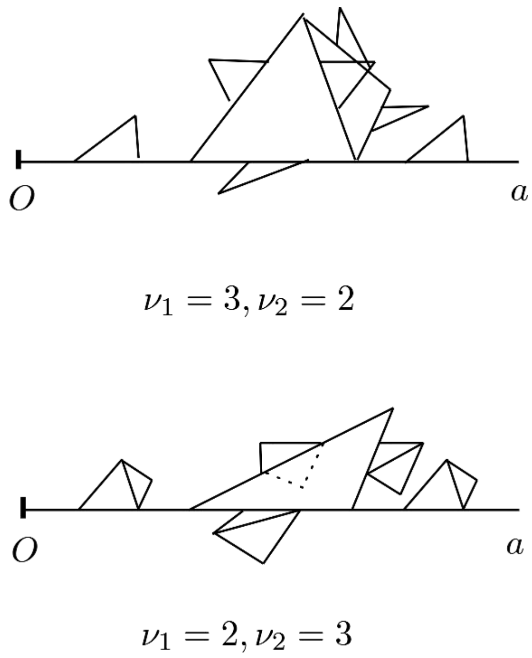

Fig.5. Some examples of F_2^3.

$$T_A(w) = \inf\{j \geqq 0 :\ w(j) \in A\},$$

where we set $\inf \emptyset = \infty$. In particular, we define a recursive sequence $\{T_i^M(w)\}_{i=0}^m$ of **hitting times of** G_M as follows: Starting from $T_0^M(w) = T_{G_M}$, for $i \geqq 1$, let

$$T_i^M(w) = \inf\{j > T_{i-1}^M(w) :\ w(j) \in G_M \setminus \{w(T_{i-1}^M(w))\}\}.$$

Here we take m to be the smallest integer such that $T_{m+1}^M(w) = \infty$. The hitting time $T_i^M(w)$ then can be interpreted as being the time (steps) taken for the path w to hit vertices in G_M for the $(i+1)$-st time, under the condition that if w hits the same vertex in G_M more than once in a row, then we only count it once.

For each $M \in \mathbb{Z}_+$, we define the **coarse-graining map** $Q_M : \bigcup_{N=M}^{\infty} W'_N \to W'_M$ by

$$(Q_M w)(i) = w(T_i^M(w)), \quad \text{for } i = 0, 1, 2, \ldots, m,$$

where m is the smallest integer such that $T_{m+1}^M(w) = \infty$. Thus,

$$Q_M w = (w(T_0^M(w)), w(T_1^M(w)), \ldots, w(T_m^M(w)))$$

is a path on the coarser graph F_M. Note that for $K < M$, $Q_K \circ Q_M = Q_K$ holds.

In the following, we will write $w(T_i^M)$ for $w(T_i^M(w))$ whenever no confusion occurs.

Define the set of finite fixed-ends paths from O to a as

$$W_N = W_N(\overline{\nu}) = \{w = (w(0), w(1), \cdots, w(n)) \in W'_N :\ w(0) = O,\ w(T_1^0(w)) = a,\ n = T_1^0(w)\}.$$

On each F_N, define a simple random walk Z_N starting at O:

$$\mathbb{P}[Z_N(0) = O] = 1.$$

$$P_{xy}^{(N)} := \mathbb{P}[Z_N(i+1) = y \mid Z_N(i) = x] = \begin{cases} \dfrac{1}{\deg x}, & \{x,y\} \in E_N, \\ 0, & \{x,y\} \notin E_N, \end{cases} \tag{3.1}$$

where $\deg x$ denotes the degree of the vertex x in (G_N, E_N).

We focus on the fixed-ends random walk on F_N that starts at O and is stopped at the first hitting time of a, which is alomost surely finite. As a result, the random walk path belongs to W_N. This correspondence induces a natural measure on W_N; that is, for each $w = (w(0), w(1), \ldots, w(n)) \in W_N$,

$$P_N[w] := \mathbb{P}[(Z_N(0), Z_N(1) \ldots, Z_N(n)) = (w(0), w(1), \ldots, w(n))\ ] = \prod_{i=1}^{n} P^{(N)}_{w(i-1)w(i)},$$

where we used $P[w : T_{\{a\}} < \infty] = 1$.

Note that a coarse-grained fixed-ends random walk is again a fixed-ends random walk on a coarser graph; that is, if $M < N$, then the distribution of $Q_M Z_N$ is equal to P_M.

4. Loop erasure by the erasing-larger-loops-first rule

For $(w(0), w(1), \cdots, w(n)) \in W_N$, if there are $c \in G_N$, and i and j, $0 \leqq i < j \leqq n$ such that $w(i) = w(j) = c$ and $w(k) \neq c$ for any k with $i < k < j$, then we call the part $[w(i), w(i+1), \ldots, w(j)]$ of the path a **loop formed at** c and define its **diameter** to be $d = \max_{i \leqq k_1 < k_2 \leqq j} |w(k_1) - w(k_2)|$, where $|\cdot|$ denotes the Euclidean distance. Note that a loop can be a part of another larger loop formed at some other vertex. By definition, the paths in W_N have no loops with $d \geqq 1$.

For each $N \in \mathbb{Z}_+$, let $\Gamma_N = \Gamma_N(\overline{\nu})$ be the set of loopless paths from O to a:

$$\Gamma_N = \{\ (w(0), w(1), \cdots, w(n)) \in W_N :\ w(i) \neq w(j),\ 0 \leqq i < j \leqq n,\ n \in \mathbb{N}\ \}.$$

We describe the loop-erasing procedure, starting with erasing loops from paths in W_1 and going down to smaller loops. A great advantage of this method is that it involves the repetition of the same operation, namely, the loop-erasure on W_1, which enables us to apply the theory of branching processes when considering the scaling limit.

Loop erasure for W_1

(i) Erase all the loops formed at the starting point (in this case, O).
(ii) Progress one step forward along the path, and erase all the loops at the new position.
(iii) Iterate this process, taking another step forward along the path and erasing the loops there, until the end point is reached (in this case, a).

Let Lw denote the resulting path, where we write $L : W_1 \to \Gamma_1$ for the loop-erasing operator. Fig.6 shows all the possible loopless paths from O to a on $F_1(\overline{\nu})$ when $\nu_1 = 3$.

So far, our loop-erasing procedure is the same as that of the chronological method defined for paths on $\mathbb{Z}^d$ in [11].

For a general N, we erase loops from the largest-scale loops down, repeatedly applying the loop-erasing procedure for W_1. To describe this procedure, we introduce a path decomposition based on the 'statistical self-similarity' and symmetry of the random pre-branched Koch curves.

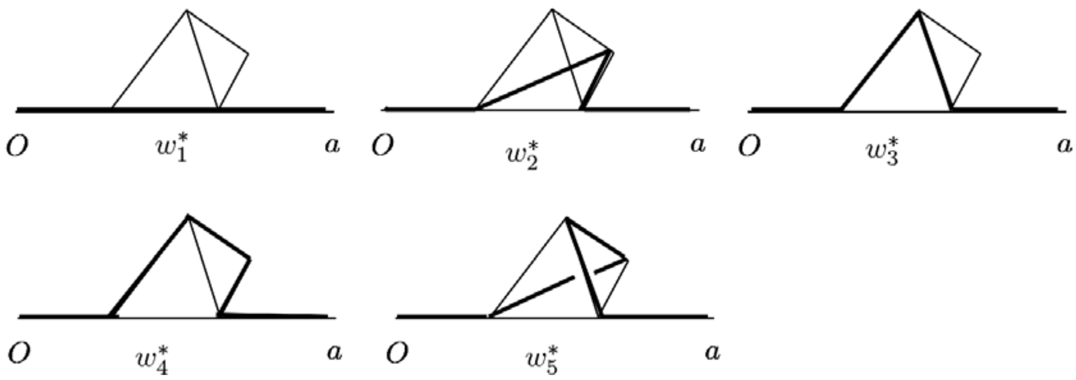

Fig.6. Loopless paths on F_1^3.

For $w \in W_N(\overline{v})$ and $M < N$, we decompose w as

$$(\tilde{w}; w_1, \cdots, w_{\ell(\tilde{w})}), \tag{4.1}$$

where $\tilde{w} = Q_M w$ and $w_i = (w(T_{i-1}^M(w)), w(T_{i-1}^M(w)+1), \cdots, w(T_i^M(w)))$, $i = 1, \cdots, \ell(\tilde{w})$.

