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EQUIVALENCE BETWEEN FOUR MODELS
OF ASSOCIAHEDRA

Somnath BASU and Sandip SAMANTA

Abstract

We present a combinatorial isomorphism between Stasheff associahedra and an inductive cone construction of those complexes given by Loday. We give an alternate description of certain polytopes, known as multiplihedra, which arise in the study of A_∞ maps. We also provide new combinatorial isomorphisms between Stasheff associahedra, collapsed multiplihedra, and graph cubeahedra for path graphs.

Contents

1. Introduction

Dov Tamari, in his 1951 thesis [30], first described associahedra (with notation $\mathcal{M}_{n-1}$) as the realization of his poset lattice of bracketings (parenthesizations) of a word with n letters. He had also pictured the 1, 2 and 3 dimensional cases (cf. figure 1a). Later these were rediscovered by Jim Stasheff [29] in his 1960 thesis on homotopy associativity and based loop spaces. Stasheff had defined these (with notation K_n) as a convex, curvilinear subset of the $(n-2)$ dimensional unit cube (cf. figure 1b) such that it is homeomorphic to the cube. Convex polytope realizations of associahedra were subsequently done by many people [16, 15, 19, 20]. These polytopes are commonly known as *associahedra* or *Stasheff complexes*.

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Ever since Stasheff's work, associahedra (and their face complexes) have continued to appear in various mathematical fields apart from its crucial role in homotopy associative algebras and its important role in discrete geometry. Indeed, the associahedron K_{n-1} appears as a fundamental tile of $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$, the compactification of the real moduli space of punctured Riemann sphere [7]. It also appears in the analysis of the compactified moduli space of *nodal disks* with markings, as described by Fukaya and Oh [14]. An important connection between associahedra (and its generalizations) and finite root systems was established in 2003 by the work of Fomin and Zelevinsky [10]. In 2006 Carr and Devadoss [5] generalized associahedra to graph associahedra $\mathcal{K}G$ for a given graph G. These appear as the tiling of minimal blow-ups of certain Coxeter complexes [5]. In particular, if G is a path graph, then $\mathcal{K}G$ is an associahedron. Bowlin and Brin [4], in 2013, gave a precise conjecture about existence of coloured paths in associahedra. They showed that this conjecture is equivalent to the four colour theorem (4CT). Earlier, in 1988, there was a celebrated work [27] of Sleator, Tarjan and Thurston on the diameter of associahedra. While working on dynamic optimality conjecture, they had used hyperbolic geometry techniques to show that the diameter of K_d is at most $2d - 8$ when $d \geq 11$, and this bound is sharp when d is large enough. Pournin [25], almost twenty five years later, showed that this bound is sharp for $d \geq 11$. Moreover, his proof was combinatorial. Even in theoretical physics, recent works [24, 2, 9] indicate that associahedron plays a key role in the theory of scattering amplitudes.

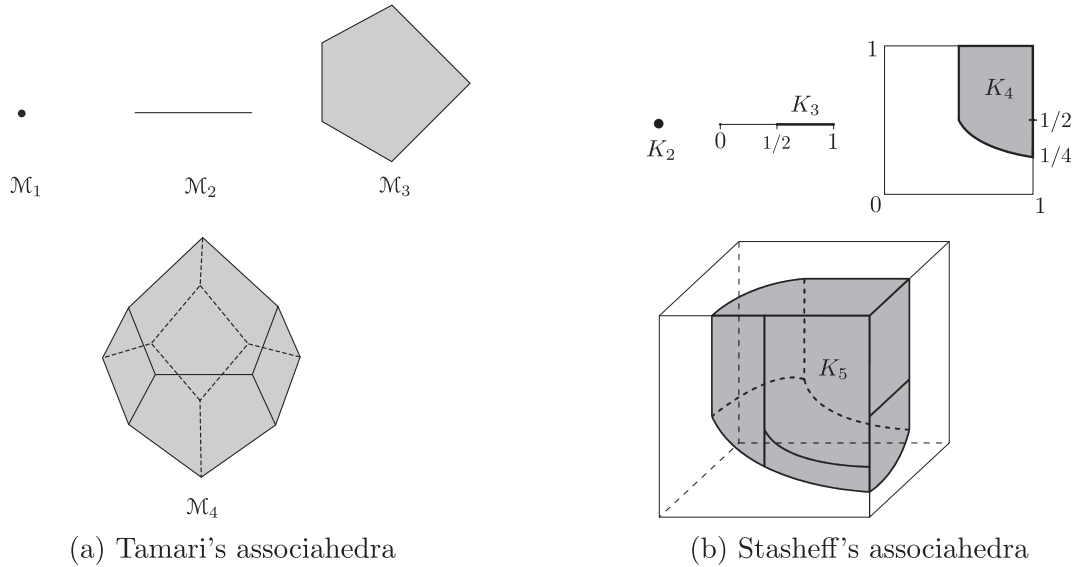(a) Tamari's associahedra   (b) Stasheff's associahedra

Fig. 1. Earliest realizations of associahedra

Let us briefly recall the construction in [29]. Stasheff, respecting Tamari's description, had sub-divided the boundary of K_n in such a way that the number of faces of codimension 1 and the adjacencies in his model matched with that in [30]. The boundary of K_n, denoted by L_n, is the union of homeomorphic images of $K_p \times_r K_q$ ($p+q = n+1$, $r = 1, 2, ..., p$), where $K_p \times_r K_q$ corresponds to the bracketing $x_1 \ldots (x_r \ldots x_{r+q-1}) \ldots x_n$. Stasheff started with K_2 as a point and defined K_n, inductively, as a cone over L_n. This definition of K_n involves K_2 through K_{n-1} all together.

As associahedra are contractible, these are of less interest as spaces in isolation. However, as combinatorial objects, the key properties of it are inherent in its description as a convex polytope. Much later, in 2005, J. L. Loday [21] gave a different inductive construction of K_n starting from K_{n-1}, leaving it to the reader to verify the details. Being a predominantly topological construction, it is not apparent why the cone construction of Loday gives rise to the known combinatorial structure on the associahedra. It is, therefore, natural to search for an explicit combinatorial isomorphism between these two constructions, leading to our first result (Theorem 3.2).

Theorem A. *Stasheff complexes are combinatorially isomorphic to Loday's cone construction of associahedra.*

There is another set of complexes $\mathcal{J}(n)$, known as *multiplihedra*, which were first introduced and pictured by Stasheff [28] in order to define A_∞ maps between A_∞ spaces, for $n \leq 4$. Mau and Woodward [22] have shown $\mathcal{J}(n)$'s to be compactification of the moduli space of quilted disks. Boardman and Vogt [3] provided a definition of $\mathcal{J}(n)$ in terms of painted trees (refer to Definition 2.6). The first detailed definition of $\mathcal{J}(n)$ and its combinatorial properties were described by Iwase and Mimura [17], while its realization as convex polytopes was achieved by Forcey [11], combining the description of Boardman-Vogt and Iwase-Mimura. Later, Devadoss and Forcey [6] generalized multiplihedra to *graph multiplihedra* $\mathcal{J}G$ for a given graph G.

In the study of A_∞ maps from an A_∞ space to a strictly associative H space (i.e., a topological monoid), multiplihedra degenerate to what we call *collapsed multiplihedra*. Stasheff [28] had pointed out that these complexes resemble associahedra. It has been observed that collapsed multiplihedra can be viewed as degeneration of graph multiplihedra for path graphs. It was long assumed that for A_∞ maps from a strictly associative H space to a A_∞ space, multiplihedra would likewise degenerate to yield associahedra. But it was Forcey [12] who realized that new polytopes were needed. These were constructed by him and named *composihedra*.

In this paper, we will give an equivalent definition (Definition 2.10) of multiplihedra, which induces a definition for collapsed multiplihedra (Definition 2.12). Using this definition, we will give a proof of the following (Proposition 3.4) by providing a new bijection of underlying posets.

Observation a. *Stasheff complexes and collapsed multiplihedra are combinatorially isomorphic.*

There is a well-known bijection bij_3 (cf. Forcey's paper [13, p. 195]; prior to Remark 2.6 and Figure 7) which is different from ours.

In 2010, Devadoss, Heath, and Vipismakul [8] defined a polytope called *graph cubeahedron* (denoted by CG) associated to a graph G. These are obtained by truncating certain faces of a cube. They gave a convex realization of these polytopes as simple convex polytopes whose face poset is isomorphic to the poset of design tubings for graphs. Graph cubeahedra for cycle graphs G (called *halohedra*) appear as the moduli space of annulus with marked points on one boundary circle. In this paper, we are mainly interested in CG for path graphs G and will prove the following (Proposition 3.5) by providing a new bijection of underlying posets.

Observation b. *The collapsed multiplihedra and graph cubeahedra for path graphs are combinatorially isomorphic.*

It turns out that bijection obtained between the posets governing Stasheff complexes and graph cubeahedra (for path graphs), by combining our bijections from Observations a and b, is the bijection of posets defined in [8, Proposition 14]. From our perspective, the bijection in Observation b is natural. Combining Theorem A, Observations a and b, we obtain the following result (Theorem 3.1).

Theorem B. *The four models of associahedra - Stasheff complexes, complexes obtained by Loday's cone construction, collapsed multiplihedra, graph cubeahedra for path graphs - are all combinatorially isomorphic.*

ORGANIZATION OF THE PAPER. The paper is organized as follows. In §2.1, we will review some of the definitions and results related to Stasheff's description of associahedra. In §2.2, the description of Loday's cone construction and some related theorems are presented while in §2.3 an equivalent definition of multiplihedra and collapsed multiplihedra is given. In §2.4 the definition of tubings, design tubings, graph cubeahedra, and related results are presented. The next section §3 contains the proof of the main result (Theorem B), which is a combination of three results. In §3.1 we prove Theorem A while §3.2 and §3.3 are devoted to the proofs of Observations a and b respectively.

2. Description of Four Models of Associahedra

An H-space is a topological space X equipped with a binary operation $m : X^2 \to X$ having a unit e. It is a natural generalization of the notion of topological groups. We can rewrite m as a map $m_2 : K_2 \times X^2 \to X$, where K_2 is a point. If m is not associative but homotopy associative (called weakly associative), then we have a map $m_3 : K_3 \times X^3 \to X$ defined through the homotopy between $m \circ (m \times 1)$ and $m \circ (1 \times m)$, where K_3 is an interval. Similarly, we can define five different maps from $X^4 \to X$ using m, and between any two such maps, there are two different homotopies (using the chosen homotopy associativity). If those two homotopies are homotopic, this defines a map $m_4 : K_4 \times X^4 \to X$, where K_4 is a filled pentagon. If we continue this process, we obtain a map $m_n : K_n \times X^n \to X$ for $n \geq 2$. These complexes K_n, called associahedra, are our main objects of interest.

