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FOURIER–MUKAI PARTNERS OF ELLIPTIC RULED SURFACES OVER ARBITRARY CHARACTERISTIC FIELDS

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Abstract

The first author explicitly describes the set of Fourier–Mukai partners of elliptic ruled surfaces over the complex number field in [30]. In this article, we generalize it over arbitrary characteristic fields. We also obtain a partial evidence of the Popa–Schnell conjecture in the proof.

1. Introduction

Let us consider the derived category of coherent sheaves $D^b(X)$ for a smooth projective variety X over an algebraically closed field k of $p := \operatorname{ch} k \geq 0$. We call a smooth projective variety Y a *Fourier–Mukai partner* of X if there exists an equivalence $D^b(X) \cong D^b(Y)$ as k-linear triangulated categories. We let $\mathrm{FM}(X)$ denote the set of isomorphism classes of Fourier–Mukai partners of X. It is a fundamental question to describe the set $\mathrm{FM}(X)$ explicitly. It is known that $|\mathrm{FM}(C)| = 1$ for any smooth projective curves C (see [13, Corollary 5.46]). On the other hand, smooth projective surfaces S may have non-trivial Fourier–Mukai partners: Namely, $|\mathrm{FM}(S)| \neq 1$ may occur. Bridgeland, Maciocia and Kawamata show in [6] and [16] that if a smooth projective surface S over $\mathbb{C}$ has a non-trivial Fourier–Mukai partner T, then both are abelian surfaces, K3 surfaces or elliptic surfaces with nonzero Kodaira dimension. There exist several known examples of surfaces S with $|\mathrm{FM}(S)| \neq 1$ ([19, 20, 29]).

In this article, we study the set $\mathrm{FM}(S)$ of elliptic ruled surfaces S defined over k. Here, an elliptic ruled surface means a smooth projective surface with a $\mathbb{P}^1$-bundle structure over an elliptic curve. We obtain the following theorem, which is a generalization of the result for $k = \mathbb{C}$ in [30] to an arbitrary algebraically closed field k.

Theorem 1.1. *Let S be an elliptic ruled surface defined over k and $\pi\colon S \to E$ be a $\mathbb{P}^1$-bundle over an elliptic curve E. If $|\mathrm{FM}(S)| \neq 1$, then S is of the form*

$$S = \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L})$$

for some $\mathcal{L} \in \operatorname{Pic}^0 E$ of order $m \geq 5$. Furthermore we have

$$\mathrm{FM}(S) = \{\mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^i) \mid i \in \mathbb{Z} \text{ with } (i, m) = 1 \text{ and } 1 \leq i < m\}/\cong,$$

and

$$|\mathrm{FM}(S)| = \varphi(m)/|H^{\mathcal{L}}_{\hat{E}}|.$$

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Here, φ is the Euler function, and we define

$$H_{\hat{E}}^{\mathcal{L}} := \{i \in (\mathbb{Z}/m\mathbb{Z})^* \mid \exists \phi \in \mathrm{Aut}_0(E) \text{ such that } \phi^*\mathcal{L} \cong \mathcal{L}^i\} \tag{1}$$

as a subgroup of $(\mathbb{Z}/m\mathbb{Z})^*$. We also have $|H_{\hat{E}}^{\mathcal{L}}| = 2, 4$ or 6, depending on the choice of E and $\mathcal{L}$.

In the case $k = \mathbb{C}$, it is known (cf. [30, Equation (3.4)]) that $S = \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L})$ is a quotient of $F_0 \times \mathbb{P}^1$ by a cyclic group action, where F_0 is an elliptic curve, and the first author uses this fact to describe the set $\mathrm{FM}(S)$ in [30]. On the other hand, in the case $p := \mathrm{ch}\, k > 0$, elliptic ruled surfaces $S = \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L})$ with $p \mid m$ do not admit a similar construction (see [28, §5.1]). Therefore, we need more general treatment to show Theorem 1.1.

In the proof of Theorem 1.1, we obtain some evidence of the Popa–Schnell conjecture in [24], which states that for any Fourier–Mukai partners X' of a given smooth projective variety X, there exists an equivalence $D^b(\mathrm{Alb}(X')) \cong D^b(\mathrm{Alb}(X))$ of derived categories of their albanese varieties.

Proposition 1.2 (=Corollary 4.7)**.** *Let $X \to A$ and $X' \to A'$ be $\mathbb{P}^n$-bundles over abelian varieties A and A' for $n = 1, 2$. If X and X' are Fourier–Mukai partners, then so are A and A'. Furthermore, the Popa–Schnell conjecture holds true in this case.*

The plan of this article is as follows. In §2, we explain some results and notation of relative moduli spaces of stable sheaves on elliptic fibrations, a main tool for the study of Fourier–Mukai partners of elliptic surfaces. We obtain a characterization of Fourier–Mukai partners of elliptic surfaces with non-zero Kodaira dimensions in Theorem 2.2 for arbitrary $p = \mathrm{ch}\, k$, which was originally proved by Bridgeland in the case $p = 0$.

In §3, we show several results on automorphisms of elliptic curves.

In §4, we first explain Theorem 4.3 by Pirozhkov, and then we apply it to show Proposition 1.2.

Finally, in §5, we first narrow down the candidates of elliptic ruled surfaces with non-trivial Fourier–Mukai partners by Proposition 1.2 and the main result in [28], and then prove Theorem 1.1.

This article is a part of the second author's doctoral thesis.

Notation and conventions. All algebraic varieties X are defined over an algebraically closed field k of characteristic $p \geq 0$. A point $x \in X$ means a closed point unless otherwise specified.

For an elliptic curve E, $\mathrm{Aut}_0(E)$ is the group of automorphisms fixing the origin.

By an *elliptic surface*, we will always mean a smooth projective surface S together with a smooth projective curve C and a relatively minimal projective morphism $\pi\colon S \to C$ whose general fiber is an elliptic curve. An *elliptic ruled surface* means a smooth projective surface with a $\mathbb{P}^1$-bundle structure over an elliptic curve.

For a morphism $\pi\colon X \to Y$ between algebraic varieties, the symbol $\mathrm{Aut}(X/Y)$ stands for the group of automorphisms of X preserving π.

2. Relative moduli spaces of sheaves on elliptic fibrations

2.1. Fourier–Mukai partners of elliptic surfaces. For a smooth projective variety X defined over an algebraically closed field k of characteristic $p \geq 0$, we denote by $D^b(X)$ the bounded derived categories of coherent sheaves on X. We call a smooth projective variety Y a *Fourier–Mukai partner* of X if $D^b(X)$ is k-linear triangulated equivalent to $D^b(Y)$. We denote by $\mathrm{FM}(X)$ the set of isomorphism classes of Fourier–Mukai partners of X.

We study the set $\mathrm{FM}(S)$ for elliptic surfaces S. Let $\pi\colon S \to C$ be an elliptic surface and denote a general fiber of π by F_π. We define

$$\lambda_\pi := \min\{D \cdot F_\pi \mid D \text{ is a horizontal effective divisor on } S\}. \tag{2}$$

Fix a polarization on S and consider the relative moduli scheme $\mathcal{M}(S/C) \to C$ of stable purely 1-dimensional sheaves[1] on the fibers π, whose existence is assured by Simpson in the case $p = 0$ in [26], and by Langer in the case of arbitrary p in [17]. For integers $a > 0$ and i with i coprime to $a\lambda_\pi$, let $J_S(a, i)$ be the union of those components of $\mathcal{M}(S/C)$ which contains a point representing a rank a, degree i vector bundle on a smooth fiber of π. Bridgeland shows in [4] that $J_S(a, i)$ is actually a smooth projective surface and the natural morphism $J_S(a, i) \to C$ is a minimal elliptic fibration.

Put $J^i(S) := J_S(1, i)$. We can also define an elliptic surface $J^j(S) \to C$ for arbitrary $j \in \mathbb{Z}$, which is not necessarily fine but the coarse moduli space of a suitable functor (see [14, §11.4]). We have $J^0(S) \cong J(S)$, the Jacobian surface associated to S, $J^1(S) \cong S$ and

$$J^i(J^j(S)) \cong J^{ij}(S) \tag{3}$$

for $i, j \in \mathbb{Z}$. See the argument after (8) for the proof of (3).

It is well-known that the following statement holds in the case $p = 0$ by [4, Theorem 1.2]. We state that it is also true for arbitrary p.

Proposition 2.1. *Elliptic surfaces S and $J^i(S)$ for some integer i with $(i, \lambda_\pi) = 1$ are derived equivalent via an integral functor $\Phi^{\mathcal{P}} := \Phi^{\mathcal{P}}_{J^i(S)\to S}$ for a universal sheaf $\mathcal{P}$ on $J^i(S)\times S$.*

Proof. To prove the statement for $p = 0$, Bridgeland first applies the Bondal–Orlov's criterion [2] (see also [13, Proposition 7.1]) for the functor $\Phi^{\mathcal{P}}$ to be fully faithful, namely he checks the strongly simpleness of $\mathcal{P}$. Then it is easy to show $\Phi^{\mathcal{P}}$ is an equivalent by checking the Bridgeland's criterion [5] for $\Phi^{\mathcal{P}}$ to be equivalent. But the Bondal–Orlov's criterion is false in the case $p > 0$ [11, Remark 1.25]. Instead, if we put an extra assumption that the Kodaira–Spencer map $\mathrm{Ext}^1_{J^i(S)}(\mathcal{O}_x, \mathcal{O}_x) \to \mathrm{Ext}^1_S(\mathcal{P}_x, \mathcal{P}_x)$ is injective, we see the proof of [2] works, and so the criterion holds (see also [13, Step 5 in the proof of Proposition 7.1]). Actually, the map is an isomorphism in our case because $\mathcal{P}$ is a universal family. This completes the proof. □

We have a nice characterization of Fourier–Mukai partners of elliptic surfaces with non-zero Kodaira dimensions.

Theorem 2.2. *Let $\pi\colon S \to C$ be an elliptic surface and T a smooth projective variety. Assume that the Kodaira dimension $\kappa(S)$ is non-zero. Then the following are equivalent.*

(i) *T is a Fourier–Mukai partner of S.*
(ii) *T is isomorphic to $J^i(S)$ for some integer i with $(i, \lambda_\pi) = 1$.*

[1] Here we consider the Gieseker stability, equivalently the slope stability for 1-dimensional sheaves. Moreover, the stability does not depend on the choice of polarizations for such sheaves.

Proof. It follows from Proposition 2.1 that (ii) implies (i). The opposite direction was proved in [6, Proposition 4.4] when $p = 0$ and S has no (-1)-curves. The most of the proof there works even for $p > 0$. So we give only a sketch of the proof.

