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A CHARACTERIZATION OF TRANSLATION PLANES
AND DUAL TRANSLATION PLANES OF
CHARACTERISTIC $\pm 2$. *

A. SOLAI RAJU

(Received October 1, 1992)

Introduction

In [7], A. Wagner has introduced a special class of finite affine planes. These are called $W$-planes by M.J. Kallaher (See [3], p. 106). These planes admit plenty of involutory homologies. A. Wagner proved (lemma 3, [7]) that a $W$-plane is either a translation plane or a dual translation plane or has certain property. In this paper, we study a finite affine plane, in which only the first of Wagner's condition is satisfied, called weak $W$-plane. We show a stronger theorem that a weak $W$-plane is a translation plane of characteristic $\pm 2$ or a dual translation plane of characteristic $\pm 2$. Towards this end, we study the implications of a weak $W$-plane admitting a $(P, l)$-transitivity for some point-line pair $(P, l)$ and prove our assertion under this extra hypothesis. In the third section of this paper, we relax this condition and show our main theorem.

1. Previous Results

For basic definitions and theorems, we refer to [2] and [3]. We also make frequent use of the following well known theorems on collineations of projective planes.

Theorem 1.1. ([2], p. 98 Corollary). Let $\Pi$ be a finite projective plane of order $n$ and let $G$ be a collineation group of $\Pi$. If $|G_{(A, l)}| > 1$ for at least two choices of $A$ on $l$, then $G_{(l, l)}$ is an elementary abelian $p$-group where $p$ is a prime divisor of $n$.

Theorem 1.2. ([2], p. 104, Corollary 1 to Theorem 4.25). Let $\pi$ be a finite projective plane and $\alpha, \beta$ be two non-trivial homologies with distinct centers $A, B$ and the same axis $l$. Then $<\alpha, \beta>$ contains an $(AB \cap l, l)$-elation mapping.

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\( A \) onto \( B \).

**Theorem 1.3.** (Gleason) ([2], p. 104, Corollary 1 to Theorem 4.26). Let \( \Pi \) be a finite projective plane and let \( G \) be a collineation group of \( \Pi \). If, for some line \( l \), \( |G_{(x,0)}| = h > 1 \) for all \( x \) on \( l \), then \( l \) is a translation line.

**Theorem 1.4.** ([2], p. 101. Lemma 4.22). Let \( \pi \) be a projective plane. If \( \alpha \) is an involutory \((A, a)\)-homology of \( \pi \) and \( \beta \) is an involutory \((B, b)\)-homology of \( \pi \) such that \( B \equiv a \) and \( A \equiv b \), then \( \alpha \beta \) is an involutory \((a \cap b, AB)\)-homology of \( \pi \).

**Theorem 1.5.** (Gleason) ([3], p. 30, Lemma 3.6). Let \( G \) be a permutation group on a finite set \( \Omega \) of order \( n > 1 \), and let \( p \) be a prime. If, for every point \( x \in \Omega \), the group \( G \) contains an element \( \alpha \) of order \( p \) fixing \( x \) and no other point of \( \Omega \), then \( G \) is transitive on \( \Omega \).

**Theorem 1.6.** ([5]). If a projective plane \( \pi \) is \((A, l)\)-transitive for every line \( l \) through \( B \) and \( A \neq B \), then the plane \( \pi \) is \((A, B)\)-transitive (i.e., \( \pi \) is \((A, l)\)-transitive for every line \( l \) through \( B \)). In particular \( AB \) is a translation line.

**Theorem 1.7.** ([4] and [6]). If a projective plane \( \pi \) is \((p_i, l_i)\)-transitive for \( i = 1, 2 \) where \( P_1 \in l_1, P_2 \in l_2 \) and \( l_1 \cap l_2 \in P_1 P_2 \), then \( \pi \) is Desarguesian or it is the plane over the near-field of order 9 or its dual.

**Theorem 1.8.** ([1]). This is no finite plane of the class \( I_6 \) or the class \( III \).

**Theorem 1.9.** (Ostrom-Wagner) ([3], p. 43, Theorem 4.3). Let \( \Sigma \) be a finite affine plane, and \( G \), a collineation group of \( \Sigma \). The following statements are equivalent:

1. The group \( G \) is transitive on the affine points of \( \Sigma \).
2. For every point \( U \in l \), the group \( G_U \) operates transitively on the affine lines of \( \Sigma \) through \( U \).