Erasure of the largest-scale loops

(1) Decompose a path $w \in W_N(\overline{v})$ into $(\tilde{w}; w_1, \cdots, w_{\ell(\tilde{w})})$, where $\tilde{w} = Q_1 w \in W_1(\overline{v})$, as in (4.1) with $M = 1$.

(2) Erase all of the loops from $\tilde{w}$ by following the loop-erasing procedure for $W_1(\overline{v})$ to obtain $L\tilde{w} \in \Gamma_1(\overline{v})$. Let $\hat{Q}_1 w$ denote this coarse, loopless path on $F_1(\overline{v})$. To be more precise, $\hat{Q}_1 w$ can be expressed as

$$\hat{Q}_1 w = (w(T_0^1), w(T_{s_1}^1), \cdots, w(T_{s_n}^1)),$$

or equivalently

$$\hat{Q}_1 w(0) = O,\ \hat{Q}_1 w(i) = w(T_{s_i}^1),\ i = 1, \ldots, n,$$

where

$$s_i = \sup\{\, j \ :\ w(T_j^1) = w(T_{s_{i-1}+1}^1)\,\}.$$

(3) Restore the original fine structures to the remaining parts to obtain a path $w' \in W_N(\overline{v})$. Specifically, for each step i of $\hat{Q}_1 w$, between $w(T_{s_i}^1)$ and $w(T_{s_{i+1}}^1)$, insert the path segment $w_{s_i+1} = (w(T_{s_i}^1), w(T_{s_i}^1 + 1), \cdots, w(T_{s_i+1}^1))$ chosen from the original decomposition in Step (1). Note that $Q_1 w' = \hat{Q}_1 w$. The resulting path w' has no loops with $d \geqq 3^{-1}$.

We repeat these three steps within each 3^{-1}–scale part to obtain a path that has no loops with $d \geqq 3^{-2}$. We then move on to each 3^{-2}–scale part, and so on, until no loops remain. We illustrate this procedure by way of the following inductive steps.

Induction steps for loop erasure

Before we list the steps, we need to elaborate on the notion of 3^{-M}-blocks. Note that if $M < N$, then each part of $F_N(\overline{v})$ that lies between two vertices $x, y \in G_M$ with $\{x, y\} \in E_M$ is similar to $F_{N-M}(\sigma^M \overline{v})$, where σ is the shift operator defined in Section 2.2. We call such a subgraph a 3^{-M}**-block** of $F_N(\overline{\nu})$. Also for $F(\overline{\nu})$, 3^{-M}**-block** is defined in the same manner, which is similar to $F(\sigma^M\overline{\nu})$.

Let $1 \leqq M < N$. Suppose that we have $w \in W_N(\overline{\nu})$ such that $Q_M w \in \Gamma_M(\overline{\nu})$ and w has no loops with $d \geqq 3^{-M}$.

1) Decompose w to obtain $(\tilde{w};\ w_1, \cdots w_k)$, with $\tilde{w} = Q_M w$, as in (4.1) (Fig.7, Fig.8 and Fig.9).
2) From each $w_i = (w(T^M_{i-1}(w)), w(T^M_{i-1}(w)+1), \cdots, w(T^M_i(w)))$, erase the largest-scale loops, that is, the loops in $Q_{M+1}w$, according to the base step procedure (1)–(3) above starting from $w(T^M_{i-1}(w))$ (instead of O) until reaching $w(T^M_i(w))$ (instead of a) to obtain $\tilde{w}_i$. Note that in Step (2), loops formed at the starting point may belong to at most three adjacent 3^{-M}-blocks. Erase all of those loops. Otherwise, step (2) is the same as above.
3) Assemble $(\tilde{w};\ \tilde{w}_1, \cdots, \tilde{w}_k)$ to obtain $w' \in W_N$, which is uniquely determined. Then $Q_{M+1}w' \in \Gamma_{M+1}(\overline{\nu})$ and w' has no loops with $d \geqq 3^{-(M+1)}$. □

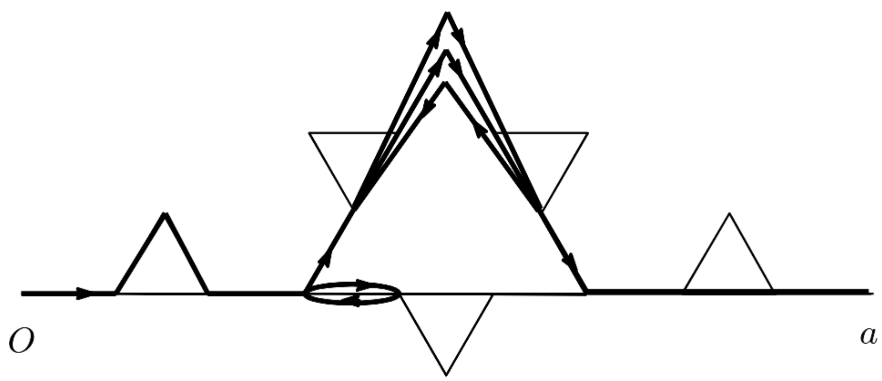

Fig.7. $w \in W_N(\overline{\nu})$; $N = 2, M = 1$.

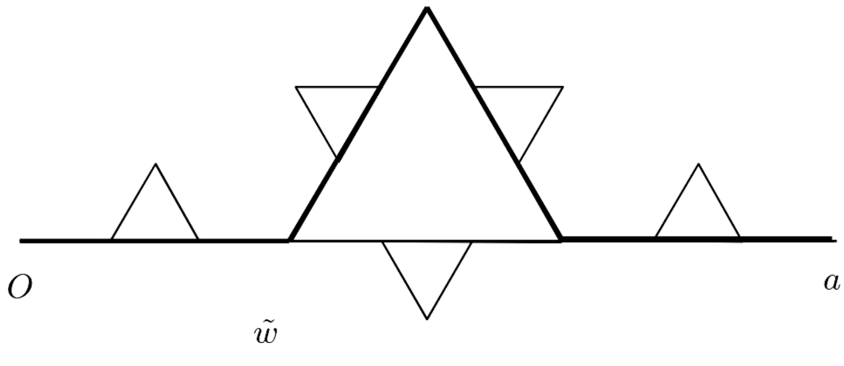

Fig.8. $\tilde{w} = Q_M w$, $\ell(\tilde{w}) = 4$.

We repeat these steps until no loops remain. Let $Lw \in \Gamma_N$ denote the resulting loopless path. In this way, the loop erasing operator L, first defined for $W_1(\overline{\nu})$, has been extended to $L : \bigcup_{N=1}^{\infty} W_N(\overline{\nu}) \to \bigcup_{N=1}^{\infty} \Gamma_N(\overline{\nu})$ with $L(W_N) = \Gamma_N$. Note that the operation described above is essentially a repetition of loop-erasing for $W_1(\overline{\nu})$.

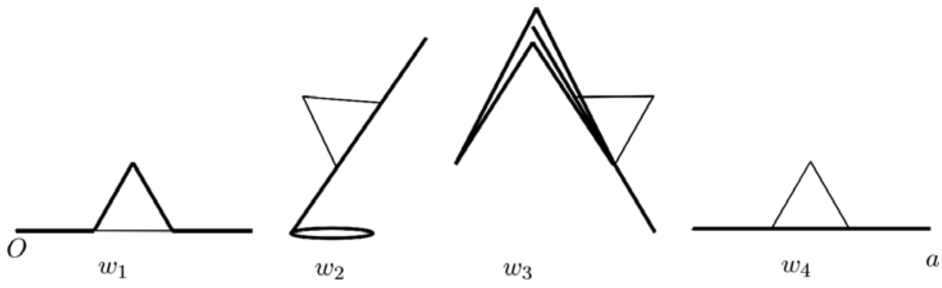

Fig.9. w_1, w_2, w_3 and w_4.