As the proof in [6, Proposition 4.4], we can show that there exists an equivalent functor $\Phi^{\mathcal{U}}\colon D^b(T) \to D^b(J^i(S))$ for some integer i with $(i, \lambda_\pi) = 1$ such that $\Phi^{\mathcal{U}}(\mathcal{O}_t) = \mathcal{O}_y$ for some $t \in T, y \in J^i(S)$. Then as in [6, Lemma 2.5], we see that there exists a rational map $f\colon T \dashrightarrow J^i(S)$ such that the kernel $\mathcal{U}$ is supported on the graph of f near the point (t, y). Because $\Phi^{\mathcal{U}}$ is an equivalence, we can avoid the possibility that f is inseparable, and hence f is a birational map. Then the proof of [6, Proposition 4.4] works in the rest (including the case that S is not minimal). □

As a consequence of Theorem 2.2, we obtain

$$\mathrm{FM}(S) = \{J^i(S) \mid i \in \mathbb{Z},\ (i, \lambda_\pi) = 1\}/\cong .$$

Moreover we see that there exist natural isomorphisms

$$J^i(S) \cong J^{i+\lambda_\pi}(S) \cong J^{-i}(S). \tag{4}$$

Hence, in order to count the cardinality of the set $\mathrm{FM}(S)$, we often regard an integer i as an element of the unit group $(\mathbb{Z}/\lambda_\pi\mathbb{Z})^*$. It follows from the isomorphisms (3) and (4) that the set

$$H_\pi := \{i \in (\mathbb{Z}/\lambda_\pi\mathbb{Z})^* \mid J^i(S) \cong S\} \tag{5}$$

forms a subgroup of $(\mathbb{Z}/\lambda_\pi\mathbb{Z})^*$. Moreover, we see from (3) that $J^i(S) \cong J^j(S)$ for $i, j \in (\mathbb{Z}/\lambda_\pi\mathbb{Z})^*$ if and only if $(S \cong) J^1(S) \cong J^{i^{-1}j}(S)$. Combining all together, we have the following.

Lemma 2.3. *For an elliptic surface $\pi\colon S \to C$ with $\kappa(S) \neq 0$, the set* $\mathrm{FM}(S)$ *is naturally identified with the group $(\mathbb{Z}/\lambda_\pi\mathbb{Z})^*/H_\pi$.*

Since H_π contains the subgroup $\{\pm 1\}$ if $\lambda_\pi \geq 3$, we see

$$|\mathrm{FM}(S)| \leq \varphi(\lambda_\pi)/2, \tag{6}$$

where φ is the Euler function.

Lemma 2.4. *Let $\pi\colon S \to C$ be an elliptic surface. Then we have the following.*

(i) *For $i \in \mathbb{Z}$ with $(i, \lambda_\pi) = 1$, consider the elliptic fibration $\pi_i\colon J^i(S) \to C$. The multiplicities of the fibers F_x and F'_x of π and π_i over a fixed point $x \in C$ coincide. Furthermore, if the fiber F_x is smooth, then it is isomorphic to F'_x.*

(ii) *Let S be an elliptic ruled surface, and take $S' \in \mathrm{FM}(S)$. Then S' is also an elliptic ruled surface with an elliptic fibration.*

Proof. (i) The first statement will be explained by using Weil–Châtelet group in §2.2. See the argument around (12). By the property of the relative moduli scheme, the fiber F'_x is the fine moduli space of line bundles of degree i on a smooth elliptic curve F_x. Consequently, the second statement follows.

(ii) Theorem 2.2 implies that there exists an integer i with $(i, \lambda_\pi) = 1$ such that $J^i(S) \cong S'$, which implies that S' has an elliptic fibration π'. The Kodaira dimension is derived invariant by [27, Corollary 4.4], and hence S' is a rational elliptic surface or an elliptic ruled surface.

Then, [12, Theorem B] implies that S' is also an elliptic ruled surface. □

2.2. Weil–Châtelet group. In this subsection, we recall the definition of the Weil–Châtelet group. For more details, see [25, Ch.X.3] and [14, Ch.11.5].

Let E_0 be an elliptic curve over a field K. A homogeneous space for E_0 is a pair (E, μ), where E is a smooth curve over K, and μ is a simply transitive algebraic group action

$$\mu\colon E \times E_0 \to E.$$

We say that two homogeneous spaces (E, μ) and (E', μ') are *equivalent* if there exists an isomorphism $\theta\colon E \to E'$ defined over K which is compatible with the action of E_0. The collection $WC(E_0)$ of equivalence classes of homogeneous spaces for E_0 has a natural group structure (cf. [25, Theorem X.3.6], [14, Proposition 11.5.1]), and it is called the *Weil–Châtelet group*.

Let $\pi\colon S \to C$ be an elliptic surface (over an algebraically closed field k). We denote the generic fiber of $\pi_i\colon J^i(S) \to C$ by J^i_η for $i \in \mathbb{Z}$. Then J^0_η is an elliptic curve over the function field of C, and we have a natural homogeneous space structure

$$\mu_i\colon J^i_\eta \times J^0_\eta \to J^i_\eta \quad (\mathcal{L}, \mathcal{M}) \mapsto \mathcal{L} \otimes \mathcal{M},$$

and hence we can regard $(J^i_\eta, \mu_i) \in WC(J^0_\eta)$. We define

$$\xi := (J^1_\eta, \mu_1) \in WC(J^0_\eta), \tag{7}$$

then, we have

$$i\xi = (J^i_\eta, \mu_i) \tag{8}$$

(cf. [14, Remark 11.5.2]) and thus

$$\operatorname{ord} \xi \mid \lambda_\pi. \tag{9}$$

It follows from (8) that the generic fibers of $J^i(J^j(S)) \to C$ and $J^{ij}(S) \to C$ are isomorphic to each other, and taking the relative smooth minimal models of compactifications of generic fibers, we obtain $J^i(J^j(S)) \cong J^{ij}(S)$ as in (3).

Take a closed point $x \in C$ and consider the henselization of the local ring $\mathcal{O}_{C,x}$ and denote it by $\mathcal{O}^h_{C,x}$. We also denote the base change of $\pi_0\colon J^0(S) \to C$ by the morphism $\operatorname{Spec} \mathcal{O}^h_{C,x} \to C$ by

$$J^0_x \to \operatorname{Spec} \mathcal{O}^h_{C,x}.$$

Then it is known by [7, Proposition 5.4.3 in p.314, Theorem 5.4.3 in p.321] that there exists an exact sequence:

$$\begin{array}{ccccccc} 0 & \to & \operatorname{Br}(J^0(S)) & \to & WC(J^0_\eta) & \to & \bigoplus_{x \in C} WC(J^0_x) \\ & & & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & & & & \xi & \mapsto & (\xi_x)_{x \in C} \end{array} \tag{10}$$

Here, we denote the image of ξ (given in (7)) in $WC(J^0_x)$ by ξ_x. It follows from [7, Proposition 5.4.2] that $m_x = \operatorname{ord} \xi_x$, where m_x is the multiplicity of the fiber of π over the point $x \in C$. Define

(11) $$\lambda'_\pi := \text{l.c.m.}_{x\in C}(m_x) = \text{ord}((\xi_x)_{x\in C}).$$

Since $\text{ord}\,\xi$ is divided by $\text{ord}((\xi_x)_{x\in C})$, we see from (9) that

$$\lambda'_\pi \mid \lambda_\pi.$$

In particular, if $i \in \mathbb{Z}$ is coprime to λ_π, then i is coprime to each m_x, and thus we have

(12) $$\text{ord}(i\xi)_x = \text{ord}\, i(\xi_x) = \text{ord}(\xi_x) = m_x.$$

Combining (12) with (8), we know that the multiplicity of the fiber of π_i over the point x is also m_x. This shows the first statement of Lemma 2.4 (i).

Define a subgroup H'_π of the group $H_\pi (:= \{i \in (\mathbb{Z}/\lambda_\pi\mathbb{Z})^* \mid J^i(S) \cong S\}$ given in (5)) to be

(13) $$H'_\pi := \{i \in H_\pi \mid i \equiv 1 \ (\text{mod}\ \lambda'_\pi)\}.$$

We use the following lemma to obtain a lower bound of the cardinality of the set $\text{FM}(S)$.

Lemma 2.5. *Let* $\pi\colon S \to C$ *be an elliptic surface with* $\text{Br}(J^0(S)) = 0$. *Then we have*

$$\left|H_\pi/H'_\pi\right| \le \left|\text{Aut}_0(J^0_\eta)\right|.$$

Proof. For each $i \in H_\pi$, fix an isomorphism $\theta_i\colon J^1_\eta \to J^i_\eta$ over the generic point $\eta \in C$. Then we obtain a structure of a homogeneous space on J^1_η by the action

$$\mu'_i := \theta_i^{-1} \circ \mu_i \circ (\theta_i \times \text{id}_{J^0_\eta})\colon J^1_\eta \times J^0_\eta \to J^1_\eta$$

such that $(J^i_\eta, \mu_i) = (J^1_\eta, \mu'_i)$ holds in $WC(J^0_\eta)$ by the definition. On the other hand, by [25, Exercise 10.4], $(J^1_\eta, \mu'_i) = (J^1_\eta, \mu_1 \circ (\text{id}_{J^1_\eta} \times \phi))$ for some $\phi \in \text{Aut}_0(J^0_\eta)$. We define an equivalence relation $\sim$ of $\text{Aut}_0(J^0_\eta)$ such that

$$\phi_1 \sim \phi_2$$

for $\phi_i \in \text{Aut}_0(J^0_\eta)$ when

$$(J^1_\eta, \mu_1 \circ (\text{id}_{J^1_\eta} \times \phi_1)) = (J^1_\eta, \mu_1 \circ (\text{id}_{J^1_\eta} \times \phi_2)).$$

Then we can define a map

$$f\colon H_\pi \to \text{Aut}_0(J^0_\eta)/\sim \quad i \mapsto \phi.$$

We see that $ij^{-1} \in H'_\pi$ if and only if $f(i) = f(j)$ as follows. First note that we have an injection

$$WC(J^0_\eta) \hookrightarrow \bigoplus_{x\in C} WC(J^0_x) \qquad \xi = (J^1_\eta, \mu_1) \mapsto (\xi_x)_{x\in C}$$

by the vanishing of the Brauer group $\text{Br}(J^0(S))$ and (10), and hence

(14) $$\text{ord}\,\xi = \lambda'_\pi (:= \text{ord}((\xi_x)_{x\in C})).$$

We observe that for $i, j \in H_\pi$, the condition $f(i) = f(j)$ is equivalent to the equality $i\xi = j\xi$ by (8), which is also equivalent to $i^{-1}j \in H'_\pi$ by (14).