We will briefly describe the four models of associahedra, one in each subsection, we are concerned with: Stasheff complexes, Loday's cone construction, collapsed multiplihedra, and graph cubeahedra for path graphs.

2.1. Stasheff complexes. Stasheff defined for each $i \geq 2$, a special cell complex K_i as a subset of I^{i-2}. It is a simplicial complex and has i degeneracy operators $s_1, ..., s_i$. Moreover, K_i has $\binom{i}{2} - 1$ faces of codimension 1. The complexes K_i, as combinatorial objects, are more complicated than the standard simplices Δ^{i-2}. According to Stasheff [29], it is defined through following intuitive content:

Consider a word with i letters, say $x_1 x_2 \ldots x_i$, and all meaningful ways of inserting one pair of parentheses. To each such insertion except for $(x_1 x_2 ... x_i)$, there corresponds a cell of L_i, the boundary of K_i. If the parentheses enclose x_k through x_{k+s-1}, we regard this cell as $K_r \times_k K_s$, the homeomorphic image of $K_r \times K_s$ under a map which we call $\partial_k(r, s)$, where $r + s = i + 1$. Two such cells intersect only on their boundaries and the 'edges' so formed

correspond to inserting two pairs of parentheses in the word. Specifically, the intersection of two cells, namely $K_r \times_k K_s$ and $K_{r'} \times_{k'} K_{s'}$, occurs if and only if one of the following conditions is satisfied: $(k \leq k' < k' + s' - 1 \leq k + s - 1)$, or $(k' \leq k < k + s - 1 \leq k' + s' - 1)$, or $(k + s - 1 < k')$, or $(k' + s' - 1 < k)$. Furthermore, if these two cells intersect, the intersection takes place along an $(i-4)$-dimensional sub-cell, which is either of the form $K_a \times_j (K_b \times_l K_c)$ appears as a cell of $K_a \times_j \partial K_{b+c-1}$, or of the form $(K_a \times_j K_b) \times_l K_c$ appears as a cell of $\partial K_{a+b-1} \times_l K_c$ for $a + b + c = i + 2$. Now the adjacency criterion is given by the following relations:

(a) $\partial_j(r, s + t - 1)(1 \times \partial_k(s,t)) = \partial_{j+k-1}(r + s - 1, t)\big(\partial_j(r,s) \times 1\big)$

(b) $\partial_{j+s-1}(r + s - 1, t)(\partial_k(r, s) \times 1) = \partial_k(r + t - 1, s)\big(\partial_j(r,t) \times 1\big)(1 \times T)$

where $T : K_s \times K_t \to K_t \times K_s$ permutes the factors. In terms of homeomorphic images of $K_r \times K_s \times K_t$, the above two relations are equivalent respectively to the identifications

(1) $$K_r \times_j (K_s \times_k K_t) = (K_r \times_j K_s) \times_{j+k-1} K_t$$

(2) $$(K_r \times_k K_s) \times_{j+s-1} K_t = (K_r \times_j K_t) \times_k K_s$$

One can easily track these relations once they are identified with 2-bracketing of the word $x_1 x_2 \ldots x_i$. The cell $K_a \times_j (K_b \times_l K_c)$ corresponds to the nested 2-bracketing

$$x_1 \ldots (x_j \ldots (x_{j+l-1} \ldots x_{j+l+c-2}) \ldots x_{j+b+c-2}) \ldots x_i,$$

for $a + b + c = i + 2$. The cell $(K_a \times_j K_b) \times_l K_c$ corresponds to $y_1 \ldots (y_j \ldots y_{j+b-1}) \ldots y_{a+b-1}$ with

$$y_u = \begin{cases} x_u & \text{if } u < l \\ (x_l \ldots x_{l+c-1}) & \text{if } u = l \\ x_{u+c-1} & \text{if } u > l \end{cases}$$

Now if $j \leq l \leq j + b$, then $(K_a \times_j K_b) \times_l K_c$ represents a nested 2-bracketing and thus corresponds to a cell of first type; this case is reflected in the identification (1) above. If $l + c < j$ or $j + b < l$, then $(K_a \times_j K_b) \times_l K_c$ represents a disjoint 2-bracketing and that corresponds to two possible cells of the second type; this case is reflected in (2).

This is enough to obtain K_i by induction. Start with $K_2 = \{*\}$ as a point. Given K_2 through K_{i-1}, construct L_i by fitting together copies of $K_r \times_k K_s$ as indicated by the above conditions, and take K_i to be the cone on L_i. Stasheff proved that these complexes are homeomorphic to cubes.

Proposition 2.1 ([29, Proposition 3]). *K_i is homeomorphic to I^{i-2} and degeneracy maps $s_j : K_i \to K_{i-1}$ for $1 \leq j \leq i$ can be defined so that the following relations hold:*

(1) $s_j s_k = s_k s_{j+1}$ *for* $k \leq j$.

(2) $s_j \partial_k(r, s) = \partial_{k-1}(r - 1, s)\big(s_j \times 1\big)$ *for* $j < k$ *and* $r > 2$.

(3) $s_j \partial_k(r, s) = \partial_k(r, s - 1)\big(1 \times s_{j-k+1}\big)$ *for* $s > 2, k \leq j < k + s$,
$s_j \partial_k(i - 1, 2) = \pi_1$ *for* $1 < j = k < i$ *and* $1 < j = k + 1 \leq i$,
$s_1 \partial_2(2, i - 1) = \pi_2$ *and* $s_i \partial_1(2, i - 1) = \pi_2$,
where π_m *for* $m = 1, 2$ *is projection onto the m-th factor.*

(4) $s_j \partial_k(r, s) = \partial_k(r - 1, s)\big(s_{j-s+1} \times 1\big)$ *for* $k + s \leq j$.

Using boundary maps $\partial_k(r, s)$ and degeneracy maps s_j, Stasheff defined the following.

DEFINITION 2.2 (A_n FORM AND A_n SPACE). An A_n form on a space X consists of a family of maps $m_i : K_i \times X^i \to X$ for $2 \leq i \leq n$ such that

(1) there exists $e \in X$ with $m_2(*, e, x) = m_2(*, x, e) = x$ for $x \in X, * = K_2$.
(2) For $\rho \in K_r, \sigma \in K_s, r + s = i + 1$, we have

$$m_i \left(\partial_k(r, s)(\rho, \sigma), x_1, \cdots, x_i \right) = m_r \left(\rho, x_1, \cdots, x_{k-1}, m_s \left(\sigma, x_k, \cdots, x_{k+s-1} \right), x_{k+s}, \cdots, x_i \right).$$

(3) For $\tau \in K_i$ and $i > 2$, we have

$$m_i \left(\tau, x_1, \cdots, x_{j-1}, e, x_{j+1}, \cdots, x_i \right) = m_{i-1} \left(s_j(\tau), x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_i \right).$$

The pair $(X, \{m_i\}_{2 \leq i \leq n})$ is called an A_n space.
If the maps m_i exist for all i, then it is called an A_∞ form, and the corresponding pair is called an A_∞ space.

Homotopy associative algebras (or A_∞ algebras), A_∞ spaces, and operads have been extensively studied. The interested reader is directed to the excellent books [23, 3, 1] and introductory notes [18].

The correspondence between the faces of Stasheff complexes (associahedra) and the bracketings indicate that these complexes can also be defined as follows.

DEFINITION 2.3 (ASSOCIAHEDRON). Let $\mathfrak{P}(n)$ be the poset of bracketings of a word with n letters, ordered such that $p < p'$ if p is obtained from p' by adding new brackets. The associahedron K_n is a convex polytope of dimension $n - 2$ whose face poset is isomorphic to $\mathfrak{P}(n)$.

This construction of the polytope K_n was first given in 1984 by Haiman in his (unpublished) manuscript [15]. In 1989, C. Lee [19, Theorem 1] proved this by considering the collection of all sets of mutually non-crossing diagonals of a polygon. Observe that the sets of mutually non-crossing diagonals of an $(n+1)$-gon are in bijective correspondence with the bracketings of a word with n letters. We will use this description later in §3.2.

2.2. Loday's Cone Construction. From the combinatorial description given by Stasheff, the associahedron K_n is a complex of dimension $n - 2$ whose vertices are in bijective correspondence with the $(n - 2)$-bracketing of the word $x_1 x_2 \ldots x_n$. But each $(n - 2)$-bracketing of the word $x_1 x_2 \ldots x_n$ corresponds to a rooted planar binary tree with $n + 1$ leaves, one of them being the root. For example, the planar rooted trees associated with $x_1(x_2(x_3 x_4))$ and $(x_1 x_2)(x_3 x_4)$ are depicted below (cf. figure 2a, 2b), the root being represented by the vertical leaf in each case.

(a) $x_1(x_2(x_3 x_4))$ (b) $(x_1 x_2)(x_3 x_4)$

Fig. 2. Correspondence between bracketing and rooted binary tree

Thus K_n can also be thought of as a complex of dimension $n - 2$ whose vertices are in bijective correspondence with planar rooted binary trees with n leaves and 1 root. Let Y_n be the set of such trees. The trees are depicted below for $2 \leq n \leq 4$.

$$Y_2 = \left\{ \bigvee \right\}, \ Y_3 = \left\{ \bigvee , \bigvee \right\}, \ Y_4 = \left\{ \bigvee , \bigvee , \bigvee , \bigvee , \bigvee \right\}$$

Any $t \in Y_n$ is defined to have *degree n*. We label the leaves (not the root) of t from left to right by $0, 1, \cdots, n-1$. Then we label the internal vertices by $1, 2, \cdots, n-1$. The i-th internal vertex is the one that falls in between the leaves $i - 1$ and i. We denote by a_i, respectively b_i, the number of leaves on the left side, respectively right side, of the i-th vertex. The product $a_i b_i$ is called the *weight* of the i-th internal vertex. To each tree $t \in Y_n$, we associate the point $M(t) \in \mathbb{R}^{n-1}$, whose i-th coordinate is the weight of the i-th vertex:

$$M(t) = (a_1 b_1, \cdots, a_i b_i, \cdots, a_{n-1} b_{n-1}) \in \mathbb{R}^{n-1}$$

For instance,

$$M \left(\bigvee \right) = (1), \ M \left(\bigvee \right) = (2, 1), \ M \left(\bigvee \right) = (1, 2),$$

$$M \left(\bigvee \right) = (1, 2, 3), \ M \left(\bigvee \right) = (1, 4, 1)$$

Observe that the weight of a vertex depends only on the sub-tree that it determines. Using these integral coordinates, Loday [20] gave a convex realization of K_{n+1} in $\mathbb{R}^n$.

