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Osaka University
A DECOMPOSITION OF $BP\langle 2 \rangle$ AND $v_1$-TORSION

DOMINIQUE ARLETTAZ AND JOHN KLIPPENSTEIN

(Received March 5, 1992)

Introduction

For any $p$-local connective spectrum $F$ (with $p$ a prime number), the first author discovered in [2] integers $\rho_j$ and maps $F \to \Sigma^{j+1} H(\pi_j F, 0)$ between $F$ and Eilenberg-MacLane spectra such that the compositions

$$F \to \Sigma^{j+1} H(\pi_j F, 0) \xrightarrow{p^s \rho_j} \Sigma^{j+1} H(\pi_j F, 0)$$

are trivial. This enabled him to prove that in the Atiyah-Hirzebruch-Dold spectral sequence for the $F$-homology of any bounded below spectrum, $p^s d_{i+1}^j = 0$ for all $j \geq 1$, $s$ and $t$. Now, let us consider the Brown-Peterson spectrum $BP$ with $BP_* = \mathbb{Z}_p[v_1, v_2, \ldots]$, where the degree of $v_k$ is $|v_k| = 2(p^k - 1)$ for $k \geq 1$, and denote as usual by $BP\langle m \rangle$ the spectrum such that $BP\langle m \rangle_* \approx \mathbb{Z}_p[v_1, v_2, \ldots, v_m]$ for any $m \geq 1$. This paper exploits a similar composite

$$BP\langle 2 \rangle/(v_1^2) \to \Sigma^{j|v_1^{j+1}} BP\langle 1 \rangle \xrightarrow{v_1^{j+1} \rho_{j+1}} \Sigma^{-|v_1^{j+1}} BP\langle 1 \rangle$$

which, as a consequence of calculations by the second author in [6] can be seen to be trivial. As a result, we can construct maps

$$f_j: BP\langle 2 \rangle \to \Sigma^{-|v_1^{j+1}} BP\langle 1 \rangle,$$

for all $j \geq 1$, which we control on the homotopy level (see Theorem 2.1). These maps induce maps between the Atiyah-Hirzebruch-Dold spectral sequences for $BP\langle 2 \rangle$ and $BP\langle 1 \rangle$-homology respectively which provide information about the differentials in the Atiyah-Hirzebruch-Dold spectral sequence for $BP\langle 2 \rangle$ (see Theorem 3.3). On the other hand, the triviality of the above composition implies torsion results on the differentials in a modified Bockstein spectral sequence for $BP\langle 2 \rangle$ analogous to the $BP$ Bockstein spectral sequence of Johnson and Wilson [5] (see Theorem 4.5).

In order to illustrate how this new information might be used in calculation,
we have considered the problem of constructing \( M(p', v_1, v_2, \ldots, v_n) \). \( M(p') \) is the Moore spectrum having all integral homology groups trivial except for \( \mathbb{Z}/p_1 \) in dimension 0. Inductively, \( M(p', v_1, v_2, \ldots, v_n) \) is the cofiber of a map \( \Sigma^{v_1+\cdots+v_n} \to M(p', v_1, v_2, \ldots, v_n)' \) which induces multiplication by \( v_n \) in BP-homology. This means that \( BP_\ast(M(p', v_1, v_2, \ldots, v_n)) \cong BP_\ast((p', v_1, v_2, \ldots, v_n)) \). The information about differentials in the spectral sequences mentioned in the previous paragraph can be used to obtain lower bounds on \( j_n \) in terms of \( i \), and \( j_2 \) in terms of \( j_1 \), if \( M(p', v_1, v_2, \ldots, v_n) \) exists.

It should be noted that better bounds than ours can be obtained from ideas in Ravenel’s book [8] (see Remark 3.6; see also the conditions on \( j_1 \) and \( j_2 \) established by Lin in [7]). Thus the main theorems of the present paper are Theorems 3.3 and 4.1 giving information about the differentials in the different spectral sequences. The results about the constructibility of \( M(p', v_1, v_2, \ldots, v_n) \) are given simply to illustrate the use of these theorems. We hope that someone else will be able to use this information in a more novel way.

The paper is organized as follows. In the first section we show how to use the first author’s results in [2] to derive the condition that if \( M(p', v_1, v_2, \ldots, v_n) \) is constructible then \( j_n \geq i \). In the second section we introduce a general construction of maps from \( BP\langle m \rangle \) to \( BP\langle m-1 \rangle \). We apply it in Section 3 to the case \( m=2 \) in order to get the interesting maps between the Atiyah-Hirzebruch-Dold spectral sequences for \( BP\langle 2 \rangle \) and \( BP\langle 1 \rangle \) respectively; we illustrate the use of these maps by proving that if \( M(p', v_1, v_2) \) is constructible, then \( j_2 > j_1 \). Finally, we discuss in the fourth section the set up and general use of the modified Bockstein spectral sequence for \( BP\langle 2 \rangle \) and obtain torsion results on its differentials.

1. The Atiyah-Hirzebruch-Dold spectral sequence for \( C(m) \)

The existence of universal bounds for the additive order of the differentials in the Atiyah-Hirzebruch-Dold spectral sequence was deduced in [2] from torsion results on the Postnikov \( k \)-invariants of spectra which produced maps between the Atiyah-Hirzebruch-Dold spectral sequence and a spectral sequence having only one-non-trivial line in its \( E^2 \)-term.