Consequently, we obtain an inclusion

$$H_\pi/H'_\pi \hookrightarrow \text{Aut}_0(J^0_\eta)/\sim$$

and the conclusion. □

3. Elliptic curves and automorphisms

Let F be an elliptic curve over an algebraically closed field k with $p = \operatorname{ch} k \geq 0$. The explicit description of the automorphism group $\operatorname{Aut}_0(F)$ fixing the origin O is well-known, and is given as follows.

Theorem 3.1 (cf. Appendix A in [25])**.** *The automorphism group* $\operatorname{Aut}_0(F)$ *is*

$$\begin{array}{ll} \mathbb{Z}/2\mathbb{Z} & \text{if } j(F) \neq 0, 1728, \\ \mathbb{Z}/4\mathbb{Z} & \text{if } j(F) = 1728 \text{ and } p \neq 2, 3, \\ \mathbb{Z}/6\mathbb{Z} & \text{if } j(F) = 0 \text{ and } p \neq 2, 3, \\ \mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z} & \text{if } j(F) = 0 = 1728 \text{ and } p = 3, \\ Q \rtimes \mathbb{Z}/3\mathbb{Z} & \text{if } j(F) = 0 = 1728 \text{ and } p = 2. \end{array}$$

Note that in the last second case, $\mathbb{Z}/4\mathbb{Z}$ *acts on* $\mathbb{Z}/3\mathbb{Z}$ *in the unique non-trivial way, and in the last case, the group is so called a binary tetrahedral group, and* Q *is the quaternion group. In the last two cases* F *is necessarily supersingular.*

For points $x_1, x_2 \in F$, to distinguish the summation of divisors and of elements in the group scheme F, we denote by $x_1 \oplus x_2$ the sum of them by the operation of F, and

$$i \cdot x_1 := x_1 \oplus \cdots \oplus x_1 \quad (i \text{ times}).$$

Furthermore, we use the symbol T_a to stand for the translation by $a \in F$:

$$T_a \colon F \to F \quad x \to a \oplus x.$$

We also denote by

$$ix_1 := x_1 + \cdots + x_1 \quad (i \text{ times})$$

the divisors on F of degree i. We denote the dual abelian variety $\operatorname{Pic}^0 F$ of F by $\hat{F}$. It is well-known that there exists a group scheme isomorphism

$$F \to \hat{F} \quad x \mapsto \mathcal{O}_F(x - O), \tag{15}$$

where O is the origin of F.

We will use the following lemma several times.

Lemma 3.2. *Take a point* $a \in F$ *with* $\operatorname{ord}(a) \geq 4$*, and* $\phi \in \operatorname{Aut}_0(F)$*. If* $\phi(a) = a$*, then* $\phi = \operatorname{id}_F$.

Proof. In any of the cases in Theorem 3.1, we have $\operatorname{ord}(\phi) \in \{1, 2, 3, 4, 6\}$. Let us first consider the case $\operatorname{ord}(\phi) = 2, 4$ or 6. In this case, $\phi^i = -\operatorname{id}_F$ for some $i \in \mathbb{Z}$, and hence we get $-1 \cdot a = a$. The condition $\operatorname{ord}(a) \geq 4$ yields a contradiction. Next, consider the case $\operatorname{ord}(\phi) = 3$. Then we have

$$(\phi - \operatorname{id}_F)(\phi^2 + \phi + \operatorname{id}_F) = 0$$

in the domain $\operatorname{End}(F)$, which implies that $\phi^2 + \phi + \operatorname{id}_F = 0$, and hence $\phi^2(a) \oplus \phi(a) \oplus a = O$.

By the assumption $\phi(a) = a$, we see that $3 \cdot a = O$. This is absurd by $\operatorname{ord}(a) \geq 4$. □

For a non-zero integer m, we define the m-torsion subgroup of F to be

$$F[m] := \{a \in F \mid m \cdot a = O\}.$$

Equivalently, $F[m]$ is the kernel of the multiplication map by m. Recall that

$$F[m] = \begin{cases} \mathbb{Z}/p^e\mathbb{Z} & \text{if } F \text{ is ordinary, } m = p^e,\ e > 0 \\ \{O\} & \text{if } F \text{ is supersingular, } m = p^e,\ e > 0 \\ \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} & \text{if } p \nmid m. \end{cases}$$

(See [25, Corollary III.6.4].) Note that these 3 cases do not exhaust all possibilities (i.e., cases where m is divisible by p but is not power of p is not covered.)

Take $a \in F$ with $\operatorname{ord}(a) = m$. In order to count Fourier–Mukai partners of elliptic ruled surfaces, we need to study the subgroup

$$H_F^a := \{i \in (\mathbb{Z}/m\mathbb{Z})^* \mid \exists \phi \in \operatorname{Aut}_0(F) \text{ such that } \phi(a) = i \cdot a\} \tag{16}$$

of $(\mathbb{Z}/m\mathbb{Z})^*$. Note that the definition of $H_{\hat{E}}^{\mathcal{L}}$ given in (1) is compatible with (16). We obtain the following result as a direct consequence of Lemma 3.2.

Lemma 3.3. *Take $a \in F$ with* $\operatorname{ord}(a) \geq 4$.

(i) *We have an injective group homomorphism*

$$\iota\colon H_F^a \hookrightarrow \operatorname{Aut}_0(F). \tag{17}$$

Furthermore, we have $|H_F^a| = 2, 4$ *or* 6.

(ii) *Suppose that $p > 0$ and* $\operatorname{ord}(a) = p^e$. *Then* (17) *is an isomorphism.*

Proof. (i) Take $i \in H_F^a$. Then there exists $\phi \in \operatorname{Aut}_0(F)$ such that $\phi(a) = i \cdot a$, and define $\iota(i)$ to be ϕ. The well-definedness of ι follows from Lemma 3.2, and ι is injective by the definition. Since H_F^a is regarded as an abelian subgroup of $\operatorname{Aut}_0(F)$ described in Theorem 3.1, and H_F^a contains $\{\pm 1\}$ as a subgroup, we obtain the second assertion.

(ii) The existence of an order p^e element in F implies that F is ordinary. Since $F[p^e] = \mathbb{Z}/p^e\mathbb{Z} = \langle a \rangle$, for any $\phi \in \operatorname{Aut}_0(F)$ we see that $\phi(a) = i \cdot a$ for some $i \in (\mathbb{Z}/p^e\mathbb{Z})^*$. Hence the injective homomorphism in (17) is surjective, and then we can confirm the statement. □

From now on, by (17) we often regard H_F^a as a subgroup of $\operatorname{Aut}_0(F)$ when $\operatorname{ord} a \geq 4$.

4. Pirozhkov's result and its application

In this section, we summarize some definitions and results in [23], and give their application to the Popa–Schnell conjecture. We also refer to [21] for fundamental notions of ∞-categories.

For a Noetherian scheme S over k, we denote by $\operatorname{Perf}(S)$ the full subcategory of $D^b(S)$ consisting of perfect complexes. A stable k-linear ∞-category $\mathcal{D}$ is said to be S-linear if there exists an action functor

$$a_{\mathcal{D}}\colon \mathcal{D} \times \operatorname{Perf}(S) \to \mathcal{D}$$

together with associativity data.

For a morphism $f\colon X \to S$ between smooth projective varieties X and S over k, the category $D^b(X)$ has a natural S-linear structure via the functor

$$D^b(X) \times D^b(S) \to D^b(X) \quad (\mathcal{E}, \mathcal{F}) \mapsto \mathcal{E} \overset{\mathbb{L}}{\otimes}_X \mathbb{L}f^*\mathcal{F}.$$

DEFINITION 4.1 ([23]). Let S be a Noetherian scheme over a field k. We say that S is *noncommutatively stably semiorthogonally indecomposable*, or *NSSI* for brevity, if for arbitrary choices of

(i) $\mathcal{D}$, an S-linear category which is proper[2] over S and has a classical generator, and
(ii) $\mathcal{A}$, a left admissible subcategory of $\mathcal{D}$ which is linear over k,

the subcategory $\mathcal{A}$ is closed under the action of $\mathrm{Perf}(S)$ on $\mathcal{D}$.

REMARK 4.2. For a quasi-compact and quasi-separated scheme S, the category $\mathrm{Perf}(S)$ has a classical generator by [3, Corollary 3.1.2]. In particular, for a smooth projective variety S, the category $D^b(S)$ has a classical generator.

Theorem 4.3 (Lemma 6.1 in [23])**.** *Let $\pi\colon X \to S$ be a smooth projective morphism which is an étale-locally trivial fibration with fiber X_0. Assume that S is a connected excellent scheme*[3]*. Then for any point $s \in S$ the base change map*

$$\left\{\begin{matrix} S\text{-linear admissible} \\ \text{subcategories} \\ \mathcal{A} \subset D^b(X) \end{matrix}\right\} \xrightarrow{\text{restriction to } X_s \cong X_0} \left\{\begin{matrix} \text{admissible subcategories} \\ \mathcal{A}_0 \subset D^b(X_0) \end{matrix}\right\}$$

is an injection.

DEFINITION 4.4. Let $\pi\colon X \to S$ be a smooth projective morphism of Noetherian schemes.

(i) An object $\mathcal{E} \in \mathrm{Perf}(X)$ is *π-exceptional* if $\mathbb{R}\pi_* \mathbb{R}\mathcal{H}om_X(\mathcal{E}, \mathcal{E}) \cong \mathcal{O}_S$.
(ii) A collection of π-exceptional objects $\mathcal{E}_1, \dots, \mathcal{E}_N \in \mathrm{Perf}(X)$ is a *π-exceptional collection* if $\mathbb{R}\pi_* \mathbb{R}\mathrm{Hom}(\mathcal{E}_j, \mathcal{E}_i) = 0$ for any $1 \leq i < j \leq N$.
(iii) A *π-exceptional pair* is a π-exceptional collection of length 2.

For a π-exceptional pair $\mathcal{E}, \mathcal{F}$, the left π-mutation $L_{\mathcal{E}}\mathcal{F}$ of $\mathcal{F}$ through $\mathcal{E}$ and the right π-mutation $R_{\mathcal{F}}\mathcal{E}$ of $\mathcal{E}$ through $\mathcal{F}$ are defined by the following distinguished triangles:

$$\pi^*\mathbb{R}\pi_* \mathbb{R}\mathcal{H}om_X(\mathcal{E}, \mathcal{F}) \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{\varepsilon} \mathcal{F} \to L_{\mathcal{E}}\mathcal{F}$$
$$R_{\mathcal{F}}\mathcal{E} \to \mathcal{E} \xrightarrow{\eta} \pi^*\mathbb{R}\pi_* \mathbb{R}\mathcal{H}om_X(\mathcal{E}, \mathcal{F})^\vee \otimes_{\mathcal{O}_X} \mathcal{F}$$

We see that mutations commute with base change.