For $w \in W_N(\overline{\nu})$, we defined $\hat{Q}_1 w$ in Step (2) for the erasure of the largest-scale loops. For later use we define $\hat{Q}_K w$ on $F_K(\overline{\nu})$ for all $K < N$ as follows. Repeat the induction steps 1)–3) K times until all loops with $d \geqq 3^{-K}$ have been erased and denote the resulting path as w'. Let $\hat{Q}_K w = Q_K w'$, that is, the coarse path before restoring fine structures.

We apply the loop-erasure to the fixed-ends random walk Z_N. The operator L then induces a measure $\hat{P}_N = P_N \circ L^{-1}$ on $\Gamma_N(\overline{\nu})$.

For $w_1^*, \cdots, w_5^*$ shown in Fig.6, let

$$p_i = \hat{P}_1(\overline{\nu})[w_i^*] \ \text{ if } \nu_1 = 2, \quad q_i = \hat{P}_1(\overline{\nu})[w_i^*] \ \text{ if } \nu_1 = 3.$$

A direct calculation gives

$$p_1 = 2/3, \ \ p_2 = 1/3, \ \ p_3 = p_4 = p_5 = 0, \tag{4.2}$$

$$q_1 = 1/2, \ \ q_2 = q_3 = 3/16, \ \ q_4 = q_5 = 1/16. \tag{4.3}$$

5. Generating functions

We define the generating functions for T_1^0, the arrival time at the other end a, by:

$$\Phi_N(\overline{\nu})(x) = \sum_{w \in \Gamma_N} \hat{P}_N(\overline{\nu})(w) x^{T_1^0(w)}, \quad x \geqq 0, \quad N \in \mathbb{N},$$

where Γ_N means $\Gamma_N(\overline{\nu})$.

A crucial observation is that in the process of erasing loops from Z_{N+1}, if we stop at the stage where we have obtained $\hat{Q}_N Z_{N+1}$, that is, before restoring the $3^{-(N+1)}$-structures, then it is nothing but the procedure for obtaining LZ_N from Z_N. This fact is expressed as

$$P_{N+1}[\{v : \hat{Q}_N v = u\}] = \hat{P}_N[u]. \tag{5.1}$$

For $M < N$, the similarity of a 3^{-M}–block of $F_N(\overline{\nu})$ to $F_{N-M}(\sigma^M \overline{\nu})$ plays an essential role in proving the recursion formula for the generating functions. Decompose $w \in W_N$ into $(\tilde{w}; w_1, \ldots, w_{\ell(\tilde{w})})$ with $\tilde{w} = Q_M w$ as in (4.1). The 3^{-M}–block of $F_N(\overline{\nu})$ with endpoints $w(T_{i-1}^M)$ and $w(T_i^M)$ is similar to $F_{N-M}(\sigma^M \overline{\nu})$. Let Δ_i denote this 3^{-M}–block of $F_N(\overline{\nu})$. We see that w_i consists of the main part going from $w(T_{i-1}^M)$ to $w(T_i^M)$ on Δ_i, and some loops, if any, leaking into adjoining 3^{-M}–blocks. Folding these leaking loops onto Δ_i by applying an appropriate rotation and reflexion gives a path similar to some path w_i' in $W_{N-M}(\sigma^M \overline{\nu})$. Recall the transition probability $P^{(N)}_{xy}$ defined in (3.1) and look closely at the factor $\prod_{j=T^M_{i-1}}^{T^M_i - 1} P^{(N)}_{w(j)w(j+1)}$. Since the multiplicity of folded loops is absorbed by $1/\deg w(T^M_{i-1})$, we have

$$\prod_{j=T^M_{i-1}}^{T^M_i - 1} P^{(N)}_{w(j)w(j+1)} = P_{N-M}(\sigma^M \overline{\nu})[w'_i]. \tag{5.2}$$

Fig.10 shows $w'_1, \dots, w'_4$ for $w_1, \dots, w_4$ in Fig.9.

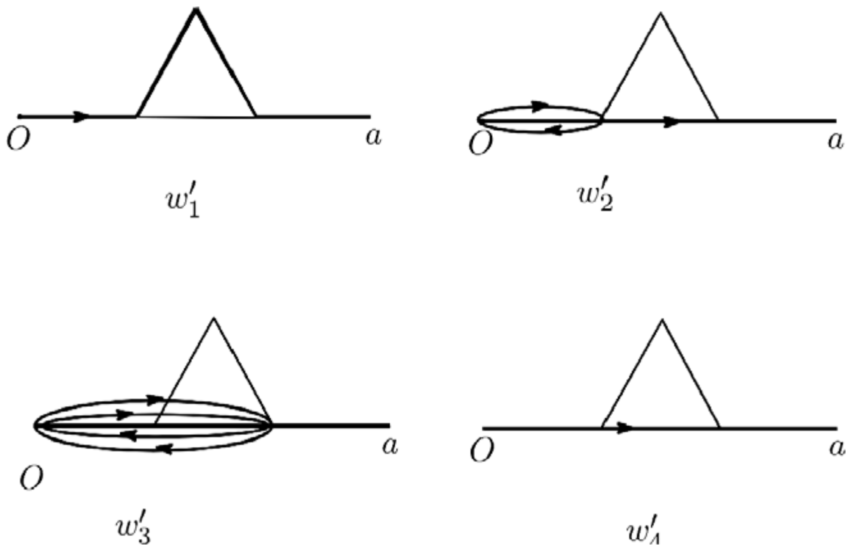

Fig. 10. w'_1, w'_2, w'_3 and w'_4.

Thus, ELLF being the repetition of loop erasure of W_1 leads to the following recursion relations for the generating functions:

Proposition 6.

$$\Phi_1(\overline{\nu})(x) = \frac{1}{3}(2x^3 + x^4) =: \Phi^{(2)}(x), \quad \textit{if } \nu_1 = 2,$$

$$\Phi_1(\overline{\nu})(x) = \frac{1}{8}(4x^3 + 3x^4 + x^5) =: \Phi^{(3)}(x), \quad \textit{if } \nu_1 = 3.$$

$$\Phi_{N+1}(\overline{\nu})(x) = \Phi_N(\overline{\nu})(\Phi^{(\nu_{N+1})}(x)). \tag{5.3}$$

$$\Phi_N(\overline{\nu})(x) = \Phi^{(\nu_1)} \circ \Phi^{(\nu_2)} \circ \cdots \Phi^{(\nu_N)}(x). \tag{5.4}$$

Proof. The first two equalities are obtained from (4.2) and (4.3).

For the recursion formula (5.3), let $(\tilde{w}; w_1, \dots w_{\ell(\tilde{w})})$, $\tilde{w} = Q_N w$ and $(\tilde{v}; v_1, \dots, v_{\ell(\tilde{v})})$, $\tilde{v} = Q_N v$ be the decompositions of $w \in \Gamma_{N+1}(\overline{\nu})$ and $v \in W_{N+1}(\overline{\nu})$, respectively. From (5.1) and (5.2), it follows that

$$\begin{aligned}
\hat{P}_{N+1}[w] &= P_{N+1}[v : Lv = w] \\
&= \sum_{u\in\Gamma_N} P_{N+1}[v : Lv = w, \hat{Q}_N v = u] \quad \text{(classification by } \hat{Q}_N v) \\
&= \sum_{u} P_{N+1}[v : Lv = w \mid \hat{Q}_N v = u] P_{N+1}[v : \hat{Q}_N v = u] \\
&= \sum_{u} P_{N+1}[v : Lv = w \mid \hat{Q}_N v = u] \hat{P}_N[u] \\
&= \sum_{u} P_{N+1}[v : Lv_i = w_i, i = 1, \dots \ell(u) \mid \hat{Q}_N v = u] \hat{P}_N[u] \\
&= \sum_{u} \left(\prod_{i=1}^{\ell(u)} P_1(\sigma^N \overline{\nu})[v : Lv = w'_i] \right) \hat{P}_N[u].
\end{aligned}$$

Hence, keeping in mind the similarity of w_i and w'_i, we have

$$\begin{aligned}
\Phi_{N+1}(\overline{\nu})(x) &= \sum_{w\in\Gamma_{N+1}} \hat{P}_{N+1}[w] x^{\ell(w)} \\
&= \sum_{u\in\Gamma_N} \sum_{w_1\in\Gamma_1} \cdots \sum_{w_{\ell(u)}\in\Gamma_1} \left(\prod_{i=1}^{\ell(u)} \hat{P}_1(\sigma^N \overline{\nu})[w'_i]] \right) \hat{P}_N[u]\, x^{\ell(w_1)+\cdots+\ell(w_{\ell(u)})} \\
&= \sum_{u\in\Gamma_N} \hat{P}_N[u] \prod_{i=1}^{\ell(u)} \left(\sum_{w_i\in\Gamma_1} \hat{P}_1(\sigma^N \overline{\nu})[w'_i] x^{\ell(w'_i)} \right) \\
&= \sum_{u\in\Gamma_N} \hat{P}_N[u] \left(\Phi^{(\nu_{N+1})}(x) \right)^{\ell(u)} \\
&= \Phi_N(\overline{\nu})(\Phi^{(\nu_{N+1})}(x)).
\end{aligned}$$

Equation (5.4) comes from the repeated use of (5.3). □

Remark. By verifying that the generating functions are the same, we can prove that the loop-erased random walk on $F_N(\overline{\nu})$ constructed here is the same as that obtained by chronological loop-erasing.