Lemma 2.4 ([20, Lemma 2.5]). *For any tree $t \in Y_{n+1}$ the coordinates of the point $M(t) = (x_1, \cdots, x_n) \in \mathbb{R}^n$ satisfy the relation*

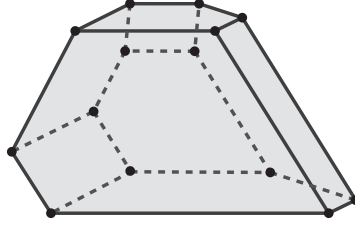$$\sum_{k=1}^{n} x_k = \tfrac{1}{2} n(n + 1).$$

Thus, it follows that

$$M(t) \in H_n = \left\{ (x_1, ..., x_n) \in \mathbb{R}^n : x_1 + x_2 + ... + x_n = \tfrac{n(n+1)}{2} \right\}.$$

Theorem 2.5 ([20, Theorem 1.1]). *The convex hull of the points $M(t) \in \mathbb{R}^n$, for $t \in Y_{n+1}$, is a realization of the Stasheff complex K_{n+1} of dimension $n - 1$.*

For example, the complex K_5 lies in the hyperplane H_4 in $\mathbb{R}^4$. Under an isometric transformation of H_4 to $\mathbb{R}^3$ (i.e., $x_4 = 0$ hyperplane), the embedded picture of K_5 is shown in figure 3.

Now starting with K_2 as a point, Loday [21, §2.4] gave a different inductive construction of the complexes K_{n+1}. The steps are as follows:

(1) Start with the associahedron K_n, which is a topological ball with the cellular sphere as the boundary. The cells of the boundary are of the form $K_p \times_r K_q$ where $p + q = n + 1$ and $r = 1, 2, ..., p$.

(2) Enlarge each cell $K_p \times_r K_q$ of the the boundary of K_n into a cell of dimension n by replacing it with $K_{p+1} \times_r K_q$ keeping the adjacency of the cells intact. Explicitly, suppose two cells $K_p \times_r K_q$ and $K_{p'} \times_{r'} K_{q'}$ are adjacent with a common boundary

Fig. 3. Loday's embedded K_5 in $\mathbb{R}^3$

sub-cell $K_a \times_j (K_b \times_l K_c)$ or $(K_a \times_j K_b) \times_l K_c$ (that are the only possibilities, check identification (1), (2) in §2.1) on the boundary of K_n. Then the cells $K_p \times_r K_q$ and $K_{p'} \times_{r'} K_{q'}$ are enlarged to $K_{p+1} \times_r K_q$ and $K_{p'+1} \times_{r'} K_{q'}$ so that they share the common enlarged boundary sub-cell $K_{a+1} \times_j (K_b \times_l K_c)$ or $(K_{a+1} \times_j K_b) \times_l K_c$. We denote the total enlarged complex by $\widehat{K}_n$. Topologically, one may think this process of enlargement as a quotient space as follows.

$$\widehat{K}_n = \left(K_n \bigsqcup_{(p,q,r) \in V_n} (K_{p+1} \times_r K_q) \right) \bigg/ \sim, \text{ where}$$

$$V_n = \{(p,q,r) \in \mathbb{N}^3 : p, q \geq 2; p + q = n + 1; 1 \leq r \leq p\} \text{ and}$$

$$\partial(K_n) \ni K_p \times_r K_q \sim (K_2 \times_1 K_p) \times_r K_q \in \partial(K_{p+1} \times_r K_q) \text{ with identification (1), (2) on}$$

$$\partial(K_{p+1} \times_r K_q) \text{ and } \partial(K_{p'+1} \times_{r'} K_{q'}) \text{ for all } (p,q,r), (p',q',r') \in V_n.$$
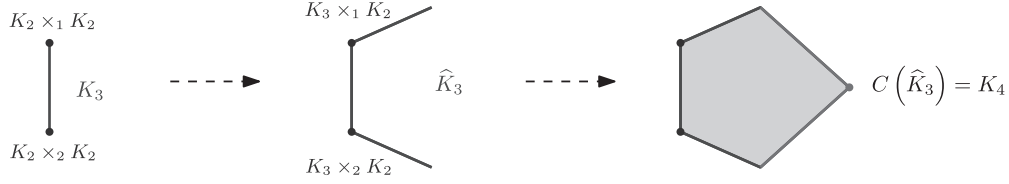
(3) Take the topological cone over the above enlargement $\widehat{K}_n$ and declare that to be K_{n+1}, i.e. $K_{n+1} := C(\widehat{K}_n) = \frac{\widehat{K}_n \times [0,1]}{\widehat{K}_n \times \{1\}}$. By regarding $[\widehat{K}_n \times \{1\}]$ as the abstract cone point x_0 (say), one may think $[(x, t)]$ in $C(\widehat{K}_n)$ as the point $tx_0 + (1-t)x$ on the segment joining x to x_0 for $t \in [0, 1]$.

Note that the above construction of K_{n+1} from K_n does not give any convex structure to it. But embedding the enlarged complex $\widehat{K}_n$ in $\mathbb{R}^{n-1}$ and choosing an appropriate cone point there, it is possible to realize $K_{n+1} = C(\widehat{K}_n)$ as a convex polytope in $\mathbb{R}^n$. This is illustrated in the following examples in low dimensions. However, for the general cases, we shall restrict ourselves to the topological part only.
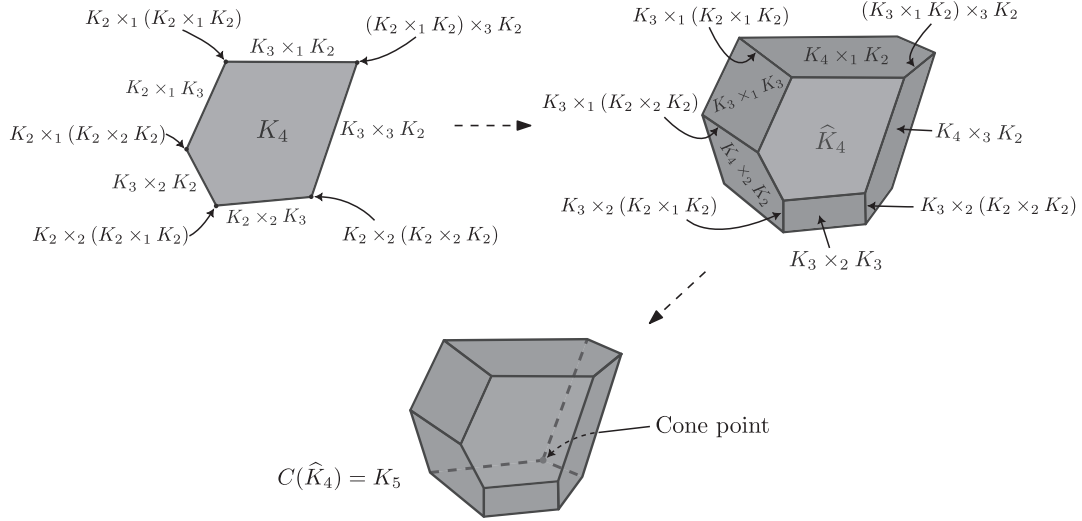
(i) To construct K_3 from K_2, form the enlarged complex $\widehat{K}_2$, which is a point (as K_2 has no boundary). Then K_3 is a cone over the point $\widehat{K}_2$, i.e., an interval.

Fig. 4. K_3 from K_2

(ii) To construct K_4 from K_3, note that K_3 has two boundary points namely $K_2 \times_1 K_2$ and $K_2 \times_2 K_2$. Thus $\widehat{K}_3$ consists of the original K_3 together with $K_3 \times_1 K_2$ and $K_3 \times_2 K_2$, which looks like an angular 'C' shape. Finally, K_4 is the cone over $\widehat{K}_3$, resulting in a filled pentagon.

(iii) To construct K_5 from K_4, see that K_4 has five boundary edges namely $K_2 \times_1 K_3$, $K_3 \times_1 K_2$, $K_3 \times_3 K_2$, $K_2 \times_2 K_3$ and $K_3 \times_2 K_2$. Each one of these shares its two boundary endpoints with the other two. See in the below picture that $K_3 \times_1 K_2$ and

Fig.5. K_4 from K_3

$K_3 \times_3 K_2$ have the common boundary point $(K_2 \times_2 K_2) \times_1 K_2 = (K_2 \times_1 K_2) \times_3 K_2$ (by (2)). Similarly, others are obtained via identification (1). Thus in $\widehat{K_4}$, each of the enlarged five cells $K_3 \times_1 K_3$, $K_4 \times_1 K_2$, $K_4 \times_3 K_2$, $K_3 \times_2 K_3$ and $K_4 \times_2 K_2$ shares two boundary edges (obtained as an enlargement of the common boundary points) with other two. Finally, K_5 is the cone over $\widehat{K_4}$.

Fig.6. K_5 from K_4

2.3. Collapsed Multiplihedra. Suppose $(X\{m_i\}), (Y, \{m_i'\})$ are two A_∞ spaces and f : $X \to Y$ is a weak homomorphism i.e., there is a homotopy between the maps $f \circ m_2$ and $m_2' \circ (f \times f)$. Such maps are called H-maps. In general, there is a notion of A_n maps in Stasheff [29, II, Def. 4.1], which satisfy $f \circ m_i = m_i' \circ (1 \times f^i)$ for $i \leq n$. Thus we have a map $f_2 : \mathcal{J}(2) \times X^2 \to Y$, where $\mathcal{J}(2)$ is an interval. To match things up, rewrite f as $f_1 : \mathcal{J}(1) \times X \to Y$, where $\mathcal{J}(1)$ is a single point. Now using m_2, m_2', f, there are six different ways (cf. figure 9) to define a map from X^3 to Y, namely $f \circ (m_2 \circ (m_2 \times 1))$, $f \circ (m_2 \circ (1 \times m_2))$, $m_2' \circ (f \times m_2)$, $m_2' \circ (m_2 \times f)$, $m_2' \circ (1 \times m_2') \circ (f \times f \times f)$, $m_2' \circ (m_2' \times 1) \circ (f \times f \times f)$. Using the weak homomorphism of f and weak associativity in X, Y (due to the existence of m_3, m_3'), one realizes that there are two different homotopies between any two of the six maps. If those two homotopies are homotopic themselves, then we have a map $f_3 : \mathcal{J}(3) \times X^3 \to Y$, where $\mathcal{J}(3)$ is a filled hexagon.

If we continue this process, we will get a map $f_n : \mathcal{J}(n) \times X^n \to Y$ for each $n \geq 1$. These complexes $\mathcal{J}(n)$ are called multiplihedra. In the figure 7b below, the dark edges collapse to a point so that the rectangular faces degenerate to edges and the pentagonal face degenerates

to a single point, giving rise to Loday's realization of K_5. There is a different degeneration from $\mathcal{J}(n)$ to K_{n+1}, as shown in [26, §5]; figure 7c exhibits this for $\mathcal{J}(4)$.