Lemma 4.5 (Lemma 2.22 in [15])**.** *Consider the following Cartesian square of finite dimensional Noetherian schemes, where π is smooth projective.*

[2]See [21] for this notion.
[3]In [23, Lemma 6.1], Pirozhkov assumes that S is a scheme over $\mathbb{Q}$, but it is not needed in its proof.

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\pi} \\ T & \xrightarrow[g]{} & S \end{array}$$

For any π-exceptional pair $(\mathcal{E}, \mathcal{F})$, it follows that $(f^*\mathcal{E}, f^*\mathcal{F})$ is an φ-exceptional pair and we have the following isomorphisms:

$$L_{f^*\mathcal{E}}(f^*\mathcal{F}) \simeq f^*(L_{\mathcal{E}}\mathcal{F})$$
$$R_{f^*\mathcal{F}}(f^*\mathcal{E}) \simeq f^*(R_{\mathcal{F}}\mathcal{E})$$

We apply Theorem 4.3 and Lemma 4.5 to obtain the following.

Proposition 4.6. *Let $\pi\colon X \to S$ be a $\mathbb{P}^n$-bundle ($n = 1, 2$) over a smooth projective variety S. Then any non-trivial S-linear admissible subcategory of $D^b(X)$ is of the following form:*

(i) *(Case $n = 1$)*

$$D^b(S)(i)(:= \pi^* D^b(S) \otimes_{\mathcal{O}_X} \mathcal{O}_X(i))$$

for some $i \in \mathbb{Z}$.

(ii) *(Case $n = 2$)*

$$\pi^* D^b(S) \otimes_{\mathcal{O}_X} \langle \mathcal{E}_1, \dots, \mathcal{E}_l \rangle,$$

where $\mathcal{E}_1, \dots, \mathcal{E}_l$ ($1 \leq l \leq n+1$) is a π-exceptional collection.

Proof. (i) Any non-trivial admissible subcategory in $D^b(\mathbb{P}^1)$ is known to be of the form $\langle \mathcal{O}_{\mathbb{P}^1}(i) \rangle$ for some $i \in \mathbb{Z}$. Since the restriction of the admissible category $D^b(S)(i)$ to a fiber is $\langle \mathcal{O}_{\mathbb{P}^1}(i) \rangle$, the injective base change map in Theorem 4.3 is surjective. Hence the result follows.

(ii) [22, Theorem 4.2] states that any non-trivial admissible subcategory $\mathcal{A}$ in $D^b(\mathbb{P}^2)$ is generated by a subcollection of successive mutations of the standard exceptional collection $\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2)$. Lemma 4.5 yields an S-linear admissible subcategory $\mathcal{A}_X$ of $D^b(X)$, which is generated by a π-exceptional subcollection obtained by successive π-mutations of the π-exceptional collection $\mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)$, and its derived restriction on a fiber is $\mathcal{A}$. This means that the injective base change map in Theorem 4.3 is surjective, and hence we obtain the result. □

The Popa–Schnell conjecture in [24] states that for any Fourier–Mukai partners X' of a given smooth projective variety X, there exists an equivalence $D^b(\mathrm{Alb}(X')) \cong D^b(\mathrm{Alb}(X))$ of derived categories.

From Proposition 4.6, we deduce that the Popa–Schnell conjecture holds true in certain situations.

Corollary 4.7. *Let $X \to A$ and $X' \to A'$ be $\mathbb{P}^n$-bundles over abelian varieties A and A' for $n = 1, 2$. If X and X' are Fourier–Mukai partners, then so are A and A'. Furthermore, the Popa–Schnell conjecture holds true in this case.*

Proof. Put $D^b(A)(i) = \pi^* D^b(A) \otimes \mathcal{O}_X(i)$, where π is the $\mathbb{P}^1$-bundle $X \to A$. Since abelian varieties are NSSI by [23, Theorem 1.4], any admissible category of $D^b(X)$ is A-linear.

Proposition 4.6 implies that any non-zero indecomposable admissible subcategory of $D^b(X)$ is equivalent to $D^b(A)$. This completes the proof of the first assertion. We see that $A \cong \mathrm{Alb}(X)$ and $A' \cong \mathrm{Alb}(X')$, and hence obtain the second. □

If X is an elliptic ruled surface over $\mathbb{C}$, namely $n = 1$ and $k = \mathbb{C}$, in Corollary 4.7, the statement follows from [30, Theorem 1.1]. The proof given above for $n = 1, 2$ and arbitrary k is more direct and natural.

REMARK 4.8. Let $X \to E$ and $X' \to E'$ be $\mathbb{P}^n$-bundles over elliptic curves E and E' for $n = 1, 2$. As a consequence of Corollary 4.7, if X and X' are Fourier–Mukai partners, then $D^b(E) \cong D^b(E')$, and hence $E \cong E'$ by [13, Corollary 5.46].

5. Fourier–Mukai partners of elliptic ruled surfaces

5.1. Singular fibers of elliptic ruled surfaces. In this subsection, we recall a result in [28]. Let $\mathcal{E}$ be a normalized, in the sense of [10, Ch. 5. §2], rank 2 vector bundle on an elliptic curve E and

$$f\colon S = \mathbb{P}(\mathcal{E}) \to E$$

be a $\mathbb{P}^1$-bundle on E defined by $\mathcal{E}$. Let us put $e := -\deg \mathcal{E}$. If S has an elliptic fibration, then $-K_S$ is nef. Then we can easily deduce $e = 0$ or -1 from [10, Corollary V.2.11, Theorems V.2.12, V.2.15]).

Theorem 5.1 (Theorem 1.1 in [28]). *Let us consider the above situation.*

(i) *For $e = 0$, we have the following possibilities:*

	$\mathcal{E}$	$\exists$ *an elliptic fibration on S?*	p
(i-1)	$\mathcal{O}_E \oplus \mathcal{O}_E$	*no multiple fibers*	$p \geq 0$
(i-2)	$\mathcal{O}_E \oplus \mathcal{L}$, ord $\mathcal{L} = m > 1$	(m, m)	$p \geq 0$
(i-3)	$\mathcal{O}_E \oplus \mathcal{L}$, ord $\mathcal{L} = \infty$	*no elliptic fibrations*	$p \geq 0$
(i-4)	*indecomposable*	*no elliptic fibrations*	$p = 0$
(i-5)	*indecomposable*	(p^*)	$p > 0$

Here $\mathcal{L}$ is an element of $\mathrm{Pic}^0 E$. *In the case S has an elliptic fibration π, for example, the notation* (m, m) *in* (i-2) *means that π has exactly two multiple fibers of multiplicities m.*

(ii) *Suppose that $e = -1$. Then the isomorphism class of such vector bundle $\mathcal{E}$ on E is unique, and S has an elliptic fibration. The list of singular fibers are as follows:*

	multiple fibers	E	p
(ii-1)	$(2, 2, 2)$		$p \neq 2$
(ii-2)	(2^*)	*supersingular*	$p = 2$
(ii-3)	$(2, 2^*)$	*ordinary*	$p = 2$

The symbol * stands for a wild fiber in the tables.

By [6] and [16], we know that if S has non-trivial Fourier–Mukai partners, S has an elliptic fibration. Hence, from now on, we suppose that S has an elliptic fibration $\pi\colon S \to \mathbb{P}^1$. Theorem 5.1 says that the multiplicities of all multiple fibers of π are the same number m.

When $e = 0$ (resp. $e = -1$), we see

(18) $$F_\pi \cdot F_f = mC_0 \cdot F_f = m \quad (\text{resp. } F_\pi \cdot C_0 = m(2C_0 - F_f) \cdot C_0 = m)$$

by [28, Remark 4.2], and hence

(19) $$\lambda_\pi = m = \lambda'_\pi$$

for both cases (recall the definitions of λ_π and λ'_π in (2) and (11) respectively). Here F_π (resp. F_f) is a fiber of π (resp. f), and C_0 stands for a section of f satisfying $C_0^2 = -e$.

Consider the case $|\mathrm{FM}(S)| \neq 1$. Then the inequality (6) yields $m = \lambda_\pi \geq 5$. Hence, S fits into either (i-2), $m \geq 5$ or (i-5), $p \geq 5$ in Theorem 5.1. Then $S' \in \mathrm{FM}(S)$ is also an elliptic ruled surface admitting an elliptic fibration π' fitting into the same case as S by Lemma 2.4.

Lemma 5.2. *Suppose that* $|\mathrm{FM}(S)| \neq 1$. *Then S fits into the case* (i-2).

Proof. It suffices to show that $|\mathrm{FM}(S)| = 1$ in the case (i-5). Suppose that S fits into the case (i-5). As we explained above, $S' \in \mathrm{FM}(S)$ is also an elliptic ruled surface in the case (i-5). In other words, S' has a $\mathbb{P}^1$-bundle structure $f' \colon \mathbb{P}(\mathcal{E}') \to E'$, where $\mathcal{E}'$ is the indecomposable vector bundle of rank 2, degree 0 on an elliptic curve E'. By Corollary 4.7, we have $E \cong E'$. Then, we see $S \cong S'$ by [10, Theorem V.2.15], in other words, $|\mathrm{FM}(S)| = 1$. □

The purpose of this paper is to describe the set $\mathrm{FM}(S)$ for elliptic ruled surfaces. Hence in the sequel, we will concentrate on the case (i-2), the unique candidate of S admitting non-trivial Fourier–Mukai partners.

5.2. Case (i-2). Take $\mathcal{L} \in \mathrm{Pic}^0 E$ with $1 < m := \mathrm{ord}\,\mathcal{L} < \infty$, and set

$$S := \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}).$$

The following lemma is elementary and useful.

Lemma 5.3. (i) *There exists an isomorphism* $S \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{M})$ *over E if and only if* $\mathcal{L} \cong \mathcal{M}^{\pm 1}$.

(ii) *For* $\phi_E \in \mathrm{Aut}(E)$, *we have an isomorphism* $f^*\phi_E$ *in the fiber product diagram:*

(20) $$\begin{CD} \mathbb{P}(\mathcal{O}_E \oplus \phi_E^*\mathcal{L}) @>{f^*\phi_E}>> S \\ @VVV \square @VV{f}V \\ E @>>{\phi_E}> E \end{CD}$$

(iii) *For some* $\mathcal{M} \in \mathrm{Pic}^0 E$, *let* $f_T \colon T := \mathbb{P}(\mathcal{O}_E \oplus \mathcal{M}) \to E$ *be the $\mathbb{P}^1$-bundle over E. Suppose that we are given an isomorphism* $\phi \colon T \to S$. *Then, if we replace ϕ appropriately, we can take* $\phi_E \in \mathrm{Aut}_0(E)$, *which makes the diagram*

(21) $$\begin{CD} T @>{\phi}>> S \\ @V{f_T}VV @VV{f}V \\ E @>>{\phi_E}> E \end{CD}$$

commutative. Moreover we have an isomorphism

$$T \cong \mathbb{P}(\mathcal{O}_E \oplus \phi_E^*\mathcal{L}) \tag{22}$$

over E, and an isomorphism

$$\mathcal{M} \cong \phi_E^*\mathcal{L}. \tag{23}$$

Proof. (i) This fact directly follows from [10, Exercise II.7.9(b)].