The purpose of this first section is to show that these results have direct consequences as soon as we concentrate our attention on specific examples. But our main goal is to motivate and illustrate the basic ideas of the method we shall develop in the next sections.

If \( m \) is a positive integer, let \( C(m) \) be the spectrum such that

\[
\pi_\ast C(m) \cong BP_\ast/(v_1, v_2, \ldots, v_{m-1}, v_{m+1}, v_{m+2}, \ldots) \cong \mathbb{Z}[v_m]
\]

(see Yagita’s [9] for the existence of \( C(m) \)). The non-trivial Postnikov \( k \)-invari-
ants of $C(m)$ are $k^{h|v_m|+1}(C(m)) \subseteq H^{h|v_m|+1}(C(m)[h|v_m| - 1]); Z(\rho)$ for $h \geq 1$, where we write $C(m)[j]$ for the $j$-th Postnikov section of $C(m)$. If we apply the method of Theorem 1.4 of [2] to the spectrum $C(m)$, we get:

**Lemma 1.1.** For any $h \geq 1$, the $k$-invariant $k^{h|v_m|+1}(C(m))$ has order dividing $p^h$.

Proof. The spectrum $C(m)[h|v_m| - 1]$ has non-trivial homotopy groups (which are isomorphic to $Z(\rho)$) only in dimensions 0, $|v_m|$, 2, $|v_m|$, ..., $(h-1)|v_m|$. Notice in particular that $C(m)[h|v_m| - 1] = C(m)[(h-1)|v_m|]$. Therefore, we can consider the cofibrations of spectra (where $H(G, n)$ denotes the Eilenberg-MacLane spectrum having all homotopy groups trivial except for $G$ in dimension $n$)

$$C(m)[d|v_m|] \to C(m)[(d-1)|v_m|] \to H(Z(\rho), d|v_m| + 1)$$

and the corresponding long exact homology sequences

$$\cdots \to H_{k|v_m|+2}H(Z(\rho), d|v_m| + 1) \to H_{k|v_m|+1}C(m)[d|v_m|] \to H_{k|v_m|+1}C(m)[(d-1)|v_m|] \to \cdots$$

for $d = 1, 2, \ldots, h-1$. According to Cartan’s [4], $H_{k|v_m|+2}H(Z(\rho), d|v_m| + 1)$ is a direct sum of copies of $Z/p$, as is $H_{k|v_m|+1}C(m)[0]$. By induction, it is then clear that $p^h H_{k|v_m|+1}C(m)[h|v_m| - 1] = 0$, and analogously that $p^h H_{k|v_m|}C(m)[h|v_m| - 1] = 0$. Finally, the universal coefficient theorem implies that the exponent of the cohomology group $H^{k|v_m|+1}(C(m)[h|v_m| - 1]; Z(\rho))$ divides $p^h$ and the proof is complete.

Now, consider the Atiyah-Hirzebruch-Dold spectral sequence for $C(m)$

$$E_{s,t}^{2} \cong H_{s}(X, \pi_t(C(m))) \cong H_{s}X \otimes \pi_tC(m) \Rightarrow C(m)_{s+t}(X),$$

where $X$ is any bounded below spectrum. The non-trivial differentials in this spectral sequence are $d^{h|v_m|+1}$ for $h \geq 1$.

**Corollary 1.2.** In the Atiyah-Hirzebruch-Dold spectral sequence for $C(m)$, the differentials satisfy

$$p^h d^{h|v_m|+1}_{s,t} = 0 \quad \text{for any } h \geq 1, s \text{ and } t.$$
isomorphism

\[ C(m) \otimes M(p', v^1, v^2, \ldots, v^m) \cong \mathbb{Z}/p'[v^m]/(v^m) \otimes \Lambda(w_1, w_2, \ldots, w_{m-1}, w_{m+1}, \ldots, w_n), \]

where \( \Lambda \) denotes an exterior algebra over \( \mathbb{Z} \) and \(|w_k| = |v_k| + 1\). On the other hand, we can calculate \( C(m) \otimes M(p', v^1, v^2, \ldots, v^m) \) via the Atiyah-Hirzebruch-Dold spectral sequence for \( C(m) \). For this, we first need to know the integral homology of \( M(p', v^1, v^2, \ldots, v^m) \): it is not hard to check that, additively,

\[ H_\ast M(p', v^1, v^2, \ldots, v^m) \cong \mathbb{Z}/p' \cdot a_0 \otimes \Lambda(a_1, a_2, \ldots, a_n), \]

where \(|a_0| = 0\) and \(|a_k| = |v_k| + 1\) for \( k = 1, 2, \ldots, n \). But in order to get the right answer for \( C(m) \otimes M(p', v^1, v^2, \ldots, v^m) \), there must be a differential which kills \( v^m \); more precisely, the differential \( d_j \) must verify

\[ d_j : \mathbb{Z}/p'[v^m] \rightarrow \mathbb{Z}/p'[v^i] , \]

where \( \lambda \) is a generator of \( \mathbb{Z}/p' \). Now, it follows from Corollary 1.2 that

\[ p' \cdot \lambda \cdot a_0 \otimes v^m = 0 \]

and consequently that \( p' \cdot \lambda \) must vanish in \( \mathbb{Z}/p' \). This implies the following assertion:

**If the spectrum** \( M(p', v^1, v^2, \ldots, v^m) \) **is constructible, then** \( j \leq i \) **for** \( m = 1, 2, \ldots, n. \)

One should say that this result is not very strong (see Remark 3.6 and notice that in the special case of \( M(p', v^1) \) for an odd prime number \( p \), the exact answer to the constructibility may be deduced from Theorem 12.1 of Adams' [1]: \( M(p', v^1) \) is constructible if and only if \( j_1 \) is divisible by \( p' - 1 \), but it is given here as an example. However, our argument produces the following more general statement: if \( X \) is a connective spectrum and \( m \) an integer such that \( C(m) \otimes X \cong \mathbb{Z}/p'[v^m]/(v^m) \otimes A \) for some \( A \) (as \( C(m) \)-modules) and \( H_\ast X \cong \mathbb{Z}/p' \otimes \Lambda(a) \otimes A \), where \(|a| = |v_m| + 1\), then \( j \geq i. \)

**2. A decomposition of** \( BP<m> \)

For every integer \( m \geq 2 \), let \( BP<m> \) denote as usual the spectrum with \( BP<m> \cong \mathbb{Z}_p[v_1, v_2, \ldots, v_m] \). Now, for \( j \) a positive integer, consider the cofibration

\[ \Sigma^{iv_m} BP<m> \rightarrow BP<m> \sigma_j \rightarrow BP<m>/(v^j), \]

where the first arrow indicates a map inducing multiplication by \( v^j \) on homotopy and where the spectrum \( BP<m>/(v^j) \) is such that \( BP<m>/(v^j) \cong \mathbb{Z} \).
This section describes a decomposition of $BP\langle m \rangle$ in terms of the spectra $BP\langle m \rangle/(v_m^*)$ for $j=1, 2, \ldots$. Look at the following commutative diagram

\[
\begin{array}{ccc}
\Sigma^{[j+1]l_{m+1}}BP\langle m \rangle & \xrightarrow{v_m^{j+1}} & \Sigma^{l_{m+1}}BP\langle m \rangle \\
\downarrow v_m^j & = & \downarrow v_m^j \\
BP\langle m \rangle & \xrightarrow{\sigma_j} & BP\langle m \rangle \\
\downarrow & \downarrow & \downarrow \\
BP\langle m \rangle/(v_m^{j+1}) & \xrightarrow{g_j} & BP\langle m \rangle/(v_m^j) \\
\downarrow l_j & & \downarrow & \\
& & \Sigma^{l_{m+1}}BP\langle m \rangle,
\end{array}
\]

in which rows and columns are cofibrations of spectra, the map $g_j$ is determined by the top left square and $l_j$ is the cofiber of $g_j$. These maps $l_j$ explain how to build the $BP\langle m \rangle/(v_m^{j+1})$'s using $BP\langle m-1 \rangle$ as the building blocks (instead of the Eilenberg-MacLane spectra in a Postnikov tower). Notice that $l_j$ is actually an element of $BP\langle m-1 \rangle/(v_m^{j+1}BP\langle m \rangle/(v_m^j))$. Especially interesting is the next result which describes connections between $BP\langle m \rangle$ and $BP\langle m-1 \rangle$.

**Theorem 2.1.** Let $j$ be any positive integer and assume that there is an integer $e_j$ such that $v_m^{e_j-1}BP\langle m-1 \rangle/(v_m^{j+1})=0$, then there exists a map

\[ f_j : BP\langle m \rangle \to \Sigma^{l_{m+1}e_j}BP\langle m-1 \rangle \]

with the property that the homomorphism $(f_j)_*$ induced by $f_j$ on homotopy acts on an element $v_m^{e_j}v_m^{e_2} \cdots v_m^{e_k} \in BP\langle m \rangle_*$ as follows:

\[ (f_j)_*(v_m^{e_j}v_m^{e_2} \cdots v_m^{e_k}) = \begin{cases} 
  v_m^{e_j}v_m^{e_2} \cdots v_m^{e_k} & \text{if } k=0, \\
  0 & \text{if } k>0.
\end{cases} \]

**Proof.** First, if we compose the last two columns of the above diagram with maps inducing multiplication by $v_m^{e_j}$ in homotopy, we obtain the following diagram in which only the columns are cofibrations:

\[
\begin{array}{ccc}
\Sigma^{l_{m+1}}BP\langle m \rangle & \xrightarrow{v_m^{e_j}} & \Sigma^{l_{m+1}}BP\langle m-1 \rangle \\
\downarrow v_m^j & & \downarrow v_m^j \\
BP\langle m \rangle & \xrightarrow{\sigma_j} & BP\langle m \rangle \\
\downarrow & \downarrow & \downarrow \\
BP\langle m \rangle/(v_m^j) & \xrightarrow{l_j} & \Sigma^{l_{m+1}}BP\langle m-1 \rangle \\
\downarrow & \downarrow & \downarrow \\
& & \Sigma^{l_{m+1}-e_j}BP\langle m-1 \rangle.
\end{array}
\]
If the hypothesis of the proposition is verified, the bottom composition $\psi_{m-1}^{*}l_{j}$ is trivial and we get a map