(a) Embedded $\mathcal{J}(4)$ in $\mathbb{R}^3$

(b) Shaded faces collapsed to get K_5
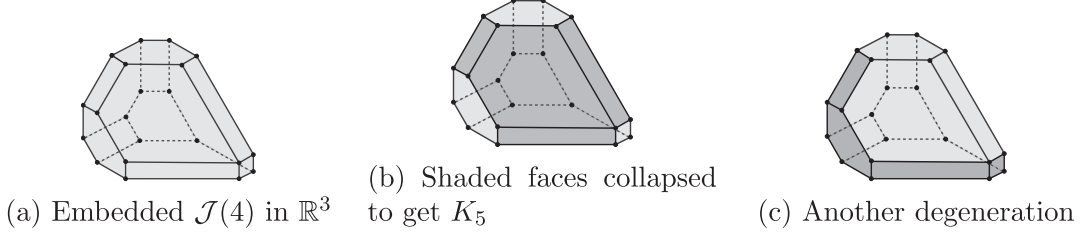
(c) Another degeneration

Fig. 7. $\mathcal{J}(4)$ and its degeneration to K_5

Multiplihedra first appeared in the work of Stasheff [28]. However, in 1986, Norio Iwase and Mamoru Mimura [17, Section 2] gave the first detailed construction of $\mathcal{J}(n)$ with face operators, and described their combinatorial properties. It was also shown that $\mathcal{J}(n)$ is homeomorphic to the unit cube of dimension $n-1$. Using this description of $\mathcal{J}(n)$, they defined A_n maps. But even before them, Boardman and Vogt [3] (around 1973) had developed several homotopy equivalent versions of a space of *painted binary trees* with interior edges of length in [0, 1] to define maps between A_∞ spaces which preserve the multiplicative structure up to homotopy. In 2008, Forcey [11, Theorem 4.1] proved that the space of painted trees with n leaves, as convex polytopes, are combinatorially equivalent to the CW-complexes described by Iwase and Mimura. Indeed, Forcey associated a coordinate to each painted binary tree, which generalized the Loday's integer coordinates associated with binary trees corresponding to the vertices of associahedra. Figure 7a of $\mathcal{J}(4)$ is drawn with such coordinates for the vertices. We shall use the definition of $\mathcal{J}(n)$, as defined in [11], in terms of painted trees.

DEFINITION 2.6. A *painted tree* is painted beginning at the root edge (the leaf edges are unpainted), and always painted in such a way that there are only following three types of nodes:

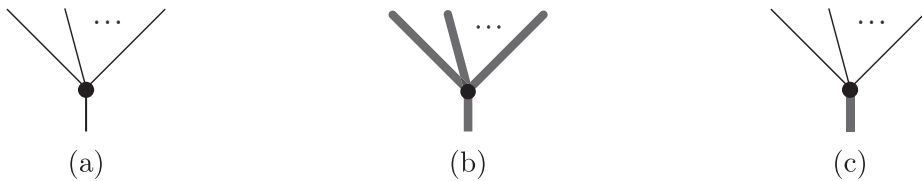(a)   (b)   (c)

Fig. 8. Admissible nodes

This limitation on nodes implies that painted regions must be connected, and that painting must proceed up every branch of a node.

Let $J(n)$ consist of all painted trees with n leaves. There is a refinement ordering defined as follows.

DEFINITION 2.7 ([11, Definition 1]). For $t, t' \in J(n)$, we say t *refines* t' and denote by $t \preccurlyeq t'$ if t' obtained from t by collapsing some of its internal edges.
We say t *minimally refines* t' if t refines t' and there is no $s \in J(n)$ such that both t refines s

and s refines t'.

Now $(J(n), \leqslant)$ is a poset with painted binary trees as smallest elements (in the sense that nothing refines them) and the painted corolla as the greatest element (in the sense that everything refines it). The n-th multiplihedra is defined as follows.

DEFINITION 2.8. The n-th multiplihedra $\mathcal{J}(n)$ is a convex polytope whose face poset is isomorphic to the poset $(J(n), \leqslant)$ of painted trees with n leaves.

The explicit inductive construction of these polytopes and the correspondence between the facets of $\mathcal{J}(n)$ and the painted trees follows from [11, Definition 4]. For instance, the vertices of $\mathcal{J}(n)$ are in bijection with the painted binary trees with n leaves; the edges are in bijection with those painted trees with n leaves which are obtained by the minimal refinement of painted binary trees with n leaves and they are glued together along the endpoints with matching associated to painted binary trees. In this way, the $(n-2)$-dimensional cells of $\mathcal{J}(n)$ are in bijection with those painted trees which refine to corolla with n leaves. They are glued together along $(n-3)$-dimensional cells with matching to associated painted trees to form the complex $\partial \mathcal{J}(n)$. Finally the $(n-1)$ dimensional complex $\mathcal{J}(n)$ is defined as the cone over $\partial \mathcal{J}(n)$ and it corresponds to the painted corolla with n leaves in the poset $J(n)$.
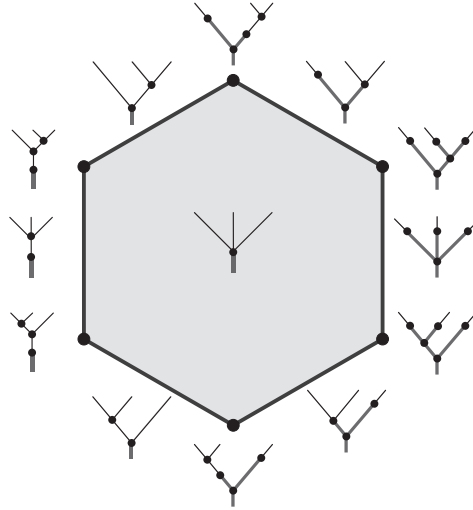

Fig.9. $\mathcal{J}(3)$ labelled by painted trees

We shall give an equivalent description of $\mathcal{J}(n)$ which reflects the promised representation of it stated at the beginning of this subsection. It is given as follows. Let $f : A \to B$ be a weak homomorphism (i.e., respects the multiplication in A and B up to homotopy) from an A_∞ space to another A_∞ space. For a given ordered collection $a_1, a_2, ..., a_n \in A$, there are three types of elements.

I. The f-image of the elements, which are obtained using different associations of the elements $a_1, a_2, ..., a_n$ in A. For example, $f(X)$, where X is some rule of association of the elements $a_1, a_2, ..., a_n$.

II. The elements obtained using f being homomorphism up to homotopy on the elements of type I and following the same association rule in B. For example, if

$X = (X_1)((X_2)(X_3))$ is some rule of association of $a_1, a_2, ..., a_n$, then elements of the form $f((X_1) \cdot ((X_2)(X_3)))$ is of this type. Here $f((X_1) \cdot ((X_2)(X_3)))$ denotes the homotopy equivalence between $f((X_1)((X_2)(X_3)))$ and $f(X_1)f((X_2)(X_3))$. Similarly, $f((X_1)\cdot((X_2)\cdot(X_3)))$, representing the homotopy equivalence between $f((X_1)(X_2 \cdot X_3))$ and $f(X_1)f((X_2) \cdot (X_3))$, is also of this type.

III. The elements that are obtained using different associations of the elements of type II in B. For example, if $X = (X_1)((X_2)((X_3)(X_4)))$ is some rule of association of $a_1, a_2, ..., a_n$, then the elements obtained using the different association of $f(X_1)$, $f(X_2)$, $f(X_3)$, $f(X_4)$ in B, namely

$$(f(X_1)f(X_2))(f(X_3)f(X_4)), \ f(X_1)f(X_2)(f(X_3)f(X_4)), \ f(X_1)(f(X_2)f(X_3))f(X_4),$$
$$(f(X_1)(f(X_2)f(X_3)))f(X_4), f(X_1)f(X_2)f(X_3)f(X_4)$$

are of this type.

DEFINITION 2.9. Let $\mathfrak{J}_n$ be the poset of all of the above three types of elements in B, ordered such that $P \prec P'$ if P is obtained from P' by at least one of the following operations:

(1) adding brackets in domain or co-domain elements.
(2) replacing '$\cdot$' by $)f($ without changing the association rule in P'.
(3) removing one or more consecutive '$\cdot$' by adding a pair of brackets that encloses all the adjacent elements to all those '$\cdot$' which are removed. In this process, ignore redundant bracketing (if obtained). The requirement of consecutive '$\cdot$' is to ensure allowable bracketing.

The above operations are to be understood in the following ways:

• For two type I (or III) elements P, P', we say $P \prec P'$ if P, P' follow above operation (1) in domain (or co-domain). For example, $f(a_1(a_2(a_3a_4))) \prec f(a_1(a_2a_3a_4))$, $f(a_1)(f(a_2)f(a_3a_4)) \prec f(a_1)f(a_2)f(a_3a_4)$.

• For two type II elements Q, Q', we say $Q \prec Q'$ if Q, Q' follow above operation (2) or (3). For example, $f(a_1)f(a_2 \cdot (a_3a_4)) \prec f(a_1 \cdot a_2 \cdot (a_3a_4))$, $f(a_1 \cdot (a_2(a_3a_4))) \prec f(a_1 \cdot (a_2 \cdot (a_3a_4)))$.

• For type I element P and type II element Q, we say $P \prec Q$ if P, Q follow above operation (3).
    For example, $f((a_1a_2)(a_3a_4)) \prec f((a_1a_2) \cdot (a_3a_4))$, $f(a_1a_2a_3a_4) \prec f(a_1 \cdot a_2 \cdot a_3 \cdot a_4)$.

• For type II element Q and type III element P, we say $P \prec Q$ if P, Q follow above operation (2) or (3). For example, $(f(a_1)f(a_2a_3))f(a_4) \prec f(a_1 \cdot (a_2a_3))f(a_4)$, $f(a_1)(f(a_2a_3)f(a_4)) \prec f(a_1)(f(a_2 \cdot a_3)f(a_4))$.

Now, depending on the poset $(\mathfrak{J}_n, \prec)$, we define another set of complexes J_n for $n \geq 1$.

DEFINITION 2.10. Define J_n to be the convex polytope of dimension $n - 1$, whose face poset is isomorphic to $(\mathfrak{J}_n, \prec)$ for $n \geq 1$.

The existence and the equivalence of these complexes with the multiplihedra follows from the following lemma.