**Theorem 1.10.** ([2]), p. 104, Theorem 4.26). Let \( \pi \) be a finite projective plane of order \( n \) and let \( G \) be a collineation group of \( \pi \). Suppose there is a line \( l \) and a point \( Q \) on \( l \) such that \( |G_{(A,0)}| = h > 1 \) for all \( A \) in \( l \), \( A \neq Q \). Then \( |G_{(Q,0)}| = n \) i.e. \( \pi \) is \((Q, l)\)-transitive.

**Theorem 1.11.** (Ostrom) ([3], p. 51, Theorem 4.6). Let \( \pi \) be a finite projective plane of order \( n \), let \( G \) be a collineation group of \( \pi \), and let \( l \) be a line of \( \pi \). If \( |G_{(A,0)}| > n \), then \( |G_{(P,0)}| > 1 \) for every point \( P \in l \).

2. Definitions and basic lemmas

**Definition:** ([3], p. 106). Let \( \Sigma \) be a finite affine plane and let \( G \) be a
collineation group of $\Sigma$. Then the plane $\Sigma$ is called a $W$-plane and $G$, a $W$-group if $G$ has the following properties:

(i) For every affine flag $(Q, l)$ of $\Sigma$, the group $G$ contains an involutory homology fixing the flag $(Q, l)$.

(ii) Let $P$ and $Q$ be any two points on the line $l_\infty$ of $\Sigma$, and let $l$ and $m$ be two affine lines of $\Sigma$ through $Q$. If $G$ contains an involutory homology fixing $P, Q$ and $l$, then $G$ contains an involutory homology fixing $P, Q$ and $m$.

**Definition.** Let $\Sigma$ be a finite affine plane and $G$ be a collineation group of $\Sigma$. If $G$ has property (i) of the above definition, then $\Sigma$ is called a weak $W$-plane with respect to $G$. We will also describe this by saying that the pair $(\Sigma, G)$ is a weak $W$-plane.

In the rest of this paper, whenever a weak $W$-plane $(\Sigma, G)$ and any collineation $\alpha$ is considered, it is tacitly assumed that $\alpha \in G$. This is to avoid repeated mention of $G$.

We may note here that any translation plane or dual translation plane of characteristic $\neq 2$ is a $W$-plane and hence a weak $W$-plane (with respect to the full collineation group).

Since we are looking at affine planes, every collineation of $G$ fixes the line $l_\infty$. Hence $l_\infty$ is fixed by every $(P, l)$-perspectivity. Hence, if further $P \in l$, then necessarily $P \in l_\infty$.

**Lemma 2.1.** Let $\Sigma$ be a weak $W$-plane with respect to a collineation group $G$ of $\Sigma$. Then either $G$ does not have any fixed point in $\Sigma$ or $\Sigma$ is a Moufang plane.

**Proof.** Let $\pi = \Sigma \cup l_\infty$ be the projective closure of $\Sigma$. Let, if possible, $Q$ be an affine point fixed by $G$.

Let $P$ be a point on $l_\infty$ and $l$ be any affine line such that $P \in l$ and $Q \in l$. Let $L$ be any affine point on $l$. Since $\Sigma$ is a weak $W$-plane, there exists an involutory homology $\alpha$ fixing the flag $(L, l)$. Now there are three possibilities, regarding the center and the axis of $\alpha$.

(i) $L$ is the center of $\alpha$ and so $l_\infty$ is the axis of $\alpha$. (ii) $l$ is the axis of $\alpha$ and some point on $l_\infty$ is the center of $\alpha$. (iii) $l \cap l_\infty$ is the center of $\alpha$ and some line through $L$ is the axis of $\alpha$.

Since $\alpha$ fixes $Q$, we conclude that $\alpha$ must be an involutory $(l \cap l_\infty, LQ)$-homology. Since $L$ is an arbitrary affine point on $l$, there exists an involutory $(l \cap l_\infty, DQ)$-homology for every affine point $X$ on $l$. By dual of (1.2), the group of all $(l \cap l_\infty, (l \cap L)Q)$-elations is transitive on the affine points of $l$. Thus the plane $\pi$ is $(P = l \cap l_\infty, (l \cap l_\infty) Q = PQ)$-transitive. Since $l$ is an arbitrary affine line, not through $Q$, $P$ can be taken to be an arbitrary point on $l_\infty$. Therefore the
plane π belongs to Lenz-Barlotti class $III_1$ with $(Q, l_\omega)$ as the distinguished point-line pair. This contradicts theorem (1.8) and our assertion stands proved.