6. The scaling limit

In this section, we investigate the limit of the loop-erased walks constructed in Section 4, as the edge length tends to 0.

Let

$$C = \{w \in C([0,\infty) \to F(\overline{\nu})) \,:\, w(0) = O,\ \lim_{t\to\infty} w(t) = a\}.$$

The space C is a complete separable metric space with the metric

$$d(u, v) = \sup_{t\in[0,\infty)} |u(t) - v(t)|,\ u, v \in C,$$

where $|x - y|$, $x, y \in \mathbb{R}^3$, denotes the Euclidean distance. Hereinafter, for $w \in \bigcup_{N=1}^{\infty} W_N$, we set

$$w(t) = a, \text{ for } t \geqq T_1^0(\omega)$$

and interpolate the path linearly,

$$w(t) = (i+1-t)w(i) + (t-i)w(i+1), \quad i \leqq t < i+1, \quad i \in \mathbb{Z}_+,$$

so that we can regard w as a continuous function on $[0, \infty)$. We will also regard Γ_N as subsets of C. The hitting times $\{T_i^M(w)\}_{i=1}^m$ are defined for $w \in C$ as in the previous sections, although the infimum is taken over continuous time:

$$T_0^M(w) = 0, \quad T_i^M(w) = \inf\{t > T_{i-1}^M(w) \ : \ w(t) \in G_M \setminus \{w(T_{i-1}^M(w))\}\}.$$

Notice that the condition $\lim_{t\to\infty} w(t) = a$ makes $\{T_i^M(w)\}_{i=0}^m$ a finite sequence.

For $N \in \mathbb{Z}_+$, we define a coarse-graining map $Q_N : C \to C$ by $(Q_N w)(i) = w(T_i^N(w))$ for $i = 0, 1, 2, \ldots, m$, and by using linear interpolation

$$(Q_N w)(t) = \begin{cases} (i+1-t)\,(Q_N w)(i) + (t-i)\,(Q_N w)(i+1), & i \leqq t < i+1, \\ \qquad\qquad i = 0, 1, 2, \ldots, m-1, & \\ a, & t \geqq m. \end{cases}$$

The loop-erasing operator is regarded as $L : \bigcup_{N=1}^{\infty} W_N \to \bigcup_{N=1}^{\infty} \Gamma_N$. $\hat{Q}_N$ is as in Section 4 with resulting paths in Γ_N.

To define an almost sure limit, we couple walks on different F_N-graphs. Let

$$\Omega = \{\omega = (\omega_0, \omega_1, \omega_2, \cdots) \ : \ \omega_0 = (O, a), \ \omega_N \in \Gamma_N, \ \omega_{N-1} = Q_{N-1}\omega_N, \ N \in \mathbb{N}\}. \tag{6.1}$$

Define the projection onto the first $N+1$ elements by

$$\pi_N \omega = (\omega_0, \omega_1, \ldots, \omega_N),$$

and define a probability measure on $\pi_N \Omega$ by

$$\tilde{P}_N[(\omega_0, \omega_1, \ldots, \omega_N)] = \hat{P}_N[\omega_N],$$

where $\hat{P}_N$ is defined in Section 4.

The following consistency condition is a direct consequence of the loop-erasing procedure:

$$\tilde{P}_N[(\omega_0, \omega_1, \ldots, \omega_N)] = \sum_{\omega'} \tilde{P}_{N+1}[(\omega_0, \omega_1, \ldots, \omega_N, \omega')], \tag{6.2}$$

where the sum is taken over all possible $\omega' \in \Gamma_{N+1}$ such that $Q_N \omega' = \omega_N$.

By virtue of (6.2) and Kolmogorov's extension theorem, there is a probability measure $P = P(\overline{\nu})$ on $\Omega_0 = C^{\mathbb{N}} = C \times C \times \cdots$ such that

$$P[\,\Omega\,] = 1$$

and

$$P \circ \pi_N^{-1} = \tilde{P}_N, \ N \in \mathbb{Z}_+.$$

Let $Y_N : \Omega_0 \to \Gamma_N \subset C$ be the projection to the $(N+1)$-st component. We regard Y_N as an $F(\overline{\nu})$-valued process $Y_N(\omega, t)$ on $(\Omega_0, \mathcal{B}, P)$, where $\mathcal{B}$ is the Borel algebra on Ω_0 generated by the cylinder sets. Then we have $P \circ Y_N^{-1} = \hat{P}_N$.

Let $\lambda_2 = \frac{d}{dx}\Phi^{(2)}(1) = 10/3$, $\lambda_3 = \frac{d}{dx}\Phi^{(3)}(1) = 29/8$, and $B_N = B_N(\overline{v}) := \prod_{i=1}^{N} \lambda_{v_i}$. Then from (5.4), we see that

$$\hat{E}_N[T_1^0(Y_N)] = \frac{d}{dx}\Phi_N(\overline{v})(1) = B_N, \tag{6.3}$$

where $\hat{E}_N$ denotes expectation with respect to $\hat{P}_N$. Define the traverse times of 3^{-N} blocks:

$$S_i^N(w) = T_i^N(w) - T_{i-1}^N(w), \tag{6.4}$$

for $i = 1, 2, \ldots, m$, where m is the smallest number such that $T_{m+1}^N(w) = \infty$.

The following proposition is proved exactly in the same way as the case of a simple random walk on the random homogeneous pre-Sierpiński gasket (see [5]).

Proposition 7. *Fix arbitrarily $v \in \Gamma_M$. For each i, $1 \leqq i \leqq T_1^0(v)$, under the conditional probability $P[\ \cdot\ |Y_M = v]$, $S_i^M(Y_{M+N})$, $N \in \mathbb{Z}_+$, is a random supercritical branching process whose N-th generation offspring distribution equals the distribution of T_1^0 under the environment $\sigma^{M+N}\overline{v}$. Here the N-th generation offspring means the number of $3^{-(M+N+1)}$-sized steps born from one step of Y_{M+N}. Furthermore,*

$$E[S_i^M(Y_{M+N})|\ Y_M = v\ ] = \frac{B_{N+M}}{B_M},$$

where E denotes expectation with respect to P. The right-hand side is independent of v. In particular,

$$E[S_1^0(Y_N)] = E[T_1^0(Y_N)] = B_N.$$

Proposition 7 suggests that we consider the time-scaled processes:

$$X_N(\ \cdot\ ) := Y_N(B_N\ \cdot\ ),\quad N \in \mathbb{Z}_+$$

so that $E[T_1^0(X_N)] = 1$.

Proposition 8. *For any $N \geqq M$,*

$$X_N(T_i^M(X_N)) = X_M(T_i^M(X_M)) = Y_M(T_i^M(Y_M)),\quad a.s. \tag{6.5}$$

This is a direct consequence of the definitions.

By similarity of the graphs, we see that $S_i^M(Y_{M+N})(\overline{v})$ has the same distribution as $S_1^0(Y_N)(\sigma^M\overline{v})$, which further implies that $S_i^M(X_{M+N})(\overline{v})(= S_i^M(Y_{M+N})(\overline{v})/B_{M+N})$ has the same distribution as $S_1^0(X_N)(\sigma^M\overline{v})/B_M$.