$$f_{j}: BP\langle m \rangle \rightarrow \Sigma^{j_{m-1}}BP\langle m-1 \rangle$$

such that $f_{j} \circ \psi_{m}^{*} = \psi_{m-1}^{*} \circ \Sigma^{j_{m-1}} \sigma_{1}$. The homomorphism $\psi_{m-1}^{*}((\Sigma^{j_{m-1}} \sigma_{1})_{*})$ induced by $\psi_{m-1}^{*} \circ \Sigma^{j_{m-1}} \sigma_{1}$ on homotopy satisfies:

$$\psi_{m-1}^{*}((\Sigma^{j_{m-1}} \sigma_{1})_{*})((\psi_{i}^{*}, \psi_{z}^{*}, \ldots, \psi_{m}^{*}^{k})) = \begin{cases} \psi_{i}^{*} \psi_{z}^{*} \ldots \psi_{m}^{*-1} \psi_{j}^{*}, & \text{if } j+k=0, \\ 0 & \text{if } j+k>0. \end{cases}$$

Consequently,

$$(f_{j})_{*}((\psi_{i}^{*}, \psi_{z}^{*}, \ldots, \psi_{m}^{*}^{k})) = \begin{cases} \psi_{i}^{*} \psi_{z}^{*} \ldots \psi_{m}^{*-1} \psi_{j}^{*}, & \text{if } k=0, \\ 0 & \text{if } k>0. \end{cases}$$

**Remark 2.2.** In the case when $m$ is 1, the fact that $p^{1} HZ_{(p)}^{j_{m+1}+1}(BP\langle 1 \rangle)(\psi_{j})=0$ (follow the argument of the proof of Lemma 1.1) makes it possible to use the decomposition of $BP\langle 1 \rangle$ and obtain maps $f_{j}: BP\langle 1 \rangle \rightarrow \Sigma^{j_{m-1}} HZ_{(p)}$, for $j \geq 1$, such that

$$(f_{j})_{*}(\psi_{l}^{*}) = \begin{cases} p^{1} & \text{if } k=0, \\ 0 & \text{if } k>0. \end{cases}$$

3. Maps between the Atiyah-Hirzebruch-Dold spectral sequences for $BP\langle 2 \rangle$ and $BP\langle 1 \rangle$

Now, let us consider the decomposition explained in the previous section in the case $m=2$ and show that the hypothesis of Theorem 2.1 is verified for any positive integer $j$.

**Proposition 3.1.** For every positive integer $j$, the $\psi_{1}$-torsion-free part of $BP\langle 1 \rangle^{*}(BP\langle 2 \rangle/(\psi_{j}))$ is concentrated in even degrees and the $\psi_{1}$-torsion part of $BP\langle 1 \rangle^{*}(BP\langle 2 \rangle/(\psi_{j}))$ is concentrated in positive degrees.

Proof. Corollary 10 of [6] implies that the assertion holds for $BP\langle 1 \rangle^{*} BP\langle 1 \rangle$. Then, use inductively the long exact sequences in $BP\langle 1 \rangle$-cohomology associated with the cofibrations

$$BP\langle 2 \rangle/(\psi_{j}+1) \xrightarrow{g_{j}} BP\langle 2 \rangle/(\psi_{j}) \xrightarrow{l_{j}} \Sigma^{j_{m+1}} BP\langle 1 \rangle$$

given by the bottom sequence in the first diagram of Section 2, for $j=1, 2, \ldots$ (and recall that $BP\langle 2 \rangle/(\psi_{1})=BP\langle 1 \rangle$):

$$\cdots \rightarrow BP\langle 1 \rangle^{*}(BP\langle 2 \rangle/(\psi_{1})) \rightarrow BP\langle 1 \rangle^{*}(BP\langle 2 \rangle/(\psi_{j}+1)) \rightarrow BP\langle 1 \rangle^{*} BP\langle 1 \rangle \rightarrow \cdots.$$
This produces the statement of the proposition for $BP\langle 1 \rangle^* (BP\langle 2 \rangle/(v_2^j))$ for all $j \geq 1$.

**Corollary 3.2.** For all positive integers $k$ and $j$, $v_t^* \cdot BP\langle 1 \rangle^{2k+1} (BP\langle 2 \rangle/(v_2^j)) = 0$ if $e_{k,j}$ is an integer $> \frac{2k+1}{|v_1|}$. In particular, $v_t^* \cdot BP\langle 1 \rangle^{l+t+1} (BP\langle 2 \rangle/(v_2^j)) = 0$ for each integer $e_{t_j} > (p+1)j$.

Proof. An element $x \in BP\langle 1 \rangle^{2k+1} (BP\langle 2 \rangle/(v_2^j))$ must be $v_1$-torsion because it is in odd degree and $v_t^* \cdot x = 0$ since its degree is negative.

This assertion enables us to apply Theorem 2.1 for $m=2$ and $e_{t_j} = (p+1)j + 1$: it produces maps

$$f_j : BP\langle 2 \rangle \to \sum_{i \geq 1} BP\langle 1 \rangle$$

for all positive integers $j$. For any bounded below spectrum $X$, let us look at the Atiyah-Hirzebruch-Dold spectral sequences for $BP\langle 2 \rangle$ and $BP\langle 1 \rangle$-homology

$$E_{s,t}^2 = H_s(X; \pi_t BP\langle 2 \rangle) = H_s X \otimes \pi_t BP\langle 2 \rangle \Rightarrow BP\langle 2 \rangle_{s+t}(X)$$

and

$$\hat{E}_{s,t}^2 = H_s(X; \pi_t BP\langle 1 \rangle) = H_s X \otimes \pi_t BP\langle 1 \rangle \Rightarrow BP\langle 1 \rangle_{s+t}(X)$$

respectively. The $f_j$'s induce homomorphisms $E_{s,t}^2 \to \hat{E}_{s,t+t+1+l}^2$ and hence maps between these spectral sequences: we then obtain immediately the next result.