Lemma 2.11. *J_n is isomorphic to the multiplihedron $\mathcal{J}(n)$ for any $n \geq 1$.*

Proof. It follows from the definitions of $\mathcal{J}(n)$ and J_n that to exhibit an isomorphism between the mentioned complexes, it is enough to provide an isomorphism at the poset level. Define a map $\Phi : J(n) \to \mathfrak{J}_n$ as follows.

i) Put a_1 through a_n from left to right above the leaves of a painted tree.

ii) If the leaves corresponding to a_k through a_l for $1 \leq k < l \leq n$ are joined to a node of type 8a or of type 8c, then associate $(a_k a_{k+1} \ldots a_l)$ (cf. figure 10a) or $f(a_k \cdot a_{k+1} \cdot \ldots \cdot a_l)$ (cf. figure 10c) respectively to that node. In case $1 \leq k = l \leq n$, then associate $f(a_k)$ to the corresponding node.

iii) Then proceed to the nodes just below the above ones. If a node is of type 8a or 8c joining X_1 through X_m as associated nodes just above, then associate $(X_1 X_2 \ldots X_m)$ or $f(X_1 \cdot X_2 \cdot \ldots \cdot X_m)$ respectively to that node. If a node is of type 8b joining $f(Y_1)$ through $f(Y_m)$ as associated nodes just above, then associate $(f(Y_1)f(Y_2) \ldots f(Y_m))$ to that node (cf. figure 10b).

iv) Continue the above step iii) till the root node of a painted tree.
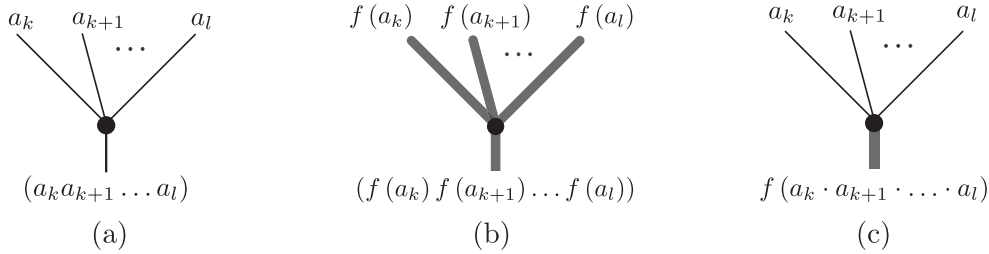

Fig. 10. Bijection between the nodes of painted tree and the elements of defined poset

The element (ignoring redundant brackets, if exist) associated to the root node of a painted tree $t \in J(n)$, is defined to be $\Phi(t) \in \mathfrak{J}_n$. For example, the Φ-image of the painted tree $t \in J(5)$ in figure 11 is $f(a_1 a_2)(f(a_3)f(a_4 \cdot a_5)) \in \mathfrak{J}_5$.

Note that each painted tree is uniquely determined by its nodes and each position of those nodes associates a unique element. Also, the image of $t \in J(n)$ under Φ is determined by the associated elements to the nodes of t. Thus Φ maps each element of $J(n)$ to a unique element of $\mathfrak{J}_n$ and hence Φ is a bijection.
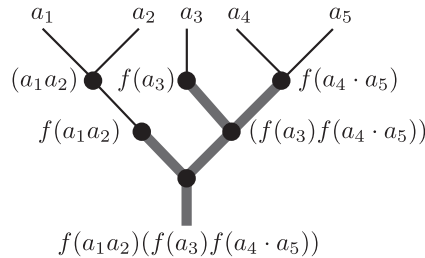

Fig. 11. Elements associated to the nodes

It remains to check that Φ preserves the partial order. By the definition of $\leqslant$, it is enough to show that $\Phi(t) < \Phi(t')$ when $t \leqslant t'$ minimally. If $t \leqslant t'$ minimally, then t' is obtained

from t by collapsing an unpainted internal edge or a painted internal edge or a bunch of painted edges. Note that collapsing an unpainted internal edge results in either the removal of brackets in the domain (operation (1) in $\mathfrak{J}_n$) or the addition of one or more $\cdot$ by removing brackets (operation (3) in $\mathfrak{J}_n$). Collapsing a painted internal edge results in the removal of brackets in the co-domain (operation (1) in $\mathfrak{J}_n$) while collapsing a bunch of painted edges results in replacing $)f($ by $\cdot$ (operation (2) in $\mathfrak{J}_n$). In all the cases $\Phi(t) \prec \Phi(t')$, completing the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Using this lemma, we consider J_n (Definition 2.10) as the n-th multiplihedron. The pictures of J_1, J_2, J_3 are depicted later in figure 12, with labelling of the faces in terms of elements of $\mathfrak{J}(1)$, $\mathfrak{J}(2)$, $\mathfrak{J}(3)$ respectively.
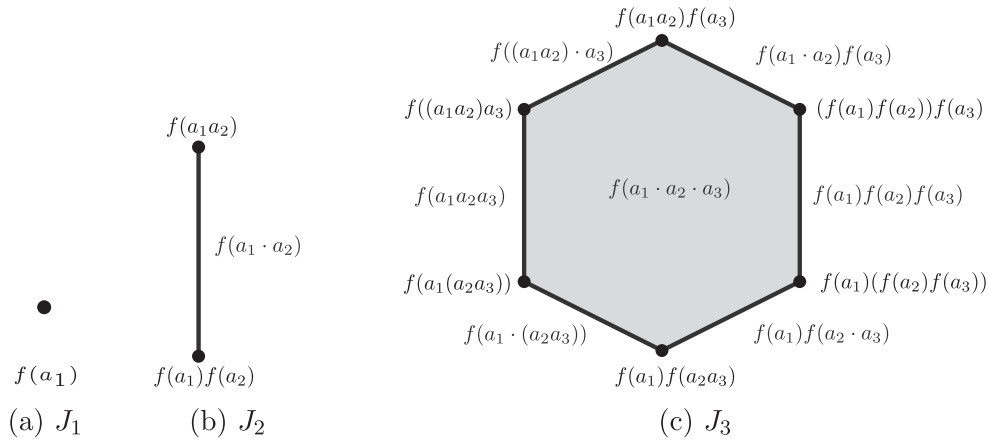

Fig. 12. Multiplihedra

Now suppose B is an associative space. Due to the associativity in B, there will be only one element of type III (as defined before) for each association rule of $a_1, a_2, ..., a_n$. For example, if $X = ((X_1 X_2)(X_3 X_4))$ is some association rule of $a_1, a_2, ..., a_n$, then there is only one element $f(X_1)f(X_2)\, f(X_3)f(X_4)$ in B using the fact that f is a homomorphism up to homotopy. We will call them degenerate type III elements.

DEFINITION 2.12. Let $\mathfrak{J}'_n$ be the poset of all type I, type II, and degenerate type III elements in B with the ordering induced from $(\mathfrak{J}'_n, \prec)$. We define the *collapsed multiplihedron* J'_n to be a cellular complex of dimension $n-1$, whose face poset is isomorphic to $\mathfrak{J}'_n$.

As the posets $\mathfrak{J}'_n$ are obtained by the degeneracy of certain elements in $\mathfrak{J}_n$, the complexes J'_n are obtained by collapsing certain faces of J_n. Thus the existence of the complexes J'_n is guaranteed by the existence of multiplihedron J_n. We will use this definition to show that J'_n is combinatorially isomorphic to the associahedron K_{n+1} in §3.2.

2.4. Graph Cubeahedra and Design Tubings. Devadoss [8] gave an alternate definition of K_n with respect to tubings on a path graph.

DEFINITION 2.13 (TUBE). Let Γ be a graph. A *tube* is a proper nonempty set of nodes of Γ whose induced graph is a proper, connected subgraph of Γ.

There are three ways that two tubes t_1 and t_2 may interact on the graph.

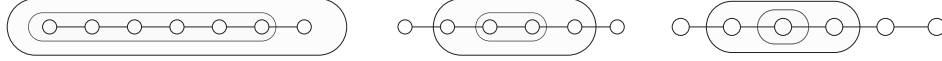- t_1 and t_2 are **nested** if $t_1 \subset t_2$ or $t_2 \subset t_1$.

Fig. 13. Nested tubes

- t_1 and t_2 **intersect** if $t_1 \cap t_2 \neq \phi$ and $t_1 \nsubseteq t_2$ and $t_2 \nsubseteq t_1$.

Fig. 14. Intersection of tubes

- t_1 and t_2 are **adjacent** if $t_1 \cap t_2 = \phi$ and $t_1 \cup t_2$ is a tube.

Fig. 15. Adjacent tubes

Two tubes are **compatible** if they are neither adjacent nor intersect i.e., t_1 and t_2 are compatible if they are nested or $t_1 \cap t_2 = \phi$ with $t_1 \cup t_2$ are not tubes.

Definition 2.14. A *tubing T* of Γ is a set of tubes of Γ such that every pair of tubes in T is compatible. A *k-tubing* is a tubing with k tubes.

A few examples of tubings are given below.

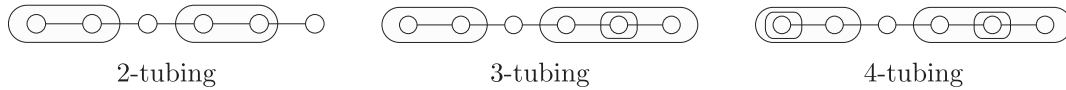2-tubing   3-tubing   4-tubing

Fig. 16. Tubings

If we think of the $n-1$ nodes of a path graph Γ as dividers between the n letters of a word and the tube as a pair of parentheses enclosing the letters, then the compatibility condition of the tubes corresponds to the permissible bracketing of the word. Now using the combinatorial description (cf. Definition 2.3) of K_n, one has the following result.

Lemma 2.15 ([5, Lemma 2.3]). *Let Γ be a path graph with $n-1$ nodes. The face poset of K_n is isomorphic to the poset of all valid tubings of Γ, ordered such that tubings $T \prec T'$ if T is obtained from T' by adding tubes.*

On a graph, Devadoss [8] defines another set of tubes called design tubes.

Definition 2.16 (Design Tube). Let G be a connected graph. A *round tube* is a set of nodes of G whose induced graph is a connected (and not necessarily proper) subgraph of G. A *square tube* is a single node of G. Then round tubes and square tubes together called *design tubes* of G.

Two design tubes are compatible if

    (1) they are both round, they are not adjacent and do not intersect;

    (2) otherwise, they are not nested.

DEFINITION 2.17 (DESIGN TUBING). A *design tubing U* of *G* is a collection of design tubes of *G* such that every pair of tubes in *U* is compatible.

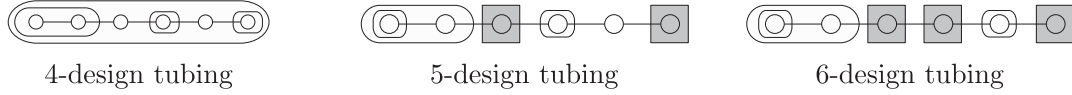4-design tubing   5-design tubing   6-design tubing

Fig. 17. Design tubings

Note that, unlike ordinary tubes, round tubes do not have to be proper subgraphs of *G*.