Note that for $N > M$

$$S_1^0(Y_N)(\overline{v}) = \sum_{i=1}^{S_1^0(Y_M)(\overline{v})} S_i^M(Y_N)(\overline{v}),$$

and

$$E[S_1^0(Y_N)(\overline{v})] = E[S_1^0(Y_M)(\overline{v})]E[S_1^M(Y_N)(\overline{v})].$$

Since $E[S_1^0(Y_N)(\overline{v})] = B_N$ and $E[S_1^0(Y_M)(\overline{v})] = B_M$, we have $E[S_i^M(Y_N)(\overline{v})] = B_N/B_M$, and hence $E[S_i^M(X_N)(\overline{v})] = 1/B_M$.

Recall that the environment $\overline{\nu}$ introduced at the beginning of Section 2.2 is a sequence of i.i.d. random variables. From the law of large numbers, it follows that

$$\lim_{n\to\infty}\Big(\prod_{i=1}^{n}\lambda_{\nu_i}\Big)^{1/n}=\lambda_2^{(1-p)}\lambda_3^{p} \tag{6.6}$$

for almost every $\overline{\nu}$. In what follows, we arbitrarily choose and fix a $\overline{\nu}$ for which (6.6) holds. Thus, 'almost surely' below means 'P-almost surely'.

Proposition 9. *Take arbitrarily $v\in\Gamma_M$. For each i, $1\leqq i\leqq T_1^0(v)$, under the conditional probability $P[\ \cdot\ |Y_M=v]$, we have the following:*

(1) *$\{S_i^M(X_{M+N}),\ N\in\mathbb{Z}_+\}$ is an L^2 bounded supermartingale and converges almost surely and in L^2 to some non-negative random variable S_i^{*M} as $N\to\infty$.*

(2) *S_i^{*M}, $i=1,\cdots,T_1^0(v)$, are independent and have an identical distribution; $S_1^{*M}(\overline{\nu})$ has the same distribution as that of $S_1^{*0}(\sigma^M\overline{\nu})/B_M$. Each $S_i^{*M}(\overline{\nu})$ is independent of v.*

(3) *The Laplace transform of $S_1^{*M}(\overline{\nu})$*

$$\phi^M(u)=E[\exp(-uS_1^{*M})],\quad u\geqq 0$$

satisfies

$$\phi^M(u)=\Phi^{(\nu_M)}(\phi^{M+1}(u/\lambda_{\nu_M})).$$

(4) *For every $\varepsilon>0$, there exists constants $c_{3.1}$ and $c_{3.2}$ independent of $\overline{\nu}$, and $c_{3.3}(\sigma^M\overline{\nu},\varepsilon)$ and $c_{3.4}=c_{3.4}(\sigma^M\overline{\nu},\varepsilon)$ such that*

$$c_{3.3}\exp(-c_{3.2}u^{\gamma+\varepsilon})\leqq\phi^M(u)\leqq c_{3.4}\exp(-c_{3.1}u^{\gamma-\varepsilon}),\quad \textit{for all } u\geqq 0,$$

where $\gamma=\dfrac{\log 3}{(1-p)\log\lambda_2+p\log\lambda_3}$.

(5) *$S_i^{*M}>0$ a.s.*

(6) *There exists a positive constant A independent of $\overline{\nu}$ such that*

$$E[(S_1^{*M})^2]\leqq\frac{A}{B_M^2}.$$

Proof. (1), (2) [4] Theorem 2.2.

(3) [4] Lemma 2.3.

(4) [4] Corollary 3.8.

(5) $P[S_i^{*M}=0]=\lim_{u\to\infty}\phi^M(u)=0$.

(6) Matching the notation of [4] to our notation gives a paraphrase of Theorem 2.2 (b):

$$Var[S_1^0(X_M)]=\sum_{i=1}^{M}\frac{Var[S^{(i)}]}{\lambda_{\nu_i}^2B_i},$$

where $S^{(2)}$ and $S^{(3)}$ are random variables with distribution of $S_1^0(X_1)$ when $\nu_1=2$ and $\nu_1=3$, respectively. Let $c=\max_{i=2,3}\dfrac{Var[S^{(i)}]}{\lambda_i^2}$, which is a constant independent of $\overline{\nu}$.

$$E[S_1^0(X_N)^2]\quad=\quad Var[S_1^0(X_N)]+E[S_1^0(X_N)]^2\leqq c\sum_{i=1}^{N}\frac{1}{B_i}+1$$

$$= \quad c \sum_{i=1}^{N} \frac{1}{\prod_{k=1}^{i} \lambda_{\nu_k}} + 1 \leqq c \sum_{i=1}^{\infty} \frac{1}{\lambda_2^i} + 1.$$

Writing A for the rightmost constant, we have $E[S_1^0(X_N)^2] \leqq A$ for any $N \in \mathbb{Z}_+$ (and any $\overline{\nu}$). This immediately gives

$$E[S_1^M(X_N)^2] \leqq \frac{A}{B_M^2},$$

for $M < N$. The L^2 convergence established in (1) completes the proof. □

By virtue of Propositions 8 and 9, we are now in a position to prove the almost sure uniform convergence for X_N.

Theorem 10. *X_N converges almost surely uniformly in $t \in [0, \infty)$ to a continuous process X as $N \to \infty$.*

Proof. Choose $\omega \in \Omega$ such that the following holds for all M: $\lim_{N\to\infty} T_i^M(X_N) = T_i^{*M}$ exists and $S_i^{*M} = T_i^{*M} - T_{i-1}^{*M} > 0$ for all $1 \leqq i \leqq k$, where $k = k_M = T_1^0(Y_M)$.

Let $R = T_1^{*0} + 1$. Then it suffices to prove that $X_N(\omega, t)$ converges uniformly in $t \in [0, R]$. In fact, if $t > R$, there is a positive integer $N_0 = N_0(\omega)$ such that $X_N(t) = a$ for all $N \geqq N_0$.

Take M arbitrarily. By expressing the arrival time at a as the sum of the traversing times of 3^{-M}–blocks, we have $T_k^M(X_N) = T_1^0(X_N)$ with $k = k_M$. Letting $N \to \infty$, we have $T_k^{*M} = T_1^{*0}$.

Our choice of ω implies that there is an $N_1 = N_1(\omega) \in \mathbb{N}$ such that for all $N \geqq N_1$,

$$\max_{1\leqq i\leqq k} |T_i^M(X_N) - T_i^{*M}| \leqq \min_{1\leqq i\leqq k}(T_i^{*M} - T_{i-1}^{*M}), \quad |T_k^M(X_N) - T_k^{*M}| < 1. \tag{6.7}$$

For $0 \leqq t \leqq T_k^{*M}$, choose $j \in \{1, 2, \ldots, k\}$ such that $T_{j-1}^{*M} \leqq t < T_j^{*M}$. Then it follows from (6.7) that for all $N \geqq N_1$, $T_{j-2}^M(X_N) \leqq t \leqq T_{j+1}^M(X_N)$ if $2 \leqq j \leqq k-1$, and $0 \leqq t \leqq T_2^M(X_N)$ for $j = 1$. Since Proposition 8 states that

$$X_N(T_j^M(X_N)) = X_M(T_j^M(X_M)), \tag{6.8}$$

for all N with $N \geqq M$, we have

$$|X_N(T_j^M(X_N)) - X_N(t)| \leqq 2 \cdot 3^{-M}. \tag{6.9}$$

For $T_k^{*M} \leqq t \leqq T_k^{*M} + 1 = R$, since $T_{k-1}^M(X_N) \leqq t$,

$$|X_N(T_k^M(X_N)) - X_N(t)| \leqq 3^{-M}. \tag{6.10}$$

By combining (6.8) – (6.10), it follows that if $N, N' \geqq \max\{N_0, N_1\}$, then for any $t \in [0, R]$,

$$\begin{aligned} |X_N(t) - X_{N'}(t)| &\leqq |X_N(T_j^M(X_N)) - X_N(t)| + |X_{N'}(T_j^M(X_{N'})) - X_{N'}(t)| \\ &\quad + |X_N(T_j^M(X_N)) - X_{N'}(T_j^M(X_{N'}))| \\ &\leqq 4 \cdot 3^{-M}. \end{aligned}$$

Since M is arbitrary, we see that $\{X_N\}$ is a Cauchy sequence in C, which implies the uniform convergence. □

Cao introduced a new method of loop-erasing from random walk paths, called partial loop-erasing (PLE), so that resulting path has the same distribution as that of chronologically loop-erased path. He showed the existence of scaling limit of the path in the space of compact sets equipped with the Hausdorff distance. Time is not considered there. However, it will be of interest to investigate the relation between ELLF and PLE.