**Lemma 2.2.** Let $(\Sigma, G)$ be a weak $W$-plane which is not a translation plane. Then $G$ has at most one fixed point $P$ on $l_\omega$ and in this case $P$ is a dual translation point.

Proof. Let $\pi=\Sigma \cup l_\omega$ the projective closure of $\Sigma$ and $P$ be a point on $l_\omega$, fixed by every collineation of $G$. Let $n$ be the order of $\pi$. Since $\pi$ admits involutory homologies, $n$ is odd.

We consider two cases separately and prove our assertion in each case.

*Case (i)* Suppose there exists an affine line $l$, not through $P$, which is not the axis of any involutory homology.

Let $L$ be an affine point on $l$. Since $\Sigma$ is a weak $W$-plane, there exists an involutory homology $\alpha$ fixing the flag $(L, \Pi)$. Now either the center of $\alpha$ is $L$ and the axis of $\alpha$ is $l_\omega$ or the center of $\alpha$ is $l \cap l_\omega$ and the axis of $\alpha$ is $lP$ (Since $\alpha$ fixes $P$ also). Let $\Omega_1$ be the set of all affine points $X$ on $l$ such that there exists an involutory $(X, l_\omega)$-homology and let $\Omega_2$ be the set of all affine points $Y$ on $l$ such that $Y \in \Omega_1$. Note that $|\Omega_1 \cup \Omega_2|=n=\text{odd}$.

For any two distinct affine points $R_1$ and $R_2 \in \Omega_1$, there exists an involutory $(R_1, l_\omega)$-homology and an involutory $(R_2, l_\omega)$-homology. Thus there exists a $(l \cap l_\omega, l_\omega)$-elation mapping $R_1$ onto $R_2$ by (1.2). Therefore the group of all $(l \cap l_\omega, l_\omega)$-elations is transitive on the points of $\Omega_1$.

For any two distinct affine points $Q_1$ and $Q_2 \in \Omega_2$, there exists an involutory $(Q_1, l_\omega)$-homology and an involutory $(Q_2, l_\omega)$-homology. By dual of (1.2), there exists a $(l \cap l_\omega, l_\omega)$-elation $\beta$ mapping $Q_1$ onto $Q_2$ and so $\beta$ maps $Q_1$ onto $Q_2$. Thus the group of all $(l \cap l_\omega, l_\omega)$-elations is transitive on the points of $\Omega_2$. Since $\Omega_1 \cap \Omega_2 = \emptyset$ the group of all $(l \cap l_\omega, l_\omega)$-elations divides the line $l$ into two orbits, namely $\Omega_1$ and $\Omega_2$.

If either $\Omega_1 = \emptyset$ or $\Omega_2 = \emptyset$, then the plane $\pi$ is $(l \cap l_\omega, l_\omega)$-transitive by (1.2) or its dual. Since $l_\omega$ has at least three distinct points, let $Z$ be a point on $l_\omega$ such that $P \neq Z \neq l \cap l_\omega$. Let $r$ be an affine line through $Z$, and $R$ be an affine point on $r$. Let $\lambda$ be an involutory homology fixing the flag $(R, r)$. Since $\lambda$ fixes $P$, $\lambda$ has to be an involutory $(R, l_\omega)$-homology. Otherwise, $\lambda$ shifts the point $l \cap l_\omega$ and fixes the line $l_\omega$ and so the plane $\pi$ is $((l \cap l_\omega) \lambda \neq l \cap l_\omega, l_\omega \lambda = l_\omega)$-transitive and thus $l_\omega$ is a translation line which contradicts our hypothesis. Since $R$ is an arbitrary affine point on $r$, the plane $\pi$ is $(Z, l \cap l_\omega)$-transitive by (1.2) and so $l_\omega$ is a translation line which again contradicts our hypothesis.