Based on design tubings, Devadoss [8] constructed a set of polytopes called graph cubeahedra. For a graph *G* with *n* nodes, define $\boxdot_G$ to be the *n-cube* where each pair of opposite facets corresponds to a particular node of *G*. Specifically, one facet in the pair represents that node as a round tube and the other represents it as a square tube. Each subset of nodes of *G*, chosen to be either round or square, corresponds to a unique face of $\boxdot_G$ defined by the intersection of the faces associated with those nodes. The empty set corresponds to the face which is the entire polytope $\boxdot_G$.

DEFINITION 2.18 (GRAPH CUBEAHEDRON). For a graph *G*, truncate faces of $\boxdot_G$ which correspond to round tubes in increasing order of dimension. The resulting polytope *CG* is the *graph cubeahedron*.

The graph cubeahedron *CG* can also be described as a convex polytope whose face poset formed through the design tubings.

Theorem 2.19 ([8, Theorem 12]). *For a graph G with n nodes, the graph cubeahedron CG is a simple convex polytope of dimension n whose face poset is isomorphic to the set of design tubings of G, ordered such that $U \prec U'$ if U is obtained from U' by adding tubes.*

In this article, we are interested in the case when *G* is a path graph. We will make use of the above theorem to show a combinatorial isomorphism between *CG* for *G* is a path graph with *n* nodes and multiplihedra J_{n+1} in §3.3.

3. Isomorphisms Between The Four Models

We prove the main result of this paper in this section.

Theorem 3.1. *The four models of associahedra: Stasheff complexes, cellular complexes obtained by Loday's cone construction, collapsed multiplihedra, and graph cubeahedra for path graphs are all combinatorially isomorphic.*

Proof. We prove the isomorphisms in the next three subsections. In §3.1 we prove that the complexes obtained via the cone construction of Loday are combinatorially isomorphic to the Stasheff complexes (Theorem 3.2). In §3.2 we prove that the Stasheff complexes and

collapsed multiplihedra are isomorphic (Proposition 3.4). Finally, in §3.3, the isomorphism between the collapsed multiplihedra and graph cubeahedra is shown (Proposition 3.5). Combining all three, we have our required result. □

3.1. Loday's construction vs Stasheff complexes. By Stasheff's description, K_{n+1} is the cone over its boundary elements $K_p \times_r K_q$ for $p+q = n+2$, $2 \leq p \leq n$ and $r = 1, 2, \ldots, p$. On the other hand, consider $C(\widehat{K}_n)$, where $\widehat{K}_n$ consists of the initial K_n together with $K_{p+1} \times_r K_q$ such that $p + q = n + 1$, $2 \leq p \leq n - 1$ and $r = 1, 2, \ldots, p$. This enlargement $\widehat{K}_n$ can be described in terms of bracketing as follows.

- K_n corresponds to 0-bracketing of the word $x_1 x_2 \ldots x_n$ i.e., the word itself or the trivial bracketing $(x_1 x_2 \ldots x_n)$. The immediate faces i.e., the boundary consists of $K_p \times_r K_q$ with $p + q = n + 1$, $2 \leq p \leq n - 1$ and $r = 1, 2, \ldots, p$. Now $K_p \times_r K_q$ corresponds to the 1-bracketing $x_1 \ldots x_{r-1}(x_r \ldots x_{r+q-1})x_{r+q} \ldots x_n$.
- The enlargement $\widehat{K}_n$ corresponds to the adding of a letter x_{n+1} to the right of the bracketing corresponding to K_n. Then the bracketing $x_1 \ldots x_{r-1}(x_r \ldots x_{r+q-1})x_{r+q} \ldots x_n$ extends to $x_1 \ldots x_{r-1}(x_r \ldots x_{r+q-1})x_{r+q} \ldots x_n x_{n+1}$ for each p, q, r such that $p + q = n + 1$, $2 \leq p \leq n - 1$, and $r = 1, 2, \ldots, p$. Also the initial K_n i.e., $(x_1 x_2 \ldots x_n)$ extends to $(x_1 x_2 \ldots x_n)x_{n+1}$, which corresponds to $K_2 \times_1 K_n$ in K_{n+1}.
- Finally one takes cone over the enlarged complex to obtain K_{n+1}.

From the above description, $\widehat{K}_n$ can be thought of as union of $K_p \times_r K_q$ with $p+q = (n+1)+1$ for $2 \leq p \leq n$ and $r = 1, 2, \ldots, p - 1$. Thus $\widehat{K}_n$ is a part of the boundary of K_{n+1} (following Stasheff's description).

Theorem 3.2. *Stasheff complexes are combinatorially isomorphic to Loday's cone construction of associahedra.*

To prove combinatorial isomorphism between the two mentioned models, we must show bijective correspondence between vertices, edges, and faces of each codimension for both models respecting the adjacencies. But the faces of codimension more than 1 are contained in the faces of codimension 1. Thus if we have an appropriate bijection between the faces of codimension 1 respecting the adjacencies for both models, then the resulting models being cone over combinatorially isomorphic codimension 1 faces, they are combinatorially isomorphic.

Proof of Theorem 3.2. It is enough to show that the boundary of K_{n+1} in Loday's construction can be subdivided to match them with the boundary elements $K_p \times_r K_q$ of K_{n+1} in Stasheff model for $p + q = n + 2$, $2 \leq p \leq n$ and $r = 1, \ldots, p$. As observed in the initial discussion, the only missing boundary part of K_{n+1} in Loday's construction is the union of $K_p \times_p K_q$ for $p + q = n + 2$ with $2 \leq p \leq n$. Note that all these missing faces are adjacent to a common vertex, which corresponds to the right to left $(n - 1)$-bracketing $x_1(x_2(\ldots (x_{n-1}(x_n x_{n+1}))\ldots))$. As there are $\binom{n-1}{n-2} = n - 1$ many choices for removing $(n - 2)$ brackets from a $(n - 1)$-bracketing (that corresponds to the vertices of K_{n+1}), each vertex of K_{n+1} is adjacent to exactly $n - 1$ faces of codimension 1 of K_{n+1} (by poset description of Stasheff's K_{n+1}). So the vertex corresponding to $x_1(x_2(\ldots (x_{n-1}(x_n x_{n+1}))\ldots))$ is not obtained in $\widehat{K}_n$. Now if we consider any other $(n - 1)$-bracketing, then there can be at most $n - 2$ parentheses after x_{n+1}. So removing those parentheses along with some others, we can get a

1-bracketing that does not enclose x_{n+1} i.e., those vertices are adjacent to some $K_p \times_r K_q$ for $p + q = n + 2$ and $r = 1, 2, \ldots, p - 1$. Thus any vertex of K_{n+1} except that corresponding to $x_1(x_2(\ldots (x_{n-1}(x_n x_{n+1}))\ldots))$ is present in $\widehat{K}_n$. We identify this missing vertex with the coning vertex of Loday's construction.

We shall prove that the missing faces of K_{n+1} in $C(\widehat{K}_n)$ can be realized as a cone over some portion of the boundary of $\widehat{K}_n$. Then we will divide the part $C(\partial \widehat{K}_n)$ accordingly to identify those with the missing faces. We will prove this together with the final result by induction on the following statements:

I. $Q_{n-3} : K_p \times_r K_q = C\left((\widehat{K}_{p-1} \times_r K_q) \cup (K_p \times_r \widehat{K}_{q-1})\right)$ if $p + q = n + 1$ and $p, q \geq 3$.

II. $P_{n-2} : K_n = C\left(\widehat{K}_{n-1}\right), n \geq 3$.

Here the equalities in the statements represent a combinatorial isomorphism. Note that Q_{n-3} is a collection of statements and the index r is superfluous. We will use the convention that $\widehat{K}_1 = \varnothing$, $C(\varnothing) = \{*\}$ and allow $p, q \geq 2$. Then Q_{n-3} contains the statement for $K_{n-1} \times_r K_2$ as well as $K_2 \times_r K_{n-1}$. Moreover, these are equivalent to the statement P_{n-3} since K_2 is a point and $K_{n-1} \times K_2$ is K_{n-1}.

The steps of induction are as follows.

Step 0: Show that P_1 holds.
Note that K_2 is a point that parametrizes the binary operation. As a point has no boundary, so $\widehat{K}_2$ is also a point, and $C(\widehat{K}_2)$ is an interval. Now K_3 parametrizes the family of 3-ary operations that relate the two ways of forming a 3-ary operation via a given binary operation. Thus, K_3 also represents an interval. Here the boundary of K_3 consists of two points $K_2 \times_1 K_2$ and $K_2 \times_2 K_2$. Let us map $K_2 \times_1 K_2$ and $K_2 \times_2 K_2$ to $\widehat{K}_2$ and the coning point in $C(\widehat{K}_2)$ respectively. Then we can map the other points of K_3 linearly to $C(\widehat{K}_2)$. Thus we get K_3 and $C(\widehat{K}_2)$ are combinatorially isomorphic. So P_1 is true.

Step 1: Assuming that P_1 through P_{n-4} hold, show that Q_{n-3} holds.
To prove it we will use the following lemma, the proof of which is given at the end of this subsection.

Lemma 3.3. *There is a natural homeomorphism*

$$C(X) \times C(Y) \equiv C\left((X \times C(Y)) \cup (C(X) \times Y)\right),$$

where x_0, y_0 are cone points for $C(X), C(Y)$ respectively and (x_0, y_0) is the cone point for $C(Z)$, where $Z = (C(X) \times Y) \cup (X \times C(Y))$.

Now assuming P_1 through P_{n-4}, we have $K_l = C(\widehat{K}_{l-1})$ for $l = 3, 4, \ldots, n - 2$. Take any $p, q \geq 3$ with $p + q = n + 1$ i.e., p, q both ranges through 3 to $n - 2$. So

$$K_p \times_r K_q = C(\widehat{K}_{p-1}) \times_r C(\widehat{K}_{q-1}) \quad \text{(by the assumption)}$$
$$= C\left((\widehat{K}_{p-1} \times_r C(\widehat{K}_{q-1})) \cup (C(\widehat{K}_{p-1}) \times_r \widehat{K}_{q-1})\right) \quad \text{(by the Lemma 3.3)}$$
$$= C((\widehat{K}_{p-1} \times_r K_q) \cup (K_p \times_r \widehat{K}_{q-1})) \quad \text{(by the assumption)}$$

This shows that Q_{n-3} is true.

Step 2: Assuming P_1 through P_{n-3}, show that P_{n-2} hold.