(ii) This assertion must be well-known. We leave the proof to readers. (For example, use [10, Proposition II.7.12].)

(iii) Since S has a unique $\mathbb{P}^1$-bundle structure, the existence of $\phi_E \in \mathrm{Aut}(E)$ fitting in (21) is assured. Next, write $\phi_E = T_a \circ \phi_E^0$ for some $\phi_E^0 \in \mathrm{Aut}_0(E)$ and $a \in E$. Since $T_a^*\mathcal{L} \cong \mathcal{L}$, the isomorphism f^*T_a (given as $f^*\phi_E$ in (20)) gives an automorphism of S. Then, if necessary, replace ϕ with $(f^*T_a)^{-1} \circ \phi$, we may assume that $\phi_E \in \mathrm{Aut}_0(E)$. By the universal property of the fiber product in (20), we obtain an isomorphism (22) over E. Then by (i) there exists an isomorphism $\mathcal{M}^{\pm 1} \cong \phi_E^*\mathcal{L}$. Since $(-\mathrm{id}_E)^*\mathcal{L} \cong \mathcal{L}^{-1}$, $f^*(-\mathrm{id}_E)$ also gives an automorphism of S. Thus, replace ϕ with $f^*(-\mathrm{id}_E) \circ \phi$ if necessary, we may assume that $\phi_E \in \mathrm{Aut}_0(E)$ and (23) holds simultaneously. □

Lemma 5.4. *For $i \in (\mathbb{Z}/m\mathbb{Z})^*$, $S \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^i)$ if and only if there exists an automorphism $\phi_E \in \mathrm{Aut}_0(E)$ such that $\phi_E^*\mathcal{L} \cong \mathcal{L}^i$. Consequently, the set*

$$\{\mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^i) \mid i \in (\mathbb{Z}/m\mathbb{Z})^*\}/\cong$$

is naturally identified with the group

$$(\mathbb{Z}/m\mathbb{Z})^*/H_{\hat{E}}^{\mathcal{L}}.$$

Here, recall that $H_{\hat{E}}^{\mathcal{L}} := \{i \in (\mathbb{Z}/m\mathbb{Z})^* \mid \exists \phi \in \mathrm{Aut}_0(E)$ such that $\phi^*\mathcal{L} \cong \mathcal{L}^i\}$.

Proof. "If" part follows from Lemma 5.3 (ii). "Only if" part follows from Lemma 5.3 (iii). □

Consider the dual morphism

$$q_1 \colon F_0 := \widehat{\hat{E}/\langle\mathcal{L}\rangle} \to E \tag{24}$$

of the quotient morphism $\hat{E} \to \hat{E}/\langle\mathcal{L}\rangle$. Then it follows from the definition of q_1 that $q_1^*\mathcal{L} \cong \mathcal{O}_{F_0}$ holds. Thus we have a diagram

$$\begin{array}{ccccc} F_0 & \xleftarrow{p_1} & F_0 \times \mathbb{P}^1 & \xrightarrow{p_2} & \mathbb{P}^1 \\ {\scriptstyle q_1}\downarrow & \square & \downarrow{\scriptstyle q_S} & & \downarrow{\scriptstyle q_2} \\ E & \xleftarrow[f]{} & S & \xrightarrow[\pi]{} & \mathbb{P}^1, \end{array} \tag{25}$$

where the left square diagram is a fiber product, and the right one is obtained by the Stein factorization of $\pi \circ q_S$. The reason why $\pi \circ q_S$ factors through p_2 is as follows. First, we have $q_S^*\omega_S \cong \omega_{F_0 \times \mathbb{P}^1}$ by [28, Lemma 2.14]. On the other hand, the elliptic fibration p_2 (resp. π) are defined by the linear system of some multiple of $-K_{F_0 \times \mathbb{P}^1}$ (resp. $-K_S$). Therefore $\pi \circ q_S$

factors through p_2.

Recall that the elliptic fibration π has exactly two multiple fibers.

Convention. By the action of $\mathrm{PGL}(1,k)$ on $\mathbb{P}^1$, we always assume below that in the case (i-2), the elliptic fibration π has multiple fibers over the points 0 and ∞ in $\mathbb{P}^1$. Furthermore, we also assume that $q_2(0) = 0$ and $q_2(\infty) = \infty$.

For $y_0 \in \mathbb{P}^1$ with $y := q_2(y_0) \in \mathbb{P}^1 \backslash \{0, \infty\}$, we denote by F_y the non-multiple fiber of π over the point y. Then it follows from $f \circ q_S = q_1 \circ p_1$ that the restriction of q_S induces the isomorphism

$$q_S|_{F_0 \times y_0} : F_0 \times y_0 \cong F_y, \tag{26}$$

since we see from (18) that $f|_{F_y}$ is a finite morphism of degree m. We tacitly identify F_0 and F_y by this isomorphism.

Take $x_0 \in F_0$ and set $x := q_1(x_0) \in E$. Then in a similar way to (26), we have an isomorphism

$$q_S|_{x_0 \times \mathbb{P}^1} : x_0 \times \mathbb{P}^1 \cong F_x, \tag{27}$$

where F_x is the fiber of f over the point x. We identify $\mathbb{P}^1$ and F_x by (27). By our convention above, we see that the two multiple fibers of π intersect with each fiber $\mathbb{P}^1$ of f at 0 and ∞ respectively.

Recall that f has two minimal sections, let's say C_0 and C_1, corresponding to the projections

$$\mathcal{O}_E \oplus \mathcal{L} \to \mathcal{O}_E \quad \text{and} \quad \mathcal{O}_E \oplus \mathcal{L} \to \mathcal{L}. \tag{28}$$

Then the multiple fibers of π are given exactly mC_0 and mC_1 (see [28, Remark 4.2]).

We use the following lemma to show Claim 5.7.

Lemma 5.5. *Let us regard the multiplicative group $\mathbb{G}_m$ as a subgroup of* $\mathrm{Aut}(\mathcal{O}_E \oplus \mathcal{L})(\cong \mathbb{G}_m \times \mathbb{G}_m)$ *by the diagonal embedding. Then there exists an injective homomorphism*

$$\iota\colon \mathbb{G}_m \cong \mathrm{Aut}(\mathcal{O}_E \oplus \mathcal{L})/\mathbb{G}_m \hookrightarrow \mathrm{Aut}(S/E).$$

Here, for $\lambda \in \mathbb{G}_m$, the automorphism $\iota(\lambda)$ of S induces the action on each fiber $\mathbb{P}^1$ of f fixing the points 0 *and* ∞.

Proof. The existence of the injection ι is assured in [9, p.202].[4] Note that since any elements of $\mathrm{Aut}(\mathcal{O}_E \oplus \mathcal{L})$ preserve the projections in (28), any $\beta \in \mathrm{Im}\,\iota$ preserves the minimal sections C_0 and C_1, and hence it gives an automorphism on each fiber $\mathbb{P}^1$ of f fixing the points 0 and ∞. □

5.3. Proof of Theorem 1.1. Let S be an elliptic ruled surface and suppose $|\mathrm{FM}(S)| \neq 1$. Lemma 5.2 implies that

$$S \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L})$$

[4] See also [18, Lemma 3]). Because Δ in ibid. is trivial, we actually see that ι gives an isomorphism.

for some $\mathcal{L} \in \operatorname{Pic}^0 E$ with $\operatorname{ord}\mathcal{L} = m \geq 5$. Now if $S' \in \operatorname{FM}(S)$, by the same reason we get $S' \cong \mathbb{P}(\mathcal{O}_{E'} \oplus \mathcal{L}')$ for some $\mathcal{L}' \in \operatorname{Pic}^0 E'$ with

$$m = \lambda_\pi = \operatorname{ord}\mathcal{L} = \operatorname{ord}\mathcal{L}'.$$

Moreover, by Corollary 4.7, we see that $E \cong E'$.

We divide the proof of Theorem 1.1 into two cases: The case $m = p^e \geq 5$ for some $e > 0$, and the case arbitrary $m \geq 5$ with $m \neq p^e$. In both cases, first we define an injective map

$$\{J^i(S) \mid i \in (\mathbb{Z}/m\mathbb{Z})^*\}/\cong \hookrightarrow \{\mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^i) \mid i \in (\mathbb{Z}/m\mathbb{Z})^*\}/\cong, \tag{29}$$

and secondly we shall see

$$|H_\pi| \leq |H^{\mathcal{L}}_{\hat{E}}|. \tag{30}$$

The cardinality of the L.H.S in (29) is $\varphi(m)/|H_\pi|$ by Lemma 2.3, and the cardinality of the R.H.S. in (29) is $\varphi(m)/|H^{\mathcal{L}}_{\hat{E}}|$ by Lemma 5.4. Therefore, combining (29) with (30), we can conclude that (29) is a bijection, and hence Theorem 2.2 yields

$$\operatorname{FM}(S) = \{\mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^i) \mid i \in (\mathbb{Z}/m\mathbb{Z})^*\}/\cong$$

as required in Theorem 1.1.

Case: $m = p^e \geq 5$ **for some** $e > 0$**.** Theorem 5.1 implies that $J^i(S) \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}_i)$ for some $\mathcal{L}_i \in \operatorname{Pic}^0 E$ with $\operatorname{ord}\mathcal{L}_i = p^e$. But in this case, E is necessarily ordinary, and hence $\hat{E}[p^e]$ is a cyclic group generated by $\mathcal{L}$. So in this case, $\mathcal{L}_i \cong \mathcal{L}^{\beta(i)}$ for some $\beta(i) \in (\mathbb{Z}/m\mathbb{Z})^*$, and thus we can define an injective map (29) by $J^i(S) \mapsto \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^{\beta(i)})$.

Denote by F_0 the elliptic curve satisfying $\hat{F}_0 = \hat{E}/\langle\mathcal{L}\rangle$ as in §5.2. Then by (26), a general fiber of the elliptic fibration $\pi\colon S \to \mathbb{P}^1$ is isomorphic to F_0.

CLAIM 5.6. The inequality (30) holds (if $m = p^e \geq 5$).