**Theorem 3.3.** There are maps of spectral sequences

$$(f_j)_* : E_{s,t}^r \to \hat{E}_{s,t+t+l+1}^r, \quad j \geq 1$$

($r \geq 2, t \geq 0$) with the following property: if $\sum_{k \geq 0} \bar{x}_k \otimes v_t^k v_1^{h+k}$ belongs to $E_{s,t}^r$ (where the sum is taken over a finite number of $k$'s, $\bar{x}_k$ is represented by an element $x_k$ of $H_s X$, $h$ is a positive integer and $t = d_s|v_1| + (h+k)|v_2|)$, then

$$(f_j)_* (\sum_{k \geq 0} \bar{x}_k \otimes v_t^k v_1^{h+k}) = 0, \quad \text{if } j < h$$

and

$$(f_h)_* (\sum_{k \geq 0} \bar{x}_k \otimes v_t^k v_1^{h+k}) \text{ is the class of } x_0 \otimes v_t^{h+(p+1)k+1} \text{ in } E_{s,t+t+l+1}^r.$$
equality

\[(f_j)_* \circ d'_{i,k}(\sum_{k \geq 0} E_k \otimes v_i^k \otimes v_i^{k+1}) = 0 \text{ in } \tilde{E}^{i-1}_{r-t+1,r+1}.
\]

**Example 3.5.** We present here a more explicit application of our comparison method. We have seen in Section 1 that if the spectrum \(M(p^i, v^i, v^j, \ldots, v^k)\) is constructible, then \(j_1\) and \(j_2\) are forced to be \(\geq i\). But now, we can prove that there must also exist a relation between \(j_1\) and \(j_2\).

If we compute \(BP<2>_m M(p^i, v^i, v^j)\) via the \(BP<2>_m\)-homology exact sequences associated with the cofibrations defining \(M(p^i, v^i, v^j)\), we find that \(BP<2>_m M(p^i, v^i, v^j) \cong \mathbb{Z}[p^i[v^i, v^j]/(v^i, v^j)].\) On the other hand, if we perform this calculation via the Atiyah-Hirzebruch-Dold spectral sequence

\[E_{s,t}^2 = H_*(M(p^i, v^i, v^j); \pi_t BP<2>); \pi_t BP<2> \Rightarrow BP<2>_s M(p^i, v^i, v^j)\]

(recall that \(H_*(M(p^i, v^i, v^j) \cong \mathbb{Z}[p^i; a_0 \otimes \Lambda(a_1, a_2)] \text{ with } |a_0| = 0 \text{ and } |a_k| = j_k v_k \text{ for } k = 1 \text{ and } 2),\) we observe as in Example 1.3 that

\[d_{j_1|z_1|}^{j_2|z_1|+1,0}(a_2 \otimes 1) = \lambda a_0 \otimes v^j_z,
\]

where \(\lambda\) is a generator of \(\mathbb{Z}/p^i\). Then, take the map \(f_{j_2}: BP<2>_m \to \Sigma^{-1}M BP<1>_m\) and notice that the homomorphism \((f_{j_2})_*\) induced by \(f_{j_2}\) on homotopy maps \(1\) onto a multiple of \(v_i\), say \(\mu v_i\) with \(\mu \in \mathbb{Z}(p^i)\), since \((f_{j_2})_* (1)\) belongs to \(\pi_1 BP<1>_m\). The homomorphism \((f_{j_2})_*: E_{j_1|z_1|+1,0}^{j_2|z_1|+1} \to \tilde{E}_{j_1|z_1|+1,0}^{j_2|z_1|+1}\) given by Theorem 3.3 provides the diagram

\[
\begin{array}{c}
\lambda a_0 \otimes v^j_z \in E_{j_1|z_1|+1,0}^{j_2|z_1|+1} \\
\downarrow d_{j_1|z_1|+1,0}^{j_2|z_1|+1} \\
a_2 \otimes 1 \in E_{j_1|z_1|+1,0}^{j_2|z_1|+1}
\end{array}
\]

Consequently, the commutativity of the diagram shows that \(d_{j_1|z_1|+1,0}^{j_2|z_1|+1}(a_2 \otimes v^j_z)\) is exactly the class of \(\lambda a_0 \otimes v^j_z(1/1)_{j_2}^z + 1\) in \(\tilde{E}_{j_1|z_1|+1,0}^{j_2|z_1|+1}\). But recall that \(BP<1>_m M(p^i, v^i, v^j) \cong \mathbb{Z}[p^i[v^i, v^j]/(v^i, v^j)] \otimes \Lambda(\mu v_i)\) (see Example 1.3). Therefore, there are only two systems of non-trivial differentials in the spectral sequence \(\tilde{E}_{j_1|z_1|}^{j_2|z_1|+1,0}\):
is 0. However, it turns out that the only differential which may kill it is \( d_{i_1|i_2+1} \) starting from \( a_1 \otimes v^{(p+1)}_{j_2+1-j_1} \) (provided \((p+1)j_2+1-j_1 \geq 0\)). This produces the following inequality in order to be in the right \( E_r \):

\[
j_2 \frac{|v_2|+1}{|v_1|} > j_1 \frac{|v_1|+1}{|v_2|},
\]
or in other words,

\[
j_2 > j_1 \frac{|v_1|}{|v_2|} = j_1 \frac{p+1}{p-1}
\]