7. Limit path properties

Theorem 11. *The sample path of the limit process X is almost surely self-avoiding. To be precise, for any $0 \leqq t_1 < t_2 \leqq T_1^{*0}$, $X(t_1) \neq X(t_2)$. In particular, $T_i^M(X) = T_i^{*M}$ for all $M \in \mathbb{Z}_+$ and for all i a.s.*

Proof. The uniform converegence of the walks that are self-avoiding and the structure of the branched-Koch-based random fractal ensure that the trajectory of X has no 'self-intersection', that is, there are no $t_1 < t_2 < t_3$ such that $X(t_1) = X(t_3) \neq X(t_2)$.

So all we need to show is that X does not stay at any point of $F(\overline{v}) \setminus \{a\}$ for a positive interval of time.

The uniform convergence also leads to $X(T_i^{*M}) = X(T_i^M(X))$. Notice that $T_i^M(X) \leqq T_i^{*M}$ from the definition of the hitting time.

Let A be the event that X does stay at some point of $F(\overline{v}) \setminus \{a\}$ for a positive interval of time, and for $m \in \mathbb{N}$, let A_m be the event that there exists t_1 with $t_1 < T_1^0(X)$ such that $X(t) = X(t_1)$ for all t in $[t_1, t_1 + 1/m]$. Then

$$P[A] = P[\bigcup_{m=1}^{\infty} A_m] \leqq \sum_{m=1}^{\infty} P[A_m].$$

The event A_m implies that for all $M > 0$ the process X stays in some adjoining 3^{-M}- blocks of $F(\overline{v})$ longer than $1/m$. For each $M > 0$ let C_M denote this event. Thus

$$P[A_m] = P[\bigcap_{M=1}^{\infty} C_M].$$

We further classify C_M by Y_M to obtain

$$\begin{aligned} P[C_M] &= \sum_{v \in \Gamma_M} P[C_M \mid Y_M = v] P[Y_M = v] \\ &\leqq \sum_{v \in \Gamma_M} P[\bigcup_{i=1}^{\ell(v)-1} \{S_i^{*M} + S_{i+1}^{*M} \geqq \frac{1}{m}\} \mid Y_M = v] P[Y_M = v], \end{aligned}$$

where $\ell(v)$ denotes the length of v. From Proposition 9 (2),

$$\begin{aligned} P[\bigcup_{i=1}^{\ell(v)-1} \{S_i^{*M} + S_{i+1}^{*M} \geqq \frac{1}{m}\} \mid Y_M = v] &\leqq \sum_{i=1}^{\ell(v)-1} P[S_i^{*M} + S_{i+1}^{*M} \geqq \frac{1}{m} \mid Y_M = v] \\ &\leqq \sum_{i=1}^{\ell(v)-1} (P[S_i^{*M} \geqq \frac{1}{2m}] + P[S_{i+1}^{*M} \geqq \frac{1}{2m}]) \\ &\leqq 2\ell(v) P[S_1^{*M} \geqq \frac{1}{2m}]. \end{aligned}$$

Hence

$$P[C_M] \leqq P[S_1^{*M} \geqq \frac{1}{2m}] \sum_{v \in \Gamma_M} 2\ell(v) P[Y_M = v] = 2P[S_1^{*M} \geqq \frac{1}{2m}]\, B_M$$
$$\leqq E[(S_1^{*M})^2] 8m^2 B_M \leqq \frac{8m^2 A}{B_M},$$

where we used Proposition 7, Chebyshev's inequality and Proposition 9 (6). Since $\{C_M\}$ is a decreasing sequence of events and $B_M \geqq \lambda_2^M$, we have

$$P[A_m] \leqq \lim_{M\to\infty} P[C_M] = 0,$$

which implies that $P[A] = 0$.

From Proposition 8 and the uniform convergence of the walks, we have $X(T_i^{*M}) = X_M(T_i^M) = X(T_i^M(X))$, which, when combined with the self-avoiding property established above, gives $T_i^M(X) = T_i^{*M}$ for all M a.s. □

From above results, we obtain the speed of convergence.

Proposition 12. *There is an integer $N_0 = N_0(\omega)$ such that for all $N \geqq N_0$*

$$\max_{t\geqq 0} |X_N(t) - X(t)| \leqq 2 \cdot 3^{-N}, \quad a.s.$$

Proof. We start with choosing $\omega \in \Omega$ and defining R, k_N and N_0 as in the proof of Theorem 10. The choice of N_0 implies that if $t \geqq R$ then $X_N(t) = X(t) = a$ for all $N \geqq N_0$.

For each $t \in [0, R]$ and $N \geqq N_0$, there exists $j = j(N, t) \in \{1, 2, \dots, k_N\}$ such that

$$T_{j-1}^{*N} \leqq t < T_j^{*N}, \tag{7.1}$$

where $T_0^{*N} = 0$.

Combining Proposition 9 (1), the uniform convergence of X_N to X and Proposition 8 gives

$$X(T_i^{*N}) = X_N(T_i^N(X_N)). \tag{7.2}$$

On the other hand, it has been shown in the proof of Theorem 10 that for any integer N,

$$X(T_i^{*N}) = X(T_i^N(X)).$$

Hence

$$X(T_i^N(X)) = X_N(T_i^N(X_N)). \tag{7.3}$$

(7.2) and (7.3), together with the self-avoiding property of X_N and X, mean that both $X_N(t)$ and $X(t)$ belong to the tetrahedron or triangle two of whose vertices are $X_N(T_{j-1}^N(X_N))$ and $X_N(T_j^N(X_N))$. Thus,

$$|X_N(T_j^N(X_N)) - X_N(t)| \leqq 3^{-N},$$

and

$$|X_N(T_j^N(X_N)) - X(t)| \leqq 3^{-N},$$

which leads to

$$|X_N(t) - X(t)| \leqq |X_N(T_j^N(X_N)) - X_N(t)| + |X_N(T_j^N(X_N)) - X(t)| \leqq 2 \cdot 3^{-N},$$

for all $N \geqq N_0$. □

Proof of Theorem 3.

Fix an environment $\overline{\nu}$ for which (6.6) holds.

Let $\tilde{X} = \tilde{X}(\omega)$ and $\tilde{Y}_N = \tilde{Y}_N(\omega)$, $N \in \mathbb{Z}_+$ denote the random sets $\{X(t) \in \mathbb{R}^3 : t \geqq 0\}$ and $\{Y_N(t) \in \mathbb{R}^3 : t \geqq 0\}$, respectively. Let V and $\overline{V}$ the open set and its closure defined in Appendix, respectively. For each $\delta > 0$ define $\mathcal{V}_\delta$ to be the collection of closed sets similar to $\overline{V}$ and of diameter δ, and for $B \subset \mathbb{R}^3$, define $N_\delta(B)$ to be the smallest number of elements of $\mathcal{V}_\delta$ that cover B.

From the structure of the random branched Koch curve, (6.1), Theorem 10 and (6.4), we see that

$$N_{3^{-N}}(\tilde{X}) = N_{3^{-N}}(\tilde{Y}_N) = S_1^0(Y_N). \tag{7.4}$$

From the choise of $\overline{\nu}$ in (6.6), we have

$$S_1^0(X_N) = \frac{1}{B_N} S_1^0(Y_N) \to S_1^{*0} \tag{7.5}$$

as $N \to \infty$ for almost all ω. Now note that

$$\frac{\log N_{3^{-N}}(\tilde{X})}{-\log 3^{-N}} = \frac{\log S_1^0(Y_N)/B_N + \log B_N}{N \log 3}.$$

Since S_1^{*0} is almost surely finite, it follows that

$$\lim_{N\to\infty} \frac{\log S_1^0(Y_N)/B_N}{N \log 3} = 0$$

for almost all ω, and (6.6) yields

$$\lim_{N\to\infty} \frac{\log B_N}{N \log 3} = \lim_{N\to\infty} \frac{1}{\log 3} \frac{1}{N} \sum_{i=1}^{N} \log \lambda_{\nu_i} = \frac{\log \lambda_2^{1-p} \lambda_3^p}{\log 3}.$$

Thus the following limit exists, which by definition ([3]) gives the path box-counting dimension.

$$\lim_{\delta \downarrow 0} \frac{\log N_\delta(\tilde{X})}{-\log \delta} = \lim_{N\to\infty} \frac{\log N_{3^{-N}}(\tilde{X})}{-\log 3^{-N}} = \frac{\log \lambda_2^{1-p} \lambda_3^p}{\log 3}. \qquad \square$$

Concerning the path Hausdorff dimension d_H, we know that $d_H \leqq d_B$ is always true ([3]), but the lower bound of d_H is still an open problem.