Hence $\Omega_1 \neq \emptyset \neq \Omega_2$.

We now observe that the plane $\pi$ is $(l \cap l_\omega, l_\omega)$-transitive if there exists a non-trivial elation which takes a point of $\Omega_1$ onto a point of $\Omega_2$. Therefore every non-trivial $(l \cap l_\omega, l_\omega)$-elation fixes $\Omega_1$ and $\Omega_2$ setwise. Since a non-trivial
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((l \cap L_0, L_0))-elation is uniquely determined by the image of any single point, not on \(L_0\), it is clear that \(|G(l \cap L_0, L_0)| = |\Omega_1|\) and \(|G(l \cap L_0, L_0)| = |\Omega_2|\). Thus \(|\Omega_1| = |\Omega_2|\).

Hence \(n\) must be even which is not possible.

Case (ii) Every affine line, not through \(P\), is the axis of an involutory homology. Since \(G\) fixes \(P\), the center of all such involutory homologies is \(P\). Thus \(P\) is a dual translation point by dual of (1.2).

**Theorem 2.3.** Let \((\Sigma, G)\) be a weak \(W\)-plane and \(\Sigma\) be \((P, l)\)-transitive for some incident point-line pair \((P, l)\). Then \(\Sigma\) is either a translation plane or a dual translation plane.

**Proof.** Let \(\pi = \Sigma \cap L_0\) be the projective closure of \(\Sigma\). We consider the following two cases separately and prove our assertion in each case.

Case (i): \(l \neq L_0\).

Case (ii): \(l = L_0\).

Case (i). Clearly \(P \in L_0\). Suppose \(\Sigma\) is a translation plane or \(P\) is a dual translation point. Then there is nothing to prove. Otherwise, by (2.2), there exists a collineation \(\alpha\) which shifts \(P\). Thus the plane \(\pi\) is \((P\alpha \neq P, l\alpha \neq l)\)-transitive. Therefore the plane \(\pi\) belongs to Lenz-Barlotti class \(\Pi_1\) with \((l \cap L_0, PP\alpha = L_0)\) as the special point-line pair. But there is no finite plane of the class \(\Pi_1\) by (1.8). So the plane \(\Sigma\) must be Moufang and hence Paipian.

Case (ii) Clearly \(P \in L_0\). If either \(\Sigma\) is a translation plane or \(P\) is a dual translation point, then our assertion stands proved. Otherwise, by (2.2), there exists a collineation \(\lambda\) which shifts \(P\). Then the plane \(\pi\) is \((P\lambda \neq P, L_0\lambda = L_0)\)-transitive. Therefore the line \(L_0\) is a translation line.

**Theorem 2.4.** Let \((\Sigma, G)\) be a weak \(W\)-plane and \(\Sigma\) be \((P, l)\)-transitive for some non-incident point-line pair \((P, l)\). Then \(\Sigma\) is the plane over a near-field or a dual near-field.

**Proof.** Let \(\pi\) be the projective closure of \(\Sigma\). We consider the following two cases separately and prove our assertion in each case.

Case (i): \(l = L_0\).

Case (ii): \(l \neq L_0\).

Case (i). If \(\Sigma\) is Moufang, we are done. Otherwise by (2.1), there exists a collineation \(\alpha\) of \(\Sigma\) such that \(P\alpha = P\). Then the plane \(\pi\) is \((P\alpha \neq P, L_0\alpha = L_0)\)-transitive. Thus the plane \(\pi\) is \((PP\alpha, L_0)\)-transitive by (1.6) and in particular \(PP\alpha \cap L_0\) is a dual translation point. Hence the plane \(\pi\) belongs to the plane over a dual near-field.

Case (ii) Subcase (a). Clearly \(P \in L_0\). Suppose \(G\) fixes \(l \cap L_0\). By (2.2), either \(l \cap L_0\) is a dual translation point or \(L_0\) is a translation line. In the first case, there exists a non-trivial \((l \cap L_0, l)\)-elation \(\beta\) which fixes \(l\) and shifts \(P\). Thus the plane
\( \pi \) is \((P\beta=P, l\beta=l)\)-transitive and so the plane \( \pi \) is \((PP\beta=l, l)\)-transitive by dual of \((1.6)\). Therefore the plane \( \pi \) belongs to the plane over a dual near-field. In second case, there exists a non-trivial \((P, l)\)-elation \( \omega \) which fixes \( P \) and shifts \( l \). Thus the plane \( \pi \) is \((P\omega=P, l\omega=l)\)-transitive. Therefore the plane \( \pi \) is \((P, l\omega=l)\)-transitive by \((1.6)\). Hence the plane \( \pi \) belongs to the plane over a near-field.