(observe that the condition \((p+1)j_2+1-j_1 \geq 0\) is then trivially verified). Therefore we have deduced very easily from our general argument the following condition on the constructibility of \( M(p', v{i_1}, v{i_2}) \):

If the spectrum \( M(p', v{i_1}, v{i_2}) \) is constructible, then \( j_1 \geq i, j_1 \geq i \) and

\[
j_2 > j_1 \frac{p+1}{p+1}.
\]

**Remark 3.6.** Stronger results may be deduced from the fact that for any ideal \( I \subset BP_* \), the realizability of \( BP_*/I \) as the \( BP \)-homology of a spectrum implies that \( I \) is an invariant ideal (see pages 138 and 319 of [8]). Thus the full power of \( BP_*=BP \) cooperations can be brought to bear on the problem. Notice also that Lin gives conditions for the realizability of \( M(p', v{i_1}, v{i_2}) \) for the case \( p \geq 5 \) in [7].

**Remark 3.7.** The method presented in the above example provides in general new information on the existence of connective spectra \( X \) such that \( BP<2> \otimes X = \mathbb{Z} \otimes [p[v_1, v_2]]/(v{i_1}, v{i_2}) \otimes A \) for some \( A \).

**Remark 3.8.** In the case when \( m \) is 1, Remark 2.2 allows us to compare the Atiyah-Hirzebruch-Dold spectral sequence for \( BP<1> \)-homology with that for ordinary homology with \( \mathbb{Z}(p) \)-coefficients (whose \( E^2 \)-term has only one non-trivial line).

**Remark 3.9.** Of course, we would like to generalize our argument and obtain maps between the Atiyah-Hirzebruch-Dold spectral sequences for \( BP<m> \) and \( BP<m-1> \) for any \( m \geq 1 \). In particular, if we were able to check the hypothesis of Theorem 2.1, i.e., to show that for any \( m \geq 2 \) some power of \( v_{m-1} \) annihilates \( BP<m-1>/(v_{m-1}) \) for appropriate \( j \)'s, then we could conclude that

\[
j_m > j_{m-1} \frac{v_{m-1}}{|v_m|} = j_{m-1} \frac{p^{m-2} + p^{m-3} + \cdots + p+1}{p^{m-1} + p^{m-2} + \cdots + p+1}
\]

for \( 2 \leq m \leq n \) in a constructible \( M(p', v{i_1}, v{i_2}, \ldots, v{i_n}) \).
4. A modified Bockstein spectral sequence for \( BP<2> \)

In Proposition 5.14 of [5], Johnson and Wilson introduce a spectral sequence coming from the \( BP \) Bockstein cofibration sequence

\[
\Sigma_{l}^{r+1} BP \langle n \rangle \to BP \langle n \rangle \to BP \langle n-1 \rangle.
\]

A similar spectral sequence can be obtained by gluing together the cofibrations in the bottom row of the first diagram in Section 2 to get the following tower of cofibrations.

The diagonal maps, \( \Delta_s \), are the cofibers of the \( l'_s \)'s and have degree one. If we smash everything with a spectrum \( X \) and then take homotopy, we get a spectral sequence with

\[
E^1_{s,t} = (\Sigma_{l}^{r+1} BP \langle 1 \rangle)(X)
\]

\[
d^r_{s,t} : E^r_{s,t} \to E^r_{s+r,t-r-1}
\]

(see Boardman's [3], Section 4). We call this the modified Bockstein spectral sequence for \( BP<2>_X \), because, as we will show later, its \( E^\ast \)-term is analogous to that of the ordinary Bockstein spectral sequence. It has the same \( E^3 \)-term, up to a change of grading, as the spectral sequence of [5]. Note that the inverse limit of the vertical maps in the tower is \( BP<2> \).

Let us analyze the convergence of this spectral sequence. For convenience we will assume that \( X \) is \((-1\)-connected and we set \( M=BP<2>_X \) and write
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for $BP\langle 2 \rangle/\langle v_2^2 \rangle X$, and for any graded module $D$, write $D_t$ to mean the elements of degree $t$ in $D$, so that we have the short exact sequence

$$0 \to (M/v_2^2 \cdot M)_t \to A_t \xrightarrow{p_t} (\text{Ker} \{ v_2^2 : M \to M \})_{t-\text{sgn}t-1} \to 0.$$  

In the terminology of [3], this is a right half plane spectral sequence so that, by Theorem 9.2 of [3], if $\text{Rlim} Z_{\ast, t} = 0$, where $Z_{\ast, t} = \Delta^t \text{(Im} \{ g_{t+1} \cdot \cdots g_{t+r-1} : A_{t+r} \to A_{t+r+1} \})$, this spectral sequence converges strongly to $\Sigma \text{lim} (BP\langle 2 \rangle/\langle v_2 \rangle \cdot X)$, filtered by $F_s = \text{Ker} \{ \text{lim} A_j \to A_s \}$. We have two jobs, show that $\text{Rlim} Z_{\ast, t} = 0$ and determine $F_s/F_{s+1}$, which is of more interest than the group being filtered, as is usual for Bockstein spectral sequences.

