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ON HOLOMORPHIC SECTIONS WITH SLOW GROWTH OF HERMITIAN LINE BUNDLES ON CERTAIN KÄHLER MANIFOLDS WITH A POLE

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1. Introduction. We call (M, o) a *Riemannian manifold with a pole* iff M is a Riemannian manifold and $\exp_o: M_o \to M$ is a global diffeomorphism. We write $r(x)$ for the distance function from o. Suppose now our (M, o) satisfies the following condition:

(1–1) There exist C^∞ functions $k, K: [0, \infty) \to [0, \infty)$ such that
   (i) $-k(r(x)) \leq$ all the radial curvature at $x \leq K(r(x))$,
   (ii) $\displaystyle\int_0^\infty tk(t)dt < \infty$,
   (iii) $\displaystyle\int_0^\infty tK(t)dt \leq 1$.

In (i) above, a *radial curvature* at an $x \in M$ denotes the sectional curvature of a 2-dimensional plane in M_x which is tangent to the unique geodesic joining the pole o of M to x (if $x = o$, then simply define a radial curvature to be a sectional curvature at o). R. Greene and H. Wu have studied general properties of Riemannian manifolds with a pole in [1]. Among other things, they have shown that Riemannian manifolds with a pole satisfying condition (1–1) give r se to a very interesting class of Riemannian manifolds. Making use of their results, we shall prove the following theorem.

Theorem 1. *Let (M, o) be an m-dimensional Kähler manifold with a pole satisfying condition (1–1) above ($m \geq 2$). Let $L \to M$ be a holomorphic line bundle over M with a hermitian fibre metric h. Suppose the Chern form $\omega = -(i/2\pi)\partial\bar\partial \log h$ of the hermitian line bundle $\{L, h\}$ satisfies one of the following conditions:*

(1–2) ω *is non positive*,

(1–3) $\|\omega(x)\| \leq v(r(x))$ $(x \in M)$,

where $v(t)$ is a nonnegative function on $[0, \infty)$ which satisfies

(1–4) $\displaystyle\int_0^\infty tv(t)dt < \infty$.

Then there exists a positive number v_0 such that if s is a non-zero holomorphic section of L over M which satisfies

$$(1\text{-}5) \qquad \qquad \|s(x)\| \leq C(1 + r(x))^v$$

on M for some constant $C > 0$ and some $v < v_0$, then s is nowhere zero on M.

When (M, o) has negative curvatures everywhere, above Theorem 1 has been proved by Greene and *Wu* (cf. Step III in the proof of Theorem *J* in [1]). Before them, Siu and Yau have proved above Theorem 1 when M is negatively curved and $k(t) = At^{-2-\varepsilon}$ $(\varepsilon > 0)$ (cf. Proposition 2-4 in [5]). Our proof of Theorem 1 will be given by generalizing the arguments in the proofs of Step III in [1] and Proposition 2-4 in [5] cited above.

It has been conjectured that an m-dimensional Kähler manifold (M, o) with a pole satisfying condition (1-1) should be biholomorphic to C^m. In fact this is true in the case where (M, o) is negatively curved and $k(t) = At^{-2-\varepsilon}$ ([5]). More generally Greene and *Wu* have verified this conjecture in the case where (M, o) is negatively curved and $k(t)$ is nondecreasing on $[\theta, \infty)$ for some $\theta > 0$ (cf. Theorem *J* in [1]). In the proofs of these results, one of the crucial steps was to prove above Theorem 1 in case (M, o) is negatively curved (Step III, [1], p. 188). Therefore our Theorem 1 will be of some use to study the conjecture mentioned above. In fact an application of our Theorem 1 to the case where (M, o) is positively curved will be published elsewhere.

2. Preliminaries. Let (M, o) be an m-dimensional Kähler manifold with a pole which satisfies condition (1-1). We recall several facts from Theorem C in [1] and Theorem in [4] as follows.

Fact 2-1. *Define C^∞ functions $p(t)$ and $q(t)$ by*

$$(2\text{-}1) \qquad \qquad p'' - kp = 0, \ p(0) = 0 \ \text{and} \ p'(0) = 1,$$

$$(2\text{-}1) \qquad \qquad q'' + Kq = 0, \ q(0) = 0 \ \text{and} \ q'(0) = 1.$$

Then the following inequalities hold on $[0, \infty)$:

$$(2\text{-}3) \qquad \qquad 1 \leq p'(t) \leq \eta \ \text{and} \ t \leq p(t) \leq \eta t,$$

$$(2\text{-}4) \qquad \qquad 1 \geq q'(t) \geq \mu \ \text{and} \ t \geq q(t) \geq \mu t,$$

where the constants η and μ are positive and satisfy

$$(2\text{-}5) \qquad \qquad 1 \leq \eta \leq \exp\left\{\int_0^\infty t k(t) dt\right\},$$

$$(2\text{-}6) \qquad \qquad 1 \geq \mu \geq 1 - \int_0^\infty t K(t) dt.$$

Fact 2–2. *Let $p(t)$ and $q(t)$ be as in* Fact 2–1. *Set $\eta^*(t)=tp'(t)/p(t)$ and $\mu^*(t)=tq'(t)/q(t)$. Then for any $t\geqq0$, we have*