Subcase (b). Suppose \( G \) does not fix \( l \cap l_\omega \). Then there exists a collineation \( \delta \) which shifts \( l \cap l_\omega \). If \( \delta \) fixes \( P \), then the plane \( \pi \) is \((P\delta=P, l\delta=l)\)-transitive and so the plane \( \pi \) is \((P, l\delta)\)-transitive by \((1.6)\) where \( l \cap l_\delta \in l_\omega \). In particular the plane \( \pi \) is \((l \cap l_\delta, l_\omega)\)-transitive. By case (i), we are done. Hence \( \delta \) does not fix \( P \). So the plane \( \pi \) is \((P\delta=P, l\delta=l)\)-transitive with \( l \cap l_\delta \in PP\delta=l_\omega \). Thus the plane \( \pi \) is Desarguesian or the plane over the near-field of order 9 or its dual by \((1.7)\).

3. On weak W-planes

**Lemma 3.1.** Let \((\Sigma, G)\) be a weak W-plane. It further \( \Sigma \) is neither a translation plane nor a dual translation plane, then a point \( X \in l_\omega \), which is the center of a non-trivial homology, is also the center of a non-trivial translation and a non-trivial affine elation (elation with affine line as axis).

**Proof.** Let \( \alpha \) be a non-trivial \((X, x)\)-homology where \( x \not\in l_\omega \). Let \( \pi \) be the projective closure of \( \Sigma \). Let \( n \) be the order of the plane \( \pi \).

Case (i) Suppose there is an affine line \( l \) through \( x \), which is not the axis of any involutory homology. Let \( \Omega_1 \) be the set of all affine points \( Q \) on \( l \) such that there exists an involutory \((Q, l_\omega)\)-homology and let \( \Omega_2 \) be the set of all affine points \( Q \) on \( l \) such that \( Q \in \Omega_2 \).

Suppose \( \Omega_1 = \emptyset \). Then for every \( Q_i \in l \), \( Q_i \not\in x \), the plane \( \pi \) has an involutory \((x, l_i)\)-homology \( \alpha_i \) for some affine line \( l_i \) through \( Q_i \), \( i = 1 \) to \( n \) and \( |\Omega_2| = n \). If any two of axes of all such involutory homologies, namely \( l_i \)'s intercept at a point on \( l_\omega \), then the group of all \((X, l_\omega)\)-elations is transitive on the affine points on \( l \) by dual of \((1.2)\). Hence the plane \( \Sigma \) is \((X, l_\omega)\)-transitive and hence \( \Sigma \) is a translation plane or a dual translation plane by \((2.3)\) which contradicts our assumption. It follows that there exists a \( Z_1 \) and \( Z_2 \in \Omega_2 \) such that \( \pi \) admits an involutory \((X, q_1)\)-homology and an involutory \((X, q_2)\)-homology with \( Z_1 \in q_1, Z_2 \in q_2, q_1 \cap q_2 \in l_\omega \). Let \((q_1 \cap q_2)X = q \). Now if \( l_i \cap l_j \in l_\omega \) for some \( i, j \in \{1, 2, \ldots, n\} \), then we are done by dual of \((1.2)\). So, we may assume that \( l_i \cap l_j \in l_\omega \) for every \( i, j \in \{1, 2, \ldots, n\} \). If for every \( i, j \in \{1, 2, \ldots, n\} \) \( l_i \cap l_j \in q \), then \( \Sigma \) is \((X, q)\)-transitive, and so we are done by \((2.3)\). Thus we may assume \( l_i \cap l_j \in q \) for some \( i, j \in \{1, 2, \ldots, n\} \). In this case, \( G \) has an affine elation \( \beta \) whose axis \( \not\in q \). Therefore \( |G(X, q)| = p^a \) for some prime \( p \) dividing \( n \) and for some integer \( a \geq 1 \) by dual of \((1.1)\). Let \( \Omega \) be the set of lines among \( l_i \)'s such that
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they are concurrent at \(q_1 \cap q_2\). Clearly \(|\Omega| = p^4\). If \(l_i\)'s meet \(q\) at distinct points, let identity \(\equiv \beta_0 \in G(X, q)\). Then \(\beta_0^{-1} \alpha_i \beta_0\) is an involutory \((X, l_i \beta_0)\)-homology. Note that \(I_i \beta_0 = l_j, j \in \{1, 2, \ldots, n\}\). So we may apply the dual of (1.2), and (1.11), and conclude that there exists a non-trivial \((X, l_i)\)-elation.