Appendix A Construction of the open set V

Construction of the open set

Let $a = (1, 0, 0)$, $d = (\frac{1}{2}, \frac{\sqrt{3}}{6}, 0)$, $e = (\frac{1}{2}, \frac{\sqrt{3}}{18}, \frac{\sqrt{6}}{9})$, $C_1 = (\frac{7}{18}, -\frac{\sqrt{3}}{54}, \frac{2\sqrt{6}}{27})$, $C_2 = (\frac{11}{18}, -\frac{\sqrt{3}}{54}, \frac{2\sqrt{6}}{27})$, $C_3 = (\frac{1}{2}, -\frac{\sqrt{3}}{18}, 0)$, $D_3 = (\frac{1}{2}, -\frac{\sqrt{3}}{54}, -\frac{\sqrt{6}}{27})$, $D_4 = (\frac{11}{18}, \frac{5\sqrt{3}}{54}, -\frac{\sqrt{6}}{27})$, $D_5 = (\frac{7}{18}, \frac{5\sqrt{3}}{54}, -\frac{\sqrt{6}}{27})$, $G_1 = (\frac{1}{3}, 0, \frac{\sqrt{6}}{12})$, $G_2 = (\frac{2}{3}, 0, \frac{\sqrt{6}}{12})$, $G_3 = (\frac{1}{2}, -\frac{\sqrt{3}}{18}, -\frac{\sqrt{6}}{36})$, $G_4 = (\frac{2}{3}, \frac{\sqrt{3}}{9}, -\frac{\sqrt{6}}{36})$, $G_5 = (\frac{1}{3}, \frac{\sqrt{3}}{9}, -\frac{\sqrt{6}}{36})$, and $G_6 = (\frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{\sqrt{6}}{12})$.

Define $\overline{V}$ to be the (closed) convex polyhedron whose vertices are O, a, d, e, C_1, C_2, C_3, D_3, D_4, D_5, G_1, G_2, G_3, G_4, G_5, and G_6, and V to be its interior (See Fig.A.1 and Fig.A.2).

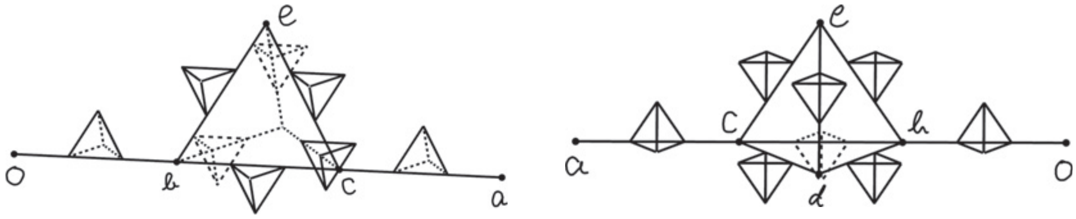

Fig. A.1.

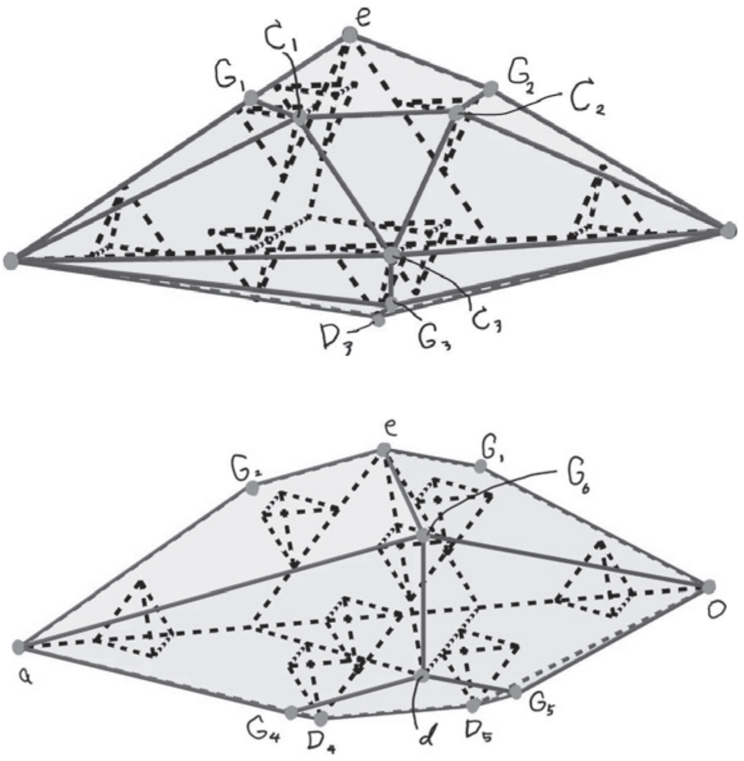

Fig. A.2.

The points a, b, c, d and e are as introduced in Fig.4 of Section 2.1, and also shown in Fig.A.1. The points C_1, C_2, C_3, D_3, D_4, and D_5 are defined as $C_1 = f_6(e)$, $C_2 = f_7(e)$, $C_3 = f_3(d)$, $D_3 = f_3(e)$, $D_4 = f_4(e)$ and $D_5 = f_5(e)$, respectively.

The vertex G_1 is determined in the following manner: Focus on the middle part of the edge $\overline{be}$. In constructing F^3 by iteration, as N in F^3_N increases, on this middle part of $\overline{be}$ grows an infinite series of tetrahedra, piling one upon another and contracting by a factor of $1/3$. Fig.A.3 shows the series of tetrahedra viewed in the direction of $\overrightarrow{be}$. Let E and F be the midpoints of the edges $\overline{cd}$ and $\overline{be}$, respectively. The straight line ℓ that goes through E and F pierces the midpoints of two edges in skew position of each tetrahedron in the series.

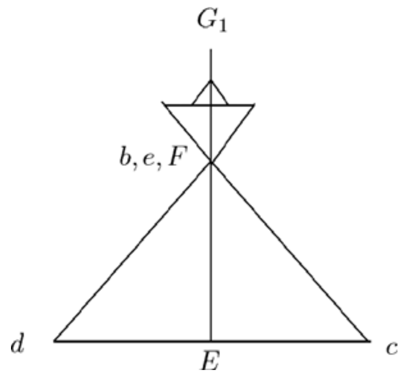

Fig. A.3.

The series converges to the point G_1 that lies on the line ℓ satisfying $|\overline{FG_1}|/|\overline{EF}| = 1/2$.

The points G_2, G_3, G_4, G_5, and G_6 are determined just in the same way, except that the edge $\overline{be}$ is replaced by $\overline{ce}$, $\overline{bc}$, $\overline{cd}$, $\overline{bd}$ and $\overline{ed}$, respectively.

Proof of $\bigcup_{i=1}^{8} f_i(V) \subseteq V$.

The polyhedron $\overline{V}$ is symmetric with respect to two planes, that is, $x = 1/2$ and the plane containing the parallel lines Oa and $f_8(d)f_8(e)$. The latter plane is shown in Fig.A.4.