As discussed earlier, to prove that P_{n-2} is true, it is enough to show $K_s \times_s K_t$ with $s + t = n + 1$ for $s, t \geq 2$ can be obtained from $C(\widehat{K}_{n-1})$. Consider $s, t \geq 2$ with $s + t = n + 1$. Then using the conventions $\widehat{K}_1 = \varnothing$ and $C(\varnothing) = \{*\}$, we can write

$$K_s \times_s K_t$$
$$= C(\widehat{K}_{s-1}) \times_s C(\widehat{K}_{t-1}) \text{ (by } P_1 \text{ through } P_{n-3})$$
$$= C\Big((\widehat{K}_{s-1} \times_s K_t) \cup (K_s \times_s \widehat{K}_{t-1})\Big) \text{ (by the Lemma 3.3)}$$
$$= C\left(\left\{\bigcup_{(p,q,r)\in V_s} \big((K_p \times_r K_q) \times_s K_t\big)\right\} \bigcup \left\{\bigcup_{(p,q,r)\in V_t} \big(K_s \times_s (K_p \times_r K_q)\big)\right\}\right) \text{(by definition of } \widehat{K}_{i-1}),$$

where $V_i = \{(a,b,c) \in \mathbb{N}^3 : 2 \leq a \leq i - 1, \ a + b = i + 1, \ 1 \leq c \leq a - 1\}, \ i = s, t$.

Now using equation (2) (in §2.1), we can write

$$(K_p \times_r K_q) \times_s K_t = (K_p \times_{s-q+1} K_t) \times_r K_q$$

(obtained by substituting $r = p, s = q, t = t, k = r, j = s - q + 1$) for the terms in the first set of unions. As $K_p \times_{s-q+1} K_t$ is a face of K_{p+t-1}, so $(K_p \times_{s-q+1} K_t) \times_r K_q$ is a face of $K_{p+t-1} \times_r K_q$, which is again a face of K_n because for $(p, q, r) \in V_s$,

$$(p + t - 1) + q = p + q + t - 1 = s + 1 + t - 1 = s + t = n + 1.$$

Thus $(K_p \times_r K_q) \times_s K_t$ is a face of $K_{p+t-1} \times_r K_q$ of codimension 1. But as $t \geq 2$ and $1 \leq r \leq p - 1$, so $r < p + t - 1$, which implies that the face $K_{p+t-1} \times_r K_q$ is already present in the enlargement $\widehat{K}_{n-1}$. Thus each term in the first set of unions is already present in $\widehat{K}_{n-1}$.

Similarly, using equation (1), we have the identification

$$K_s \times_s (K_p \times_r K_q) = (K_s \times_s K_p) \times_{s+r-1} K_q$$

(obtained by substituting $r = s, s = p, t = q, k = r, j = s$) for the terms in the second set of unions. Here $(K_s \times_s K_p) \times_{s+r-1} K_q$ is a face of $K_{s+p-1} \times_{s+r-1} K_q$, which is a face of K_n because for $(p, q, r) \in V_t$,

$$(s + p - 1) + q = s - 1 + (p + q) = s - 1 + t + 1 = s + t = n + 1.$$

Thus $(K_s \times_s K_p) \times_{s+r-1} K_q$ is a face of $K_{s+p-1} \times_{s+r-1} K_q$ of codimension 1. But $r \leq p - 1 < p$ implies $s + r - 1 < s + p - 1$, which further implies that the face $K_{s+p-1} \times_{s+r-1} K_q$ is already present in the enlargement $\widehat{K}_{n-1}$. Thus each term in the second set of unions is also present in $\widehat{K}_{n-1}$.

It follows that all the parts in the unions are present as a part of the boundary of $\widehat{K}_{n-1}$. Thus the cone over that particular part of the boundary of $\widehat{K}_{n-1}$, we will get $K_s \times_s K_t$ for all $s, t \geq 2$ (with $s + t = n + 1$). Also, these are present as a part of boundary of $C(\widehat{K}_{n-1})$. Therefore we get a bijection between the faces (of codimension 1) of K_n and $\widehat{K}_{n-1}$. Consequently, they are combinatorially isomorphic. So P_{n-2} is true. This completes the induction step as well as

the proof of the theorem.   □

REMARK 3.1. In the above isomorphism, we mapped the starting K_n to $K_2 \times_1 K_n$ and the extension of the boundary element $K_p \times_r K_q$ to $K_{p+1} \times_r K_q$. Similarly we could map the starting K_n to $K_2 \times_2 K_n$ and the extension of the boundary $K_p \times_r K_q$ to $K_{p+1} \times_{r+1} K_q$. But if we want to map the starting K_n to $K_n \times_r K_2$ $(r = 1, 2, ..., n)$, the corresponding extension of boundary $K_p \times_t K_q$ should map to

$$
\begin{cases}
K_p \times_t K_{q+1} & \text{if } t \leq r \leq t + q - 1 \\
K_{p+1} \times_t K_q & \text{if } r > t + q - 1 \\
K_{p+1} \times_{t+1} K_q & \text{if } r < t.
\end{cases}
$$

With a slight modification in the above proof, one can similarly prove that this produces an isomorphism. This, in turn, implies that the faces $K_n \times_r K_2$ or $K_2 \times_r K_n$ of K_{n+1} are all equivalent from the point of view of Loday's construction.

We end this subsection with the proof of Lemma 3.3.

Proof of Lemma 3.3.   We will prove the equality by showing both inclusions. First suppose $(x, y) = t(x_0, y_0) + (1 - t)(x_1, y_1) \in C(Z)$, where $t \in [0, 1]$ and $(x_1, y_1) \in Z$. Without loss of generality suppose $(x_1, y_1) \in C(X) \times Y$ i.e., $x_1 = t'x_0 + (1 - t')x_1'$ for some $t' \in [0, 1]$ and $x_1' \in X$. So

$$
\begin{aligned}
(x, y) &= (tx_0 + (1 - t)x_1, ty_0 + (1 - t)y_1) \\
&= (tx_0 + (1 - t)t'x_0 + (1 - t)(1 - t')x_1', ty_0 + (1 - t)y_1) \\
&= ((1 - (1 - t)(1 - t'))x_0 + (1 - t)(1 - t')x_1', ty_0 + (1 - t)y_1) \\
&= (t_1 x_0 + (1 - t_1)x_1', ty_0 + (1 - t)y_1) \in C(X) \times C(Y)
\end{aligned}
$$

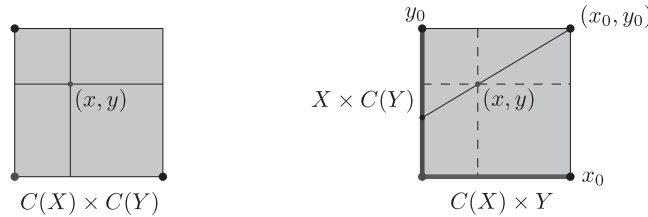and $t_1 = 1 - (1 - t)(1 - t')$. This implies that $C(Z) \subseteq C(X) \times C(Y)$.

Fig. 18. Visual proof when $X = Y =$ point

Conversely let $(x, y) = (t_1 x_0 + (1 - t_1)x_1, t_2 y_0 + (1 - t_2)y_1) \in C(X) \times C(Y)$ for some $x_1 \in X$, $y_1 \in Y$ and $t_1, t_2 \in [0, 1]$. Now consider the following cases
Case I: $t_1 = t_2 = t$.

$$(x, y) = t(x_0, y_0) + (1 - t)(x_1, y_1) \in C(Z).$$

Case II: $t_1 > t_2$.

$$(x, y) = t_2(x_0, y_0) + (1 - t_2)\left(\tfrac{t_1 - t_2}{1 - t_2} x_0 + \tfrac{1 - t_1}{1 - t_2} x_1, y_1\right)$$

$$= t_2(x_0, y_0) + (1 - t_2)(t'x_0 + (1 - t')x_1, y_1) \in C(Z), \quad \text{where } t' = \tfrac{t_1 - t_2}{1 - t_2}.$$

Case III: $t_1 < t_2$.

$$(x, y) = t_1(x_0, y_0) + (1 - t_1)\left(x_1, \tfrac{t_2 - t_1}{1 - t_1}y_0 + \tfrac{1 - t_2}{1 - t_1}y_1\right)$$
$$= t_1(x_0, y_0) + (1 - t_1)(x_1, t'y_0 + (1 - t')y_1) \in C(Z), \quad \text{where } t' = \tfrac{t_2 - t_1}{1 - t_1}.$$

Combining all three cases, we conclude that $(x, y) \in C(Z)$ and consequently $C(X) \times C(Y) \subseteq C(Z)$. $\qquad\square$

3.2. Stasheff complexes vs Collapsed Multiplihedra. We shall use the Definition 2.3 for Stasheff complexes. Similarly, due to Lemma 2.11, we will use Definition 2.12 for collapsed multiplihedra.

Proposition 3.4. *Stasheff complexes K_{n+1} and collapsed multiplihedra J'_n are combinatorially isomorphic.*

Proof. Both K_{n+1} and J'_n are cellular complexes whose face posets are isomorphic to $\mathfrak{P}(n + 1)$ and $\mathfrak{J}'_n$ respectively. Therefore, in order to exhibit an isomorphism between J'_n and K_{n+1}, it suffices to find a bijection between $\mathfrak{P}(n + 1)$ and $\mathfrak{J}'_n$ as posets.

Define $\phi : \mathfrak{J}'_n \to \mathfrak{P}(n + 1)$ as follows

$$f(X_1) \mapsto f(X_1)a_{n+1} := (X_1)a_{n+1}$$

$$f((X_1) \cdot \ldots \cdot (X_{k-1}) \cdot (X_k)) \mapsto ((X_1) \cdot \ldots \cdot (X_{k-1}) \cdot (X_k))a_{n+1} := (X_1)\ldots(X_{k-1})(X_k)a_{n+1}$$

$$\phi\left(f(X_1)\ldots f(X_{k-1})f(X_k)\right) = f(X_1)\ldots f(X_{k-1})f(X_k)a_{n+1}$$
$$= f(X_1)\ldots f(X_{k-1})((X_k)a_{n+1})$$
$$= f(X_1)\ldots f(X_{k-2})((X_{k-1})((X_k)a_{n+1}))$$
$$= \cdots$$
$$= (X_1)(\ldots((X_{k-1})((X_k)a_{n+1}))\ldots),$$

$$\phi\left(f((X_1) \cdot (X_2))f((X_3) \cdot (X_4) \cdot (X_5))\right) = f((X_1) \cdot (X_2))(((X_3) \cdot (X_4) \cdot (X_5))a_{n+1})$$
$$= ((X_1) \cdot (X_2))((X_3)(X_4)(X_5)a_{n+1})$$
$$= (X_1)(X_2)((X_3)(X_4)(X_5)a_{n+1}).$$

Here X_i's are some rule of association of the elements $a_1, a_2, ..., a_n$ in A of some length such that the total length of all X_i's is n and a_{n+1} is some different element in A. In the above correspondence, note that the bracketing in X_i's are not changed. We only include some pair of brackets removing f's or remove $\cdot$ and keep it as it is with an extra letter a_{n+1} on the right to get a bracketing of the word $a_1a_2 \ldots a_{n+1}$. Also, note that each parenthesis right to the letter a_{n+1} determines the number of f and their position as well, where no parentheses mean only single f with the $\cdot$'s in between the associated words. Thus, the position of each f and $\cdot$ gives a unique bracketing of the word $a_1a_2 \ldots a_{n+1}$ and the process can also be reversed. So ϕ is bijective. Now in order to check ϕ preserves the poset relation, we need to show $\phi(P \prec P') \implies \phi(P) \prec \phi(P')$. There are three possible ways (cf. operation (1), (2), (3)) by which P can be related to P'.