Proof. [7, Propositions 5.3.3, 5.3.6] implies that $\kappa(J^0(S)) = -\infty$. Combining this fact with [7, Corollary 5.3.5], we see that $J^0(S)$ is an elliptic ruled surface with a section. Therefore, by the classification in Theorem 5.1 and [7, Theorem 5.3.1 (i)], we have $J^0(S) \cong F_0 \times \mathbb{P}^1$. Then we have $\operatorname{Br}(J^0(S)) = 0$ by [8, Proposition 2.1]. Moreover we have $\lambda_\pi = p^e = \lambda'_\pi$ by (19), and hence the group H'_π in Lemma 2.5 is trivial. Therefore Lemma 2.5 yields

$$\left|H_\pi\right| \leq \left|\operatorname{Aut}_0(J^0_\eta)\right|.$$

Recall that $H^{\mathcal{L}}_{\hat{E}} = \operatorname{Aut}_0(E)$ by Lemma 3.3 (ii) in the case $m = p^e \geq 5$. Hence, to obtain the conclusion, it suffices to check that $|\operatorname{Aut}_0(J^0_\eta)| \leq |\operatorname{Aut}_0(E)|$. Thus we may assume $2 < |\operatorname{Aut}_0(J^0_\eta)|$. Note that we have a surjective homomorphism

$$\operatorname{Aut}_0(J^0(S)/\mathbb{P}^1) \to \operatorname{Aut}_0(J^0_\eta),$$

where $\operatorname{Aut}_0(J^0(S)/\mathbb{P}^1)$ means the automorphism group of $J^0(S)(\cong F_0 \times \mathbb{P}^1)$ over $\mathbb{P}^1$, fixing the 0-section. Thus, we have an isomorphism $\operatorname{Aut}_0(J^0(S)/\mathbb{P}^1) \cong \operatorname{Aut}_0(F_0)$, and moreover obtain

$$2 < |\operatorname{Aut}_0(J^0_\eta)| = |\operatorname{Aut}_0(J^0(S)/\mathbb{P}^1)| = |\operatorname{Aut}_0(F_0)|.$$

This yields $j(F_0) = 0$ or 1728. Since the morphism $q_1 \colon F_0 \to E$ obtained in (24) is a composition of relative Frobenius morphisms (cf. [25, Theorem V.3.1]), [10, Exercise IV.4.20(a)] produces the isomorphism $E \cong F_0$, which completes the proof. $\square$

Claim 5.6 completes the proof of Theorem 1.1 in the case $m = p^e \geq 5$.

Case: Arbitrary $m \geq 5$ **with** $m \neq p^e$ **for any** $e > 0$**.** We may put $m = np^e$ with $e \geq 0$, $n > 1$, $p \nmid n$. We generalize the method of [30] below.

Recall that $S \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L})$, and define elliptic curves F_0 and F as $\hat{F}_0 := \hat{E}/\langle \mathcal{L} \rangle$ and $\hat{F} := \hat{E}/\left\langle \mathcal{L}^{p^e} \right\rangle$. Denote by

$$q_E \colon F \to E$$

the dual morphism of the quotient morphism $\hat{E} \to \hat{F} = \hat{E}/\left\langle \mathcal{L}^{p^e} \right\rangle$. Set

$$\mathcal{M} := q_E^* \mathcal{L} \ \text{ and } \ T := \mathbb{P}(\mathcal{O}_F \oplus \mathcal{M}).$$

Then we see $\hat{F}_0 = \hat{F}/\langle \mathcal{M} \rangle$ and $\operatorname{ord} \mathcal{M} = p^e$. Moreover if $e > 0$, the existence of a nonzero element $\mathcal{M}$ of $\hat{F}[p^e]$ implies that F is ordinary, and the dual morphism of the quotient morphism

$$\hat{F} \to \hat{F}_0 = \hat{F}/\langle \mathcal{M} \rangle .$$

is the e-th iteration of the relative Frobenius morphisms (cf. [25, Theorem V.3.1]). Then we obtain the following commutative diagram:

(31)
$$\begin{array}{ccccc}
F_0 & \xleftarrow{p_1} & F_0 \times \mathbb{P}^1 & \xrightarrow{p_2} & \mathbb{P}^1 \\
{\scriptstyle \mathrm{Fr}^e}\downarrow & \square & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle \mathrm{Fr}^e_{\mathbb{P}^1}} \\
F & \xleftarrow[f_1]{} & T & \xrightarrow[\pi_1]{} & \mathbb{P}^1 \\
{\scriptstyle q_E}\downarrow & \square & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q_{\mathbb{P}^1}} \\
E & \xleftarrow[f]{} & S & \xrightarrow[\pi]{} & \mathbb{P}^1
\end{array}$$

Both of the left squares are fiber product diagrams, and the right squares are obtained by the Stein factorizations of $\pi_1 \circ h_1$ and $\pi \circ q$ respectively. Moreover we have

$$\deg q_E = \deg q = \deg q_{\mathbb{P}^1} = n.$$

Take

(32) $$i \in \mathbb{Z} \text{ with } 1 \leq i < m, \quad (i, m) = 1.$$

Note that this condition implies that $(i, p^e) = (i, n) = 1$, and hence we sometimes regard $i \in (\mathbb{Z}/p^e\mathbb{Z})^*$ or $i \in (\mathbb{Z}/n\mathbb{Z})^*$ below.

Recall that we have already proved Theorem 1.1 for line bundles whose order is p-th power. By applying it to $\mathcal{M}$, we obtain

(33) $$J^i(T) \cong \mathbb{P}(\mathcal{O}_F \oplus \mathcal{M}^{\beta(i)})$$

for some $\beta(i) \in (\mathbb{Z}/p^e\mathbb{Z})^*$. Moreover, since $(\mathrm{Fr}^e)^* \mathcal{M} \cong \mathcal{O}_{F_0}$, we have a diagram

(34)
$$\begin{array}{ccccc} F_0 & \xleftarrow{p_1} & F_0 \times \mathbb{P}^1 & \xrightarrow{p_2} & \mathbb{P}^1 \\ \Big\downarrow{\scriptstyle \mathrm{Fr}^e} & \square & \Big\downarrow{\scriptstyle h_i} & & \Big\downarrow{\scriptstyle \mathrm{Fr}^e_{\mathbb{P}^1}} \\ F & \xleftarrow[f_i]{} & J^i(T) & \xrightarrow[\pi_i]{} & \mathbb{P}^1 \end{array}$$

as in (25). Here f_i is a $\mathbb{P}^1$-bundle defined by using the $\mathbb{P}^1$-bundle structure on $\mathbb{P}(\mathcal{O}_F \oplus \mathcal{M}^{\beta(i)})$ and the isomorphism (33).

Fix an n-th primitive root of unity ζ. Consider the multiplication on $\mathbb{G}_m$ by ζ, and extend it to the automorphism of $\mathbb{P}^1$. Denote it by $g_{\mathbb{P}^1}$. Because we see that $q_{\mathbb{P}^1}$ in (31) fixes points 0 and ∞ in $\mathbb{P}^1$, it turns out that the morphism $q_{\mathbb{P}^1}$ is the quotient morphism by the action of the group $\langle g_{\mathbb{P}^1} \rangle \cong \mathbb{Z}/n\mathbb{Z}$ on $\mathbb{P}^1$.

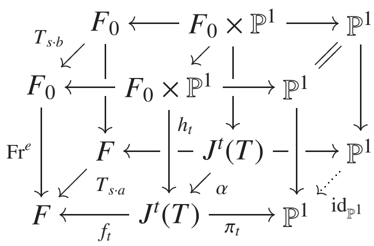
Take $a \in F$ such that $E \cong F/\langle a \rangle$ and $\operatorname{ord} a (= \operatorname{ord} \mathcal{L}^{p^e}) = n$. Then we can construct an action of the group $G := \mathbb{Z}/n\mathbb{Z}$ on $J^i(T)$ as follows.

Claim 5.7. For each $s \in (\mathbb{Z}/n\mathbb{Z})^*$ and $t \in (\mathbb{Z}/p^e\mathbb{Z})^*$, there exists an automorphism g_s of $J^t(T)$ which induces the translation $T_{s \cdot a}$ of F and the automorphism $g_{\mathbb{P}^1}$ of $\mathbb{P}^1$.

Proof. Since $T^*_{s \cdot a}\mathcal{M} \cong \mathcal{M}$, there exists an automorphism

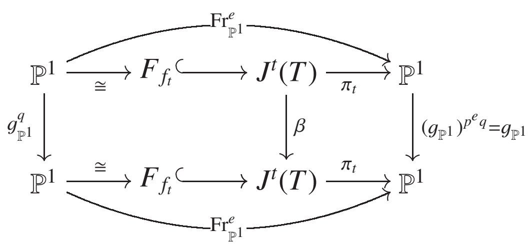
$$\alpha \in \operatorname{Aut}(J^t(T)) (\overset{(33)}{\cong} \operatorname{Aut}(\mathbb{P}(\mathcal{O}_F \oplus \mathcal{M}^{\beta(t)})))$$

compatible with $T_{s \cdot a}$ on F. Note that $T_{s \cdot a}$ lifts a translation $T_{s \cdot b}$ on F_0 for some $b \in F_0$ with $\mathrm{Fr}^e(b) = a$, and hence α lifts to $T_{s \cdot b} \times \mathrm{id}_{\mathbb{P}^1}$ on $F_0 \times \mathbb{P}^1$.



Therefore, α respects the elliptic fibration π_t, i.e. $\alpha \in \operatorname{Aut}(J^t(T)/\mathbb{P}^1)$.

Next take an integer q with $p^e q = 1$ in $(\mathbb{Z}/n\mathbb{Z})^*$. It follows from Lemma 5.5 that there exists an automorphism $\beta \in \operatorname{Aut}(J^t(T)/F)$ which induces the automorphism $g^q_{\mathbb{P}^1}$ on each fiber F_{f_t} (which we identify with $\mathbb{P}^1$ by (27)) of the $\mathbb{P}^1$-bundle f_t. Combining (27) with the commutativity of the right square in (34), we see that $\pi_t|_{F_{f_t}} : F_{f_t} \to \mathbb{P}^1$ coincides with $\mathrm{Fr}^e_{\mathbb{P}^1}$, and then β induces the automorphism $(g_{\mathbb{P}^1})^{p^e q} = g_{\mathbb{P}^1}$ on $\mathbb{P}^1$, the base space of π_t.