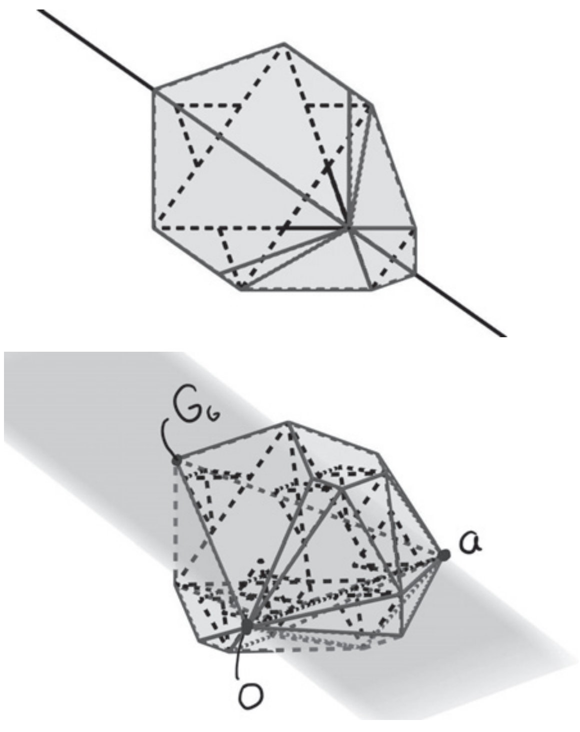

Fig. A.4.

Thus, it suffices to show that $f_1(V)$, $f_3(V)$, $f_6(V)$, $f_8(V) \subseteq V$ (see Fig.A.5). Since f_1 is the similitude with center O and of factor $1/3$, we immediately have $f_1(V) \subseteq V$. Note that from the convexity of $\overline{V}$, it is sufficient to show that the images of all the vertices of $\overline{V}$ belong to

$\overline{V}$. Clearly, $f_i(O), f_i(a) \in \overline{V}$ for $i = 3, 6, 8$.

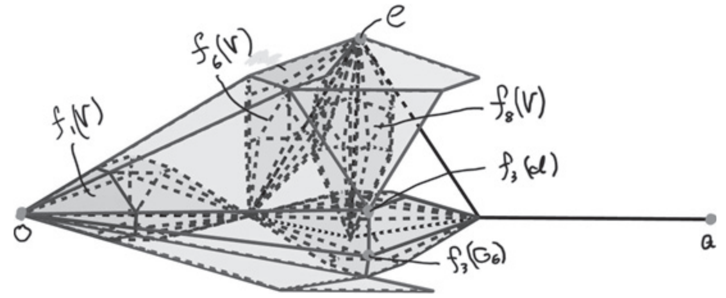

Fig. A.5(a). $f_3(V)$.

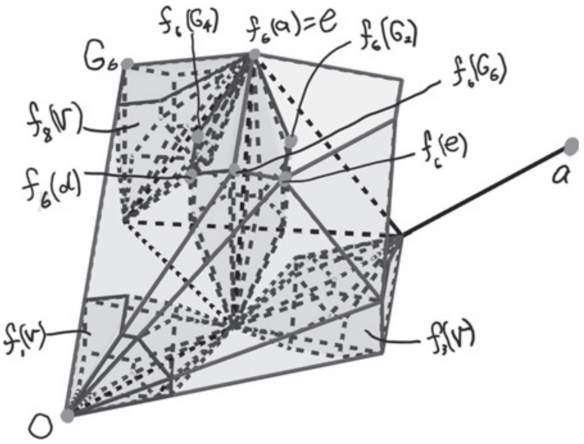

Fig. A.5(b). $f_6(V)$.

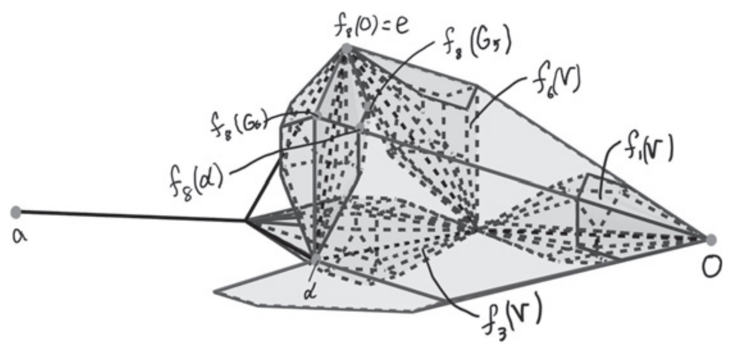

Fig. A.5(c). $f_8(V)$.

Concerning $f_3(V)$, since both V and $f_3(V)$ are symmetric with respect to the two planes described above, it is sufficient to show that $f_3(x) \subseteq \overline{V}$ for $x \in \{d, e, C_1, C_3, D_3, D_4, G_1, G_3, G_6\}$. We see that $f_3(d) = C_3$, $f_3(e) = D_3$ and $f_3(G_6) = G_3$. For the rest of the vertices, we have checked by direct calculation that their images are contained in $\overline{V}$, as is seen from Fig.A.5.

Concerning $f_6(V)$ and $f_8(V)$, we see that

$f_6(e) = C_1$, $f_6(G_6) = G_1$, $f_6(G_2) \in \overline{G_1G_2}$, and $f_6(G_4), f_6(d) \in \triangle OG_1G_6$.

$f_8(G_6) = G_6$, $f_8(d), f_8(G_5) \in \overline{OG_6}$. As for the images of the other vertices by f_6 and f_8, we have checked that they belong to $\overline{V}$. □

Proof of $f_i(V) \cap f_j(V) = \emptyset$ and $f_i(\overline{V}) \cap f_j(\overline{V}) \subset G_1^3$ for $i \neq j$..

It is sufficient to show that there is a plane that separates adjoining pairs of images of V, say, $f_3(V)$ and $f_6(V)$. Here we only show that $f_3(V) \cap f_6(V) = \emptyset$, because proofs for other pair can be done exactly in the same way. Let S be the plane that is perpendicular to $\overrightarrow{ce}$ and contains b. Note that $f_3(V)$ and $f_6(V)$ are symmetric with respect to S (See Fig.A.6).

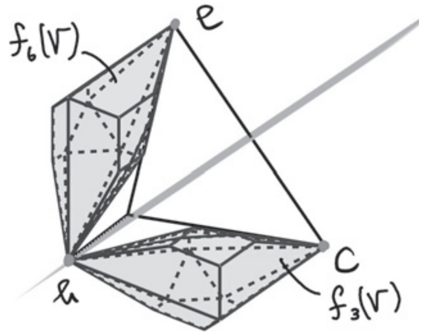

Fig. A.6.

Thus, if it is shown that $f_6(V)$ lies above S, then it automatically holds that $f_3(V)$ lies below S. We show that the line segments $\overline{bf_6(C_3)}$ and $\overline{bf_6(G_3)}$ lie above S except the end point b. To this end, we calculate inner products

$$\overrightarrow{bf_6(C_3)} \cdot \overrightarrow{ce} = \left(\frac{1}{9}, \frac{2\sqrt{3}}{81}, \frac{4\sqrt{6}}{81}\right) \cdot \left(-\frac{1}{6}, \frac{\sqrt{3}}{18}, \frac{\sqrt{6}}{9}\right) = \frac{1}{54} > 0,$$

and

$$\overrightarrow{bf_6(G_3)} \cdot \overrightarrow{ce} = \left(\frac{1}{9}, \frac{\sqrt{3}}{27}, \frac{5\sqrt{6}}{108}\right) \cdot \left(-\frac{1}{6}, \frac{\sqrt{3}}{18}, \frac{\sqrt{6}}{9}\right) = \frac{1}{54} > 0.$$

This result shows that these two vectors have positive components in the direction of $\overrightarrow{ce}$, and so the line segments $\overline{bf_6(C_3)}$ and $\overline{bf_6(G_3)}$ lie above the plane S. In exactly the same manner, we can show that the other vertices of $f_6(V)$ are above S. □

Proof of $F^3 \subseteq \overline{V}$. Recall that F_2^3 is the set shown in Fig.3 of Section 2.1. The fact that $F_2^3 \subseteq \overline{V}$ leads to $f_i(F_2^3) \subseteq f_i(\overline{V})$ for $i = 1, \ldots, 8$, which further gives $F_3^3 = f^{(3)}(F_2^3) = \bigcup_{i=1}^{8} f_i(F_2^3) \subseteq f^{(3)}(\overline{V}) \subseteq \overline{V}$. It follows inductively that $F_N^3 \subseteq \overline{V}$ for all $N \in \mathbb{N}$, that is, $\bigcup_{N=1}^{\infty} F_N^3 \subseteq \overline{V}$. Taking closure, we have $F^3 \subseteq \overline{V}$. □

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