(1) P is obtained from P' by adding brackets in domain. Since ϕ do not interact with

the brackets in domain, $\phi(P)$ is also obtained from $\phi(P')$ by adding brackets i.e., $\phi(P) < \phi(P')$.

(2) P is obtained from P' by replacing $\cdot$ by ')f('. Thus P contains more f than P'. But from the correspondence, we know each f corresponds to a pair of brackets, so $\phi(P)$ must be obtained from $\phi(P')$ by adding brackets i.e., $\phi(P) < \phi(P')$.

(3) P is obtained from P' by removing one or more consecutive $\cdot$ by adding a pair of brackets that encloses all the adjacent elements to those $\cdot$. To obtain P, this process adds brackets to P' and ϕ does not change the parent bracketing. So so $\phi(P)$ must be obtained from $\phi(P')$ by adding brackets i.e., $\phi(P) < \phi(P')$.

Thus ϕ defines a bijection of the posets $\mathfrak{J}'_n$ and $\mathfrak{P}(n+1)$. Hence J'_n and K_{n+1} are combinatorially isomorphic. $\square$

3.3. Collapsed Multiplihedra vs Graph Cubeahedra.

Proposition 3.5. *Collapsed multiplihedra J'_{n+1} and graph cubeahedra CP_n for path graph P_n with n nodes are combinatorially isomorphic.*

Proof. Recall from Theorem 2.19 that the graph cubeahedron CP_n is a convex polytope of dimension n whose face poset is isomorphic to the set of design tubings of P_n. Recall that the collapsed multiplihedra J'_{n+1} is a cellular complex of dimension n whose face poset is isomorphic to $\mathfrak{J}'_{n+1}$. Thus, to describe an isomorphism, it is enough to prove a bijection at the poset level.

A bijection between the design tubings and the elements of $\mathfrak{J}'_{n+1}$ is defined through the following correspondences:

- Put a_1 through a_{n+1} starting from the left of the left-most node to the right of the right-most node of the graph:
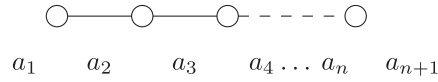

Fig. 19. Initial step

- Each round tube corresponds to a pair of parentheses. If the round tube includes k-th and $(k+r-1)$-th node of the graph, then the corresponding parentheses include a_k through a_{k+r}.
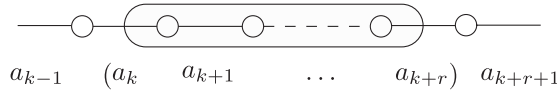

Fig. 20. Correspondence of round tube

- Each square tube corresponds to the inclusion of ')f(' in the string $f(a_1a_2\ldots a_{n+1})$. If the square tube include k-th node of the graph, then ')f(' will be included in between a_k and a_{k+1}.
- An empty node in a tubing corresponds to '$\cdot$' i.e. if k-th node of the graph is not included by any tube of the given tubing, then put a '$\cdot$' between a_k and a_{k+1}.
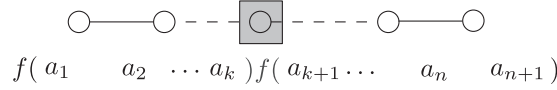
$$f(\ a_1 \qquad a_2 \quad \cdots a_k \ )f(\ a_{k+1}\cdots \qquad a_n \qquad a_{n+1}\ )$$

Fig. 21. Correspondence of square tube

$$f(\ a_1 \bullet \quad a_2 \ )f(\ldots a_k \ \bullet \ a_{k+1}\ldots)f(\ a_n \qquad a_{n+1}\ )$$
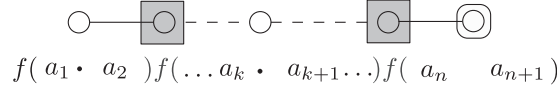
Fig. 22. Correspondence of empty node

Finally, as the position of each tube and its appearance give a unique element of $\mathfrak{J}'_{n+1}$, we get a bijective correspondence between design tubings and elements of $\mathfrak{J}'_{n+1}$. An example, assuming $n = 6$, is given below.

$$f((a_1 \quad a_2\ ) \quad a_3 \ )f((a_4 \quad a_5\ ) \ \bullet \ a_6 \ )f(\ a_7\ ) \qquad f((a_1 \quad a_2 \quad a_3) \quad (a_4 \quad a_5) \quad (a_6 \quad a_7))$$
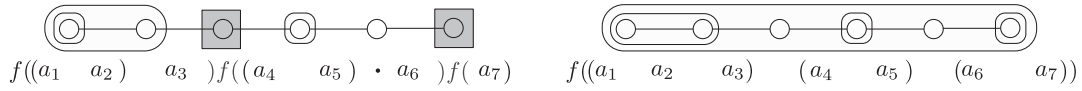
Fig. 23. Bijection between design tubings and multiplihedra

It follows from the correspondence that the removal of a round tube corresponds to the removal of a pair of parentheses or adding '·' and the removal of a square tube corresponds to replacing ')f(' by '·'. This shows that the poset relation between design tubings matches with the poset relation in $\mathfrak{J}'_{n+1}$. As the two posets are isomorphic, this finishes the proof. □

References

[1] J.F. Adams: Infinite Loop Spaces, Annals of Mathematics Studies, No. 90, Princeton University Press, Princeton, N.J., University of Tokyo Press, Tokyo, 1978.

[2] N. Arkani-Hamed, Y. Bai, S. He and G. Yan: *Scattering forms and the positive geometry of kinematics, color and the worldsheet*, J. High Energy Phys. (2018), 096, front matter+75.

[3] J.M. Boardman and R.M. Vogt: Homotopy Invariant Algebraic Structures on Topological Spaces, Lecture Notes in Mathematics, Vol. 347, Springer-Verlag, Berlin-New York, 1973.

[4] G. Bowlin and M.G. Brin: *Coloring planar graphs via colored paths in the associahedra*, Internat. J. Algebra Comput. **23** (2013), 1337–1418.

[5] M.P. Carr and S.L. Devadoss: *Coxeter complexes and graph-associahedra*, Topology Appl. **153** (2006), 2155–2168.

[6] S. Devadoss and S. Forcey: *Marked tubes and the graph multiplihedron*, Algebr. Geom. Topol. **8** (2008), 2081–2108.

[7] S.L. Devadoss: *Tessellations of moduli spaces and the mosaic operad*; in Homotopy invariant algebraic structures, Contemporary Mathematics 239, American Mathematical Society, Providence, RI, 1999, 91–114.

[8] S.L. Devadoss, T. Heath and W. Vipismakul: *Deformations of bordered surfaces and convex polytopes*, Notices Amer. Math. Soc. **58** (2011), 530–541.

[9] L. Ferro and T. Łukowski: *Amplituhedra, and beyond*, J. Phys. A **54** (2021), Paper No. 033001, 42 pp.

[10] S. Fomin and A. Zelevinsky: *Y-systems and generalized associahedra*, Ann. of Math. **158** (2003), 977–1018.

[11] S. Forcey: *Convex hull realizations of the multiplihedra*, Topology Appl. **156** (2008), 326–347.

[12] S. Forcey: *Quotients of the multiplihedron as categorified associahedra*, Homology Homotopy Appl. **10** (2008), 227–256.

[13] S. Forcey: *Extending the Tamari lattice to some compositions of species*; in Associahedra, Tamari Lattices and Related Structures, vol. 299 of Progr. Math., Birkhäuser/Springer, Basel, 2012, 187–210.

[14] K. Fukaya and Y.-G. Oh: *Zero-loop open strings in the cotangent bundle and Morse homotopy*, Asian J. Math. **1** (1997), 96–180.

[15] M. Haiman: *Constructing the associahedron*, 1984. available at `https://math.berkeley.edu/~mhaiman/ftp/assoc/manuscript.pdf`.

[16] D. Huguet and D. Tamari: *La structure polyédrale des complexes de parenthésages*, J. Combin. Inform. System Sci. **3** (1978), 69–81.

[17] N. Iwase and M. Mimura: *Higher homotopy associativity*; in Algebraic topology (Arcata, CA, 1986), vol. 1370 of Lecture Notes in Math., Springer, Berlin, 1989, 193–220.

[18] B. Keller: *Introduction to A-infinity algebras and modules*, Homology Homotopy Appl. **3** (2001), 1–35.

[19] C.W. Lee: *The associahedron and triangulations of the n-gon*, European J. Combin., **10** (1989), pp. 551–560.

[20] J.-L. Loday: *Realization of the Stasheff polytope*, Arch. Math. **83** (2004), 267–278.

[21] J.-L. Loday: *The multiple facets of the associahedron*, 2005. available at `https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.297.9348&rep=rep1&type=pdf`.

[22] S. Mau and C. Woodward: *Geometric realizations of the multiplihedron and its complexification*, 2009.

[23] J.P. May: The Geometry of Iterated Loop Spaces, Lecture Notes in Mathematics, Vol. 271, Springer-Verlag, Berlin-New York, 1972.

[24] S. Mizera: *Combinatorics and topology of Kawai-Lewellen-Tye relations*, J. High Energy Phys. (2017), 97.

[25] L. Pournin: *The diameter of associahedra*, Adv. Math. **259** (2014), 13–42.

[26] S. Saneblidze and R. Umble: *Diagonals on the permutahedra, multiplihedra and associahedra*, Homology Homotopy Appl. **6** (2004), 363–411.

[27] D.D. Sleator, R.E. Tarjan and W.P. Thurston: *Rotation distance, triangulations, and hyperbolic geometry*, J. Amer. Math. Soc. **1** (1988), 647–681.

[28] J. Stasheff: H-Spaces from a Homotopy Point of View, Lecture Notes in Mathematics, Vol. 161, Springer-Verlag, Berlin-New York, 1970.

[29] J.D. Stasheff: *Homotopy associativity of H-spaces. I, II*, Trans. Amer. Math. Soc. **108** (1963), 275–292; ibid., **108** (1963), 293–312.

[30] D. Tamari: *Monoïdes préordonnés et chaînes de malcev*, Bull. Soc. Math. France **82** (1954), 53–96.

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