Hence, the automorphism $g_s := \alpha \circ \beta$ has the desired property. $\square$

Denote by g a generator of the cyclic group $G = \mathbb{Z}/n\mathbb{Z}$, and define the action of G on $J^t(T)$ by

$$\rho_{s,t}\colon G \to \operatorname{Aut}(J^t(T)) \quad g \mapsto g_s. \tag{35}$$

For the integer i given in (32), regard $i \in (\mathbb{Z}/n\mathbb{Z})^*$ and $i \in (\mathbb{Z}/p^e\mathbb{Z})^*$, and set $\rho_i := \rho_{i,i}$. We define the quotient variety to be

$$S_i := J^i(T)/_{\rho_i}G \tag{36}$$

by the action ρ_i, and denote the quotient morphism by

$$q_i\colon J^i(T) \to S_i.$$

It is easy to see that S is the quotient of $T = J^1(T)$ by the action $\rho_{s,1}$ for some s. Replace $a \in F$ with $s \cdot a$, and redefine g_s and $\rho_{s,t}$ by this new a, so that $S = S_1$ holds. After this replacement, we consider only the action ρ_i, but not general $\rho_{s,t}$.

We set

$$g_i^0 := T_{i\cdot b} \times g_{\mathbb{P}^1}^q \in \operatorname{Aut}(F_0 \times \mathbb{P}^1).$$

Then we see that $\operatorname{ord} g_i^0 = \operatorname{ord} T_{i\cdot b} = \operatorname{ord} g_{\mathbb{P}^1}^q = n$ and it is compatible with $g_i \in \operatorname{Aut}(J^i(T))$ defined in Claim 5.7:

$$h_i \circ g_i^0 = g_i \circ h_i. \tag{37}$$

We also define the action on $F_0 \times \mathbb{P}^1$ by

$$\rho_i^0\colon G \to \operatorname{Aut}(F_0 \times \mathbb{P}^1) \quad g \mapsto g_i^0 \tag{38}$$

for each i.

Take an integer j with $1 \le j < m$, $(j, m) = 1$ and $ij = 1$ in $(\mathbb{Z}/m\mathbb{Z})^*$. For the projection

$$p_{13}\colon F_0 \times \Delta_{\mathbb{P}^1} \times F_0 \to F_0 \times F_0,$$

define a line bundle

$$\mathcal{U}_0 := p_{13}^*\mathcal{O}_{F_0\times F_0}(\Delta_{F_0} + (j-1)F_0 \times O + (i-1)O \times F_0)$$

on

$$F_0 \times \Delta_{\mathbb{P}^1} \times F_0 (\cong (F_0 \times \mathbb{P}^1) \times_{\mathbb{P}^1} (F_0 \times \mathbb{P}^1)).$$

Then $F_0 \times \mathbb{P}^1$ in the second factor in R.H.S. serves as $J^i(F_0 \times \mathbb{P}^1)$ where $\mathcal{U}_0$ plays the role of a universal sheaf, and moreover it is shown in [30, page 3229] that it satisfies

$$(\rho_1^0(g) \times \rho_i^0(g))^*\mathcal{U}_0 \cong \mathcal{U}_0. \tag{39}$$

On the other hand, it follows from [4, Theorem 5.3] that we can take a universal sheaf $\mathcal{U}'$ on $T \times_{\mathbb{P}^1} J^i(T)$, which satisfies that $\mathcal{U}'|_{z\times J^i(T)}$ is a line bundle of degree j on F_0 for general $z \in T$. For a point $(x, y) \in F_0 \times (\mathbb{P}^1\backslash\{0, \infty\})$, there exists an isomorphism

$$((h_1 \times h_i)^*\mathcal{U}')|_{(F_0\times\mathbb{P}^1)\times_{\mathbb{P}^1}(x,y)} \cong \mathcal{U}'|_{T\times_{\mathbb{P}^1} h_i((x,y))}, \tag{40}$$

since the restriction of $h_1 \times h_i$ gives

$$(F_0 \times \mathbb{P}^1) \times_{\mathbb{P}^1} (x, y) \cong F_0 \times y \cong F_y \cong T \times_{\mathbb{P}^1} h_i((x, y)),$$

where the second isomorphism comes from (26). Hence, we see that the L.H.S. in (40) is a line bundle of degree i on F_0. Then, by the universal property of $\mathcal{U}_0$, there exists an automorphism $\phi_0 \in \operatorname{Aut}(F_0)$ such that

$$(\mathrm{id}_{F_0 \times \Delta_{\mathbb{P}^1}} \times \phi_0)^* \mathcal{U}_0 \cong (h_1 \times h_i)^* \mathcal{U}' \otimes p_3^* \mathcal{N}_0$$

for some $\mathcal{N}_0 \in \operatorname{Pic}^0 F_0$.

We shall construct an elliptic ruled surface T' and (iso)morphisms ϕ_F, ϕ, h' which make the following diagrams commutative:

(41)
$$\begin{array}{ccccc}
 & & F_0 & \longleftarrow & F_0 \times \mathbb{P}^1 \\
 & \swarrow^{\phi_0} & \big| & \swarrow & \big\downarrow h_i \\
F_0 & \longleftarrow & F_0 \times \mathbb{P}^1 & & \\
\big\downarrow \mathrm{Fr}^e & & \mathrm{Fr}^e \big\downarrow \quad F & \longleftarrow \quad h' \big| & J^i(T) \\
 & \swarrow^{\phi_F} & & \big\downarrow \swarrow^{\phi} & \\
F & \longleftarrow & T' & &
\end{array}$$

First, ϕ_0 descends to $\phi_F \in \operatorname{Aut}(F)$ via $\mathrm{Fr}^e \colon F_0 \to F$ by [25, Corollary II.2.12], and ϕ_F induces an isomorphism

$$\phi\colon J^i(T) \cong \mathbb{P}(\mathcal{O}_F \oplus \mathcal{M}^{\beta(i)}) \to T' := \mathbb{P}(\mathcal{O}_F \oplus \phi_{F*}\mathcal{M}^{\beta(i)}).$$

Note that $\phi_{F*} \in \operatorname{Aut}_0(\hat{F})$ preserves the subgroup $\ker \widehat{\mathrm{Fr}^e} = \hat{F}[p^e] = \langle \mathcal{M} \rangle$ of $\hat{F}$, and thus $\phi_{F*}\mathcal{M}^{\beta(i)} \in \langle \mathcal{M} \rangle$. Hence we obtain a morphism

$$h'\colon F_0 \times \mathbb{P}^1 \cong \mathbb{P}(\mathcal{O}_{F_0} \oplus \mathcal{O}_{F_0}) \to T' \cong \mathbb{P}(\mathcal{O}_F \oplus \phi_{F*}\mathcal{M}^{\beta(i)}),$$

which fits into the diagram in (41). Moreover we have the following commutative diagram:

$$\begin{array}{ccccc}
F_0 \times \Delta_{\mathbb{P}^1} \times F_0 & \xleftarrow{(\mathrm{id}_{F_0 \times \Delta_{\mathbb{P}^1}}) \times \phi_0} & F_0 \times \Delta_{\mathbb{P}^1} \times F_0 & \xrightarrow{p_3} & F_0 \\
{\scriptstyle h_1 \times h'} \big\downarrow & & \big\downarrow {\scriptstyle h_1 \times h_i} & & \big\downarrow {\scriptstyle \mathrm{Fr}^e} \\
T \times_{\mathbb{P}^1} T' & \xleftarrow[\mathrm{id}_T \times \phi]{} & T \times_{\mathbb{P}^1} J^i(T) & \xrightarrow[f_i \circ p_2]{} & F
\end{array}$$

Take $\mathcal{N} \in \operatorname{Pic}^0 F$ such that $(\mathrm{Fr}^e)^* \mathcal{N} = \mathcal{N}_0$, and define a line bundle

$$\mathcal{U} := (\mathrm{id}_T \times \phi)_* (\mathcal{U}' \otimes (f_i \circ p_2)^* \mathcal{N})$$

on $T \times_{\mathbb{P}^1} T'$ so that

$$\mathcal{U}_0 \cong (h_1 \times h')^* \mathcal{U} \tag{42}$$

holds. The pair $(T', \mathcal{U})$ serves as $J^i(T)$ and its universal sheaf, and thus we redefine T' to be $J^i(T)$.

CLAIM 5.8. The universal sheaf $\mathcal{U}$ on $T \times_{\mathbb{P}^1} J^i(T)$ satisfies

$$(\rho_1(g) \times \rho_i(g))^* \mathcal{U} \cong \mathcal{U}.$$

Proof. Take $y_0 \in \mathbb{P}^1 \backslash \{0, \infty\}$ with $y := \mathrm{Fr}^e(y_0) \in \mathbb{P}^1 \backslash \{0, \infty\}$. Denote by $F_y \times F'_y$ the fiber of

$\pi_1 \times \pi_i \colon T \times_{\mathbb{P}^1} J^i(T) \to \mathbb{P}^1$ over the point y. Pull back the isomorphism (42) to the subscheme $F_0 \times y_0 \times F_0$, which is isomorphic to $F_y \times F'_y$ by (26), and combine (37) and (39) with it, and then we have isomorphisms

$$((\rho_1(g) \times \rho_i(g))^* \mathcal{U})|_{F_y \times F_y} \cong ((\rho_1^0(g) \times \rho_i^0(g))^* \mathcal{U}_0)|_{F_0 \times y_0 \times F_0} \cong \mathcal{U}_0|_{F_0 \times y_0 \times F_0} \cong \mathcal{U}|_{F_y \times F_y}.$$

$$\begin{array}{ccccc} F_0 \times y_0 \times F_0 & \hookrightarrow & F_0 \times \Delta_{\mathbb{P}^1} \times F_0 & \xrightarrow{p_2} & \mathbb{P}^1 \ni y_0 \\ \cong \downarrow & & \downarrow h_1 \times h_i & & \downarrow \mathrm{Fr}^e_{\mathbb{P}^1} \\ F_y \times F'_y & \hookrightarrow & T \times_{\mathbb{P}^1} J^i(T) & \xrightarrow[\pi_1 \times \pi_i]{} & \mathbb{P}^1 \ni y \end{array}$$

This yields that the line bundle $L := (\rho_1(g) \times \rho_i(g))^* \mathcal{U} \otimes \mathcal{U}^{-1}$ is trivial over the open set $(\pi_1 \times \pi_i)^{-1}(\mathbb{P}^1 \backslash \{0, \infty\})$ by [10, Exercise III.12.4]. We also see by (37), (39) and (42) that $(h_1 \times h_i)^* L$ is trivial over $\mathbb{P}^1 \backslash \{0, \infty\}$, and thus

$$L \cong \mathcal{O}_{T \times_{\mathbb{P}^1} J^i(T)}(b(D_0 \times D'_0 - D_\infty \times D'_\infty)) \tag{43}$$

for some $b \in \mathbb{Z}$, where $p^e D_0$ and $p^e D'_0$ (resp. $p^e D_\infty$ and $p^e D'_\infty$) are the multiple fibers over $0 \in \mathbb{P}^1$ (resp. ∞) of π_1 and π_i. Note that $\operatorname{ord} L$ divides p^e, the multiplicity of the multiple fibers. Since $\operatorname{ord}(\rho_1(g) \times \rho_i(g)) = n$ and the R.H.S. in (43) is $(\rho_1(g) \times \rho_i(g))$-invariant, we see that

$$\mathcal{U} \cong (\rho_1(g) \times \rho_i(g))^{n*} \mathcal{U} \cong (\rho_1(g) \times \rho_i(g))^{(n-1)*} \mathcal{U} \otimes L \cong \cdots \cong \mathcal{U} \otimes L^{\otimes n},$$

and hence $\operatorname{ord} L \mid n$. Since $p \nmid n$, we have $\operatorname{ord} L = 1$, as it is required. □

Recall that we have the following commutative diagram by the definition of S_i in (36):

$$\begin{array}{ccccc} F & \xleftarrow{f_i} & J^i(T) & \xrightarrow{\pi_i} & \mathbb{P}^1 \\ q_E \downarrow & \square & \downarrow q_i & & \downarrow q_{\mathbb{P}^1} \\ E & \longleftarrow & S_i & \xrightarrow[\pi_{S_i}]{} & \mathbb{P}^1 \end{array}$$

Here, q_E and $q_{\mathbb{P}^1}$ are the same one appeared in (31), and π_{S_i} is an elliptic fibration.