Hence we may assume that there exists an affine point \(Z \in q\), and \(Z \equiv q \cap l_i, i \in \{1, 2, \ldots, n\}\). We consider the flag \((Z, q)\). Let \(\alpha_0\) be the involutory homology fixing the flag \((Z, q)\). Then it is clear that \(\alpha_0 = \alpha_i\) for any \(i\). If \(X\) is the center of \(\alpha_0\), then the axis of \(\alpha_0\) passes through \(Z\), and hence is different from any \(l_i\). So we may apply the dual of (1.2), and (1.11). If \(\alpha_0\) is an involutory \((Z, l_\omega)\)-homology, then \(\beta^{-1} \alpha_0 \beta\) is also an involutory \((Z, \beta + Z, l_\omega \beta = l_\omega)\)-homology. So \(|G(X, l_\omega)| > 1\) by (1.2). Finally suppose \(\alpha_0\) is an involutory \((D, q)\)-homology where \(D \in l_\omega\). If \(D \in l_i\) for some \(i\) where \(l_i\) is an element of \(\Omega\), then \(\alpha_i \alpha_i\) is an involutory \((q \cap l_i, DX = l_\omega)\)-homology. Then \(\beta^{-1} \alpha_0 \alpha_i \beta\) is also an involutory \(((q \cap l_i) \beta + f \cap l_\omega, l_\omega)\)-homology. Thus \(G\) contains a non-trivial \((X, l_\omega)\)-elation.

Therefore we may assume \(D \in l_i\) for any \(l_i \in \Omega\). Here we observe that fixes \(\Omega\) setwise and note that \(\alpha\) does not fix any element in \(\Omega\). Therefore 2 divides \(|\Omega| = p^4\). This implies \(2 = p\), \(p\) divides \(n\) implies that 2 divides \(n\), a contradiction.

Hence \(|\Omega| > 1\) by (2.1). We observe that \(|G(X, l_\omega)| > 1\) by (1.2). If \(\Omega_2 = 0\), then \(|\Omega_2| = n\) and for every point \(P\) on \(l\), there exists an involutory \((P, l_\omega)\)-homology. By (1.2), the group of all \((X, l_\omega)\)-elations is transitive on the affine points of \(l\). So the plane \(\pi\) is \((X, l_\omega)\)-transitive which contradicts our assumption by (2.3). So \(|\Omega_2| > 1\). Also for every point \(Q \in \Omega_2\), there exists an involutory \((X, q)\)-homology with \(Q \in g\). If any two axes of all such involutory homologies intercept at a point on \(l_\omega\), then the group of all \((X, l_\omega)\)-elations divides the line \(l\) into exactly two orbits, namely \(\Omega_1\) and \(\Omega_2\) by (1.2) and its dual. Following the same method as (2.2), we observe that \(n\) is even. But the plane \(\pi\) has involutory homologies and so \(n\) has to be odd. Hence this case cannot arise.

So, there exists \(Q_1\) and \(Q_2 \in \Omega_2\) such that \(\pi\) admits an involutory \((X, q_1)\)-homology and an involutory \((X, q_2)\)-homology with \(Q_1 \in q_1, Q_2 \in q_2\) and \(q_1 \cap q_2 \subseteq l_\omega\). By dual of (1.2), there exists a non-trivial \((X, q_1 \cap q_2)\)-elation which proves our assertion in this case.