(2–7)
$$1\leqq\eta^*(t)\leqq\eta\,,$$

(2–8)
$$1\geqq\mu^*(t)\geqq\mu\,.$$

If D^2r (resp. D^2r^2) denotes the Hessian of the function r (resp. the function r^2), then the following inequalities hold on $M-\{o\}$:

(2–9)
$$\frac{\mu^*(r)}{r}\,(g-dr\otimes dr)\leqq D^2r\leqq\frac{\eta^*(r)}{r}\,(g-dr\otimes dr)\,,$$

(2–10)
$$2\mu g\leqq D^2r^2\leqq2\eta g\,,$$

where $g=2\sum g_{j\bar{k}}dz^jd\bar{z}^k$ is the Kähler metric of (M,o).

As usual the associated Kähler form Ω of the Kähler metric $g=2\sum g_{j\bar{k}}dz^j\cdot d\bar{z}^k$ is defined by $\Omega=i\sum g_{j\bar{k}}dz^j\wedge d\bar{z}^k$.

Lemma 2–1. *The following inequalities hold in $M-\{o\}$:*

(2–11)
$$\frac{\eta^*(r)}{r}\,(\Omega-i\partial r\wedge\bar{\partial}r)\geqq i\partial\bar{\partial}r\geqq\frac{\mu^*(r)}{r}\,(\Omega-i\partial r\wedge\bar{\partial}r)\,,$$

(2–12)
$$2\eta\Omega\geqq i\partial\bar{\partial}r^2\geqq2\mu\Omega\,,$$

(2–13)
$$\Omega\geqq2i\partial r\wedge\bar{\partial}r\,.$$

Proof. Let J be the natural almost complex structure on M. Define J-invariant symmetric covariant two tensors h_1 and h_2 by

$$h_1(X,Y)=(dr\otimes dr)(X,Y)+(dr\otimes dr)(JX,JY)\,,$$
$$h_2(X,Y)=D^2r(X,Y)+D^2r(JX,JY)\,.$$

Then $h_1(JX,Y)=2i\partial r\wedge\bar{\partial}r(X,Y)$. Since $h_1(X,X)\leqq g(X,X)$, we have (2–13). On the other hand, since g is a Kähler metric, we have $h_2(x,x)=(2i\partial\bar{\partial}r)(X,JX)$. Then (2–9) and (2–13) imply (2–11). Finally (2–10) implies (2–12).   q.e.d.

We need a few more facts from [1]. A Riemannian manifold with a pole (N,e) is called a *model* iff the linear isotropy group of isometries at the pole e is the full orthogonal group. Then for a point $x\in N$, all the radial curvature at x are the same. Hence there exists a C^∞ function $K_N:[0,\infty)\to R$ such that for a point x, any radial curvature at x is equal to $k_N(r_N(x))$, where $r_N:N\to[0,\infty)$ is the distance function from e. The k_N is called the *radial curvature function* of the model (N,e). Moreover the metric g_N of N relative to geodesic polar coordinate centered at e assumes the form

(2–14)
$$g_N=dt^2+f(t)^2d\theta^2\,,$$

where f is a C^∞ function on $[0, \infty)$ which satisfies $f > 0$ on $(0, \infty)$ and

(2-15) $f'' + k_N f = 0$ with $f(0) = 0$ and $f'(0) = 1$.

Conversely for any C^∞ function $f(t)$ on $[0, \infty)$ satisfying $f > 0$ on $(0, \infty)$, $f(0) = 0$ and $f'(0) = 1$, there exists uniquely (up to isometry) a model (N, e) such that (2-14) holds. Then the radial curvature function k_N is equal to $-f''/f$. Therefore by (2-3) we have the following fact (cf. p. 60 of [1]).

Fact 2-3. *Consider the function $p(t)$ defined in Fact 2-1. Then there exists a 2 m dimensional model (N, e) whose metric relative to geodesic polar coordinates centered at e is given by*

$$g_N = dt^2 + p(t)^2 d\theta^2 ,$$

and the radial curvature function k_N is exactly $-k$.

Now by Proposition 2.15 (Laplacian Comparison Theorem) of [1], we have the following fact.

Fact 2-4. *Let (M, o) be a Kähler manifold with a pole satisfying condition (1-1) and (N, r) the model constructed in Fact 2-3. Let $f(t)$ be a nondecreasing C^∞ function on $(0, \infty)$. Then for every $x \in M - \{o\}$ and $y \in N - \{e\}$ such that $r(x) = r_N(y)$, we have*

$$\Delta f(r) (x) \leqq \Delta f(r_N) (y) .$$

Lemma 2-2. *Let (M, o) be a Kähler manifold with a pole satisfying condition (1-1). Let $p(x)$ be the C^∞ function defined in Fact 2-1. For a positive number $R > 0$, define a C^∞ function f_R on $(0, \infty)$ by*

$$f_R(t) = \int_t^R \frac{ds}{p(s)^{2m-1}} .$$

Then we have $\Delta f_R(r) \geqq 0$ on $M - \{o\}$.

Proof. Let (N, e) be the model constructed in Fact 2-3. Let $\{x