Claim 5.9. For each i, there exists $\alpha(i) \in (\mathbb{Z}/m\mathbb{Z})^*$ such that we have an isomorphism

$$S_i \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^{\alpha(i)})$$

over E.

Proof. First of all, we know by Theorem 5.1 that there exisits an isomorphism $S_i \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}_i)$ over E for some $\mathcal{L}_i \in \operatorname{Pic}^0 E$ with $\operatorname{ord} \mathcal{L}_i = m$. Then the result follows from

$$\mathcal{L}_i \in \ker(\widehat{\mathrm{Fr}^e} \circ \widehat{q_E}) = \langle \mathcal{L} \rangle \cong \mathbb{Z}/m\mathbb{Z}.$$ □

Recall that $S = S_1$ below.

Claim 5.10. There exists an isomorphism $J^i(S) \cong S_i$.

Proof. First, we shall show that there exists a coherent sheaf $\mathcal{U}_i$ on $S \times S_i$ such that

$$(q_1 \times \mathrm{id}_{J^i(T)})_* \mathcal{U} \cong (\mathrm{id}_S \times q_i)^* \mathcal{U}_i \tag{44}$$

for the morphisms

$$T \times J^i(T) \overset{q_1 \times \mathrm{id}_{J^i(T)}}{\to} S \times J^i(T) \overset{\mathrm{id}_S \times q_i}{\to} S \times S_i.$$

Claim 5.8 implies that

$$(\rho_1(g) \times \mathrm{id}_{J^i(T)})^* \mathcal{U} \cong (\mathrm{id}_T \times \rho_i(g)^{-1})^* \mathcal{U}.$$

Push forward the both sides by the morphism $q_1 \times \mathrm{id}_{J^i(T)}$ and then we obtain

$$(q_1 \times \mathrm{id}_{J^i(T)})_* \mathcal{U} \cong (\mathrm{id}_S \times \rho_i(g)^{-1})^* (q_1 \times \mathrm{id}_{J^i(T)})_* \mathcal{U},$$

that is, the sheaf $(q_1 \times \mathrm{id}_{J^i(T)})_* \mathcal{U}$ is G-invariant with respect to the diagonal action of G on $S \times J^i(T)$, where G acts on S trivially. Since $G = \langle g \rangle$ is a finite cyclic group, the G-invariance of coherent sheaves is equivalent to the G-equivariance, and hence there exists a coherent sheaf $\mathcal{U}_i$ on $S \times S_i$ satisfying (44).

For $z \in J^i(T)$, we have

$$\mathcal{U}_i|_{S \times q_i(z)} \cong ((q_1 \times \mathrm{id}_{J^i(T)})_* \mathcal{U})|_{S \times z} \cong q_{1*}(\mathcal{U}|_{T \times z}).$$

Here, the second isomorphism follows from [2, Lemma 1.3] and the smoothness of q_1. Suppose that z is not contained in multiple fibers of π_i, that is, $y := \pi_i(z) \in \mathbb{P}^1 \backslash \{0, \infty\}$ by the convention stated in §5.2. Then $\mathcal{U}|_{T \times z}$ is actually a sheaf on $F_y \times z$, and the restriction $q_1|_{F_y \times z}$ is an isomorphism by (26). It turns out that $\mathcal{U}_i|_{S \times q_i(z)}$ is also a line bundle of degree i on $F_{q_{\mathbb{P}^1}(y)} \times q_i(z)$.

Then, by the universal property of $J^i(S)$, there exists a morphism from

$$\pi_{S_i}^{-1}(\mathbb{P}^1 \backslash \{0, \infty\})(\subset S_i) \to \pi_{J^i(S)}^{-1}(\mathbb{P}^1 \backslash \{0, \infty\})(\subset J^i(S))$$

over $\mathbb{P}^1 \backslash \{0, \infty\}$, where π_{S_i} and $\pi_{J^i(S)}$ are the elliptic fibrations on S_i and $J^i(S)$ respectively. Since $\mathcal{U}_i|_{S \times q_i(z_1)} \not\cong \mathcal{U}_i|_{S \times q_i(z_2)}$ on F_y for $z_1 \neq z_2 \in J^i(T)$, this morphism is injective, and hence S_i and $J^i(S)$ are birational over $\mathbb{P}^1$. Then, [1, Proposition III.8.4] implies that $S_i \cong J^i(S)$. □

Combining Claims 5.9 and 5.10, we obtain the inclusion (29) by the map

$$J^i(S) \mapsto \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^{\alpha(i)}).$$

The next aim is to show (30).

CLAIM 5.11. There exists an injective group homomorphism

$$\overline{\alpha} \colon H_\pi/\{\pm 1\} \to H_{\hat{E}}^{\mathcal{L}}/\{\pm 1\}.$$

Proof. Take $i \in H_\pi (:= \{i \in (\mathbb{Z}/m\mathbb{Z})^* \mid J^i(S) \cong S\})$. We have $\alpha(i) \in (\mathbb{Z}/m\mathbb{Z})^*$ so that there exists an isomorphism

$$\psi \colon \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^{\alpha(i)}) \overset{\cong}{\to} S_i \overset{\cong}{\to} J^i(S)$$

by Claims 5.9 and 5.10. We use ψ and the $\mathbb{P}^1$-bundle structure on $\mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^{\alpha(i)})$ to fix a

$\mathbb{P}^1$-bundle structure on $J^i(S)$:

$$f_{J^i(S)}\colon J^i(S) \to E$$

Then Lemma 5.3 (iii) implies that there exist an isomorphism φ and an automorphism $\varphi_E \in \mathrm{Aut}_0(E)$ fitting in the commutative diagram

$$\begin{array}{ccccc} \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^{\alpha(i)}) & \xrightarrow{\psi} & J^i(S) & \xrightarrow{\varphi} & S \\ \downarrow & & \downarrow{\scriptstyle f_{J^i(S)}} & & \downarrow{\scriptstyle f} \\ E & = & E & \xrightarrow[\varphi_E]{} & E \end{array} \tag{45}$$

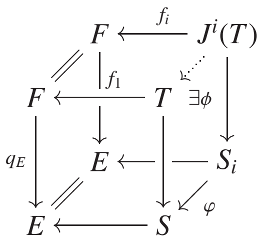
and $\varphi_E^*\mathcal{L} \cong \mathcal{L}^{\alpha(i)}$ is satisfied.

Take another isomorphism $\varphi'\colon J^i(S) \to S$. Then since $\varphi' \circ \varphi^{-1}$ is an automorphism of $\mathbb{P}(\mathcal{O}_E \oplus \mathcal{L})$, we have $(\varphi'_E \circ \varphi_E^{-1})^*\mathcal{L} \cong \mathcal{L}^{\pm 1}$ by Lemma 5.3 (i) and (ii). Thus we obtain the group homomorphism

$$\alpha\colon H_\pi \to H^{\mathcal{L}}_{\hat{E}}/\{\pm 1\}(:= \{i \in (\mathbb{Z}/m\mathbb{Z})^* \mid \exists \phi \in \mathrm{Aut}_0(E) \text{ s.t. } \phi^*\mathcal{L} \cong \mathcal{L}^i\}/\{\pm 1\}.)$$

Thus it suffices to prove $\mathrm{Ker}\,\alpha = \{\pm 1\}$. Suppose $i \in \mathrm{Ker}\,\alpha$. Since $\varphi_E^*\mathcal{L} \cong \mathcal{L}^{\pm 1}$ holds in this case, Lemma 3.2 implies that φ_E fitting in the diagram (45) is either id_E or $-\mathrm{id}_E$. Replace φ with $f^*(-\mathrm{id}_E) \circ \varphi$ (see the notation in Lemma 5.3 (ii) and the proof of ibid. (iii)) if necessary, and then we may assume that $\varphi_E = \mathrm{id}_E$. We have the following commutative diagram [5]:

(46)



Because the front and the back squares in (46) are the fiber product diagrams, there exists an isomorphism $\phi\colon J^i(T) \to T$ which makes the right square the fiber product.

Since ϕ descends to $\varphi\colon S_i = J^i(T)/_{\rho_i}G \to S = T/_{\rho_1}G$ for $G = \mathbb{Z}/n\mathbb{Z} = \langle g \rangle$, we have

$$\rho_1(g) \circ \phi = \phi \circ \rho_i(g)^l$$

for some l. Recall that both of $\rho_1(g)$ and $\rho_i(g)$ induce the same automorphism $g_{\mathbb{P}^1}$ on the base curve $\mathbb{P}^1$ of the elliptic fibrations on T and $J^i(T)$ (see Claim 5.7 and (35)). Then we see $l = \pm 1$. Next recall $\rho_1(g)$ (resp. $\rho_i(g)$) induces the automorphism T_a (resp. $T_{i\cdot a}$) on F, the base curve of the $\mathbb{P}^1$-bundle f_1 (resp. f_i). Then we know that

$$T_a = (T_{i\cdot a})^l = T_{li\cdot a},$$

and hence, $1 = il$ in $(\mathbb{Z}/n\mathbb{Z})^*$. Therefore we have $i = \pm 1$, and hence $\mathrm{Ker}\,\alpha \subset \{\pm 1\}$. The other direction is obvious. □

By Claim 5.11, we conclude that $|H_\pi| \le |H^{\mathcal{L}}_E|$ as is required in (30).

[5]Here, we identify S_i and $J^i(S)$ by Claim 5.10.

Therefore, we complete the proof of the first statement in Theorem 1.1 for arbitrary $m \geq 5$. The second follows from Lemma 3.3 (ii).

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