The whole strategy for both computations, is to work with the much simpler systems, $C_s = M/v_2^2 \cdot M$ and $B_s = \text{Ker} \{ v_2^2 : M \to M \}$ and then use the short exact sequence of inverse systems

$$0 \to C_{s+1} \xrightarrow{i_{s+1}} A_{s+1} \xrightarrow{p_{s+1}} B_{s+1} \to 0$$

$$0 \to C_s \xrightarrow{i_s} A_s \xrightarrow{p_s} B_s \to 0$$

to derive the results for $A_s$.

**Lemma 4.1.** $\lim \left\langle \left. A_s \right| \text{lim} C_s \right\rangle$.

**Proof.** If $x$ is an element of $B_{s+1}$, then $g''(x) = v_2^s x \in B_s$ because of the definition of $g_s$ given by the first diagram of Section 2. Now use the exact sequence (see 1.8 of [3])

$$0 \to \lim B_s \xrightarrow{d} \Pi B_s \xrightarrow{d} \Pi B_s \xrightarrow{R \lim B_s} 0,$$  

where $(dx)_s = x_s - g''(x_{s+1}) = x_s - v_2^s x_{s+1}$. But since $M_\ast = 0$ for $\ast < 0$, for any $x \in M_t$, there exists an integer $k$ (with $0 \leq k \leq \frac{t}{|v_2^t|}$) and an element $y \in M_{t-\text{sgn}t}$ such that $x = v_2^k y$ and $y \in \text{Im} \{ v_2 \}$. This shows that $d$ is injective and that $\lim B_s = 0$. Finally, from the six term exact sequence

$$0 \to \lim C_s \xrightarrow{i_s} \lim A_s \xrightarrow{p_s} \text{Rlim} C_s \xrightarrow{d} \text{Rlim} A_s \xrightarrow{R \lim B_s} 0,$$

we get the desired isomorphism.

Now we want to identify $F_s/F_{s+1}$. First, define $F'_s = \text{Ker} \{ \lim C_j \to C_s \}$.  

A diagram chase around

\[
\begin{array}{cccccc}
  & & & & & 0 \\
  & & & & \downarrow & \\
 F'_s \hookrightarrow \lim_j C_j & \xrightarrow{\sim} & C_s & \xrightarrow{\sim} & \lim_j A_j & \hookrightarrow A_s \\
  \downarrow & & \downarrow & & \downarrow & \\
 F_s \hookrightarrow \lim_j A_j & \xrightarrow{\sim} & A_s & & & \\
\end{array}
\]

(where the two rows and the last column are exact), shows that \( F_s \cong F'_s \): so it is sufficient to identify \( F'_s/F'_{s+1} \). If \( x \in M \), let us call \( x'_s \) the class of \( x \) in \( C_s \): it is then clear that \( g'_s(x'_{s+1}) = x'_s \). Now from the sequence (*) for \( \lim_j C_j \), we see that \( \lim C_s \) is the usual \( (v_2) \)-adic completion of \( M \). Thus \( F'_s/F'_{s+1} = v_2^{-s}M/v_2^{s+1}M \) and we have proved the following

**Lemma 4.2.** \( F'_s/F'_{s+1} = v_2^{-s}M/v_2^{s+1}M \).

Finally we have to show that the spectral sequence converges. Although we only really need weak convergence, Theorem 9.2 of [3] implies that our filtration makes weak convergence equivalent to strong convergence and that we may verify the convergence of the spectral sequence to \( \varprojlim \Sigma \lim_j (BP \langle 2 \rangle \langle (v_2) \rangle \langle X \rangle) \cong \Sigma BP \langle 2 \rangle \langle X \rangle \) (see Lemma 4.1) by checking that \( \text{Rlim } Z''_{i,t} = 0 \). Here we definitely need to fix a value of \( t \) before taking these limits. Since the maps in the \( B \) system change \( t \), we are going to regrade \( B \), taking \( B_{s,t} \) to mean the image of \( A_{s,t} \), i.e. \( B_{s,t} \) means elements of degree \( t-s \mid v_2 \mid -1 \) annihilated by multiplication by \( v_2^{t} \) so that \( B_{s,t} = 0 \) for \( t \leq s \mid v_2 \mid \). Having fixed \( s \) and \( t \), pick \( r \) to be any integer with \( (r+s)\mid v_2 \mid \geq t+1 \). Now, a non-zero element \( x \) in \( Z'_{i,t} \) corresponds to an element \( x'_{s+r} \) in \( A_{s+r,t} \) with \( y = g_{s+1} \cdots g_{s+r-1}(x'_{s+r}) \not= 0 \) but \( g_s(y) = 0 \). Let \( Z''_{i,t} \) be defined similarly: \( Z''_{i,t} = \Delta^{-1}(\text{Im } (i_{s+1} \cdots g_{s+r-1}) : C_{s+r,t} \to C_{s+r-1,t}) \). For \( x' \not= 0 \in Z''_{i,t} \) we have an element \( x'_{s+r} \) in \( C_{s+r,t} \) with analogous conditions about the action of the \( g_j \)'s: \( y' = i_{s+1} \cdots g_{s+r-1}(x'_{s+r}) \not= 0 \) but \( g_s(y') = 0 \). But since \( B_{s+r,t} = 0 \), we deduce that \( i_{s+r} : C_{s+r,t} \to A_{s+r,t} \) is an isomorphism and it follows from the fact that \( g_j \circ i_{j+1} = i_j \circ g_j \) for all \( j \) that

\[
Z''_{i,t} \cong Z''_{i,t} \text{ for all } r \geq \frac{t+1}{\mid v_2 \mid} - s .
\]