Case (ii) Let us assume that every affine line through \(X\) is the axis of an involutory homology. If the center of all such involutory homologies is the point \(x \cap l_\omega\), then the plane \(\pi\) is \((x \cap l_\omega, l_\omega)\)-transitive by dual of (1.2) which cannot happen by (2.3). Therefore \(\pi\) has an involutory \((D, z)\)-homology \(\gamma\) with \(D \neq x \cap l_\omega\) and \(X \in z\). Then \(\alpha^{-1} \gamma \alpha\) is an involutory \((D \alpha = D, z \alpha = z)\)-homology. Then the group \(<\alpha, \alpha^{-1} \gamma \alpha>\) contains a non-trivial \((X, z)\)-elation, say \(\beta\) by a dual of (1.2). Then let \(q\) be an affine line through \(X\). Then by our assumption, \(\pi\) has an involutory \((D, q)\)-homology \(\delta\) for some point \(X \in D_1 \in l_\omega\). Thus the group \(<\delta, \beta^{-1} \delta \beta>\) has a non-trivial \((X, q)\)-elation by dual of (1.2). Since \(q\) is an
arbitrary affine line through $X$, we observe that $|G_{(X,X)}| > n$. By (1.11), $|G_{(X,t_0)}| > 1$.

**Corollary 3.2.** Let $(Σ, G)$ be a weak $W$-plane and $π=Σ∪l_∞$ the projective closure of $Σ$. Then there exists a point on $l_∞$ which is not the center of any involutory homology if $Σ$ is neither a translation plane nor a dual translation plane.

**Proof.** Suppose $π$ is neither a translation plane nor a dual translation plane.

Suppose every point on $l_∞$ is the center of an involutory homology. By (3.1), for every point $X$ on $l_∞$, we observe that $|G_{(X,l∞)}| > 1$ and $|G_{(X,l)}| > 1$ for at least one affine line $l$ through $X$. By (1.1), $G_{(t_0,l_∞)}$ is an elementary abelian $p$-group for some prime number $p$ dividing $n$ and for every point $X$ on $l_∞$, $|G_{(X,X)}| > 1$ is an elementary abelian $p$-group by dual of (1.1). Let $X$ be a point on $l_∞$. By (3.1), there exists a non-trivial $(X, l)$-relation $β_1$ of order $p$ and the collineation $β_1$ fixes the point $X$ and acts on other points on $l_∞$. Since $X$ is an arbitrary point on $l_∞$, the collineation group $G$ is transitive on $l_∞$ by (1.5). Also $|G_{(X,l∞)}| > 1$ for every point $X$ on $l_∞$. Therefore $|G_{(X,l∞)}| > 1$ is independent of $X$. Then $|G_{(X,l_∞)}|=h>1$ for all points $X$ on $l_∞$. By (1.3), $l_∞$ is a translation line which contradicts our hypothesis. Hence our assertion stands proved.

**Corollary 3.3.** Any weak $W$-plane, which contains an involutory homology with an affine center, is a translation plane or dual translation plane.

**Proof.** Let $(Σ, G)$ be a weak $W$-plane and $π=Σ∪l_∞$ the projective closure of $Σ$.

Let $α$ be an involutory $(Z, l_∞)$-homology. We prove that the plane $Σ$ has to be a translation plane or a dual translation plane.

Let $Ω_1$ be the set of all points $X$ on $l_∞$ such that $|G_{(X,l_∞)}| > 1$ and $|G_{(X,l)}| > 1$ for at least one affine line $l$ through $X$ and let $Ω_2$ be the set of all points $Y$ on $l_∞$ such that $Y ∉ Ω_1$. We claim that $Ω_2=∅$.

**Suppose** $Y∈Ω_2$. Consider the line $YZ$. If $YZ$ is not the axis of any involutory homology, then the involutory homology $δ$ fixing the affine flag $(Q, YZ)$ is an involutory $(Q, l_∞)$-homology. Otherwise, $Y$ is the center of $δ$ or $δ$ is an involutory $(D, YZ)$-homology. In the second case, $αδ$ is an involutory $(Y, DZ)$-homology by (1.4). So in each case, $Y$ is the center of some involutory homology. Thus, by (3.1), $Y∈Ω_1$ which contradicts our assumption. It follows that for every affine point $Q∈YZ$ there exists an involutory $(Q, l_∞)$-homology. By (1.2), the group of all $(Y, l∞)$ ellations is transitive on the affine points of $YZ$. Hence the plane $π$ is $(Y, l∞)$-transitive from which our assertion follows by (2.3).