Now, because \( C_{r+1} \) maps surjectively onto \( C_r \), \( Z''_{i,t+1} \) maps onto \( Z''_{i,t} \) so that from (*) we get

**Lemma 4.3.** \( \text{Rlim } Z''_{i,t} \cong \text{Rlim } Z''_{i,t+1} = 0 \).
Finally, what we have proved is

**Theorem 4.4.** The modified Bockstein spectral sequence for $BP\langle 2 \rangle \wedge \mathbb{Z} X$, where $X$ is $(-1)$-connected, converges strongly to $BP\langle 2 \rangle \wedge \mathbb{Z} X$ filtered by $F_{s,t} = \ker \left( \lim_{i \to \infty} \mathbb{Z} P(2)^{L}(v_i) \to BP\langle 2 \rangle^{L}(v_i) X \right)$ with

$$E_{r+1}^{s,t} = F_{s,t}/F_{s+1,t} = v_{i}^{s} \cdot BP\langle 2 \rangle^{L}(v_{i}^{s+1}) \cdot BP\langle 2 \rangle^{L}(v_{i}^{s+1}) X.$$

Being a kind of Bockstein spectral sequence, although it is meant to give us a schema for computing $BP\langle 2 \rangle$-homology from $BP\langle 1 \rangle$-homology, in practice what is of interest are the differentials. Since

$$v^{(p+1)/s+1} \cdot l_{s} = 0$$

(see Corollary 3.2), we get:

**Theorem 4.5.** In the modified Bockstein spectral sequence for $BP\langle 2 \rangle \wedge \mathbb{Z} X$,

$$v^{(p+1)(r+s)+1} \cdot d_{r,s} = 0.$$

Proof. Notice that the horizontal arrows, $l_{s}$, in the tower of cofibrations are elements of the groups which Corollary 3.2 tells us are killed by appropriate powers of $v_{i}$. Since the definition of the differential $d_{r,s}$ involves composing a certain class of maps in $BP\langle 2 \rangle^{L}(v_{i}^{r+s}) X$ with the homomorphism induced by

$$l_{s+r} : BP\langle 2 \rangle^{L}(v_{i}^{r+s}) \to \Sigma^{r+s} BP\langle 1 \rangle,$$

we get the desired result. Notice that the same result holds for the cohomology version of this spectral sequence.

**Example 4.6.** Again we apply this result to the study of the constructibility of $M(p', v_{i}^{r}, v_{i}^{s})$ and not surprisingly get a similar result. First, setting $J = \mathbb{Z} p'_{[v_{i}]}/(v_{i}^{l})$ so that $BP\langle 1 \rangle^{L}M(p', v_{i}^{r}, v_{i}^{s}) = J \oplus \Sigma^{s} p^{s+1} J$, we find that

$$E_{r,s}^{1} = \Sigma^{s} p^{s+1} J \cdot \alpha_{s} \oplus \Sigma^{s} p^{s+1} J \cdot \beta_{s}.$$

Then, for degree reasons, $d_{r,s}^{*}(\alpha_{s}) = 0$ for all $r$ and $s$ as well as $d_{r,s}^{*} = 0$ for all $r > j_{2}$. Since $BP\langle 2 \rangle^{L}M(p', v_{i}^{r}, v_{i}^{s}) = \mathbb{Z} p'_{[v_{i}]}/(v_{i}^{r}, v_{i}^{s})$, we get:

$$E_{r,s}^{*} = v_{i}^{s} \cdot BP\langle 2 \rangle^{L}M(p', v_{i}^{r}, v_{i}^{s})/v_{i}^{s+1} \cdot BP\langle 2 \rangle^{L}M(p', v_{i}^{r}, v_{i}^{s}) = \begin{cases} \Sigma^{s} p^{s+1} J, & \text{if } s < j_{2} \\ 0, & \text{if } s \geq j_{2} \end{cases}.$$

The first case tells us that $d_{r,s}^{*} = 0$ for $s < j_{2}$ (i.e. no multiple of $\alpha_{s}$ is in the image of a differential when $s < j_{2}$), while the second case coupled with the fact that $d_{r,s}^{*} = 0$ for $r > j_{2}$ tells us that $d_{r,s}^{*}(\beta_{s}) = \lambda \alpha_{s}$ where $\lambda$ generates $\mathbb{Z}/p'$. Finally, by Theorem 4.5 we have
Therefore, we must have 
\[(p+1){j_2} + 1 > j_1 - 1\], i.e., \[j_2 > \frac{j_1 - 2}{p+1}\] in a constructible \[M(p', v^i_1, v^i_2)\].

References