**Hence** $Ω_2=∅$ and so $|Ω_1| = n$. Following the same proof as in (3.2), we see that $l_∞$ is a translation line these by proving our theorem.
Theorem 3.4. Under the assumption of (3.1), the collineation group G of Σ is transitive on the affine points of Σ.

Proof. Let Ω₁ be the set of all points X on lₙ such that X is the center of an involutory homology and let Ω₁ be the set of all points Y on lₙ such that Y ∈ Ω₁.

We observe that for every point X ∈ Ω₁, |G(X, lₙ)| > 1 and |G(X, lₙ)| > 1 for at least one affine line l through X by (3.1). If Ω₂ = ∅, by the same argument as in (3.1), we observe that lₙ is a translation line which contradicts our assumption. Thus |Ω₂| > 1 by (2.2). Let Y ∈ Ω₂. Let m be an affine line through Y and let M be an affine point on m. Let α be an involutory homology fixing the flag (M, m). By (3.3), M is not the center of α. Since Y ∈ Ω₂, Y is not the center of α. Therefore m is the axis of α. Also since m is an arbitrary affine line through Y, every affine line through Y is the axis of an involutory homology. We observe that α fixes Ω₂ setwise. Further the collineation α also fixes Y in Ω₂ and fixes no other point in Ω₂. So 2 divides |Ω₂|.

Clearly Ω₁ ≠ ∅ and so |Ω₁| > 1 by (2.2). Let X ∈ Ω₁. Let l be an affine line through X and T be an affine point on l. Let γ be an involutory homology fixing the flag (T, l). By (3.3), the point T is not the center of γ. If the axis of γ is the line l, then the center of γ is a point in Ω₁ by our assumption. We observe the collineation γ fixes Ω₂ setwise and fixes no point in Ω₂. So 2 divides |Ω₂| which contradicts to 2 divides (|Ω₂| − 1). Hence l ∩ lₙ = X is the center of γ and the axis of γ is an affine line lₙ through T and further lₙ ∩ lₙ ∈ Ω₂. Since T is an arbitrary point on l, for every affine point Z ∈ l, there exists an involutory (X, z)-homology where z is an affine line through Z. Let Z₁ and Z₂ ∈ l. Then there exists an involutory (X, z₁)-homology and an involutory (X, z₂)-homology with Z₁ ∈ z₁ and Z₂ ∈ z₂. Then there exists a non-trivial (X, (z₁ ∩ z₂)X)-elation mapping z₁ onto z₂ (hence mapping Z₁ onto Z₂) by dual of (1.2). Thus the group G(X, z₁) is transitive on the affine points of l. Since l is an arbitrary affine line through X, the group G(X, x) is transitive on the affine points of l for every affine line l through X. The above assertion follows for every point X on Ω₁. Since |Ω₁| > 1, there exists X₁, X₂ ∈ Ω₁ and X₁ ≠ X₂. We clear that <G(X₁, x₁), G(X₂, x₂)> is transitive on the affine points of Σ.

We now come to our main theorem.

Theorem 3.5. A weak W-plane is either a translation plane or a dual translation plane of characteristic ≠ 2.

Proof. Let (Σ, G) be a weak W-plane. Let π = Σ ∪ lₙ the projective closure of Σ.

Suppose the plane π is neither a translation plane nor a dual translation plane. Then the group G is transitive on the affine points of Σ by (3.4). Let Ω₁ and
\( \Omega_2 \) be subsets of points on \( L_u \), defined as in (3.4). Also we observe that for every point \( X \in \Omega_1 \), \( |G_{(x,l)}| > 1 \) for at least one affine line \( l \) through \( X \). By (1.9), for every point \( U \in L_v \), the group \( G_U \) operates transitively on the affine lines of \( \Sigma \) through \( U \). Therefore for every point \( X \in \Omega_1 \), \( |G_{(x,l)}| = h > 1 \) for all affine line \( l \) through \( X \). So the plane \( \pi \) is \((X, L_u)\)-transitive by (1.10) for every point \( X \in \Omega_1 \). By (2.3), the plane \( \pi \) is either a translation plane or a dual translation plane which contradicts our assumption.

The assertion about characteristic easily follows and so it stands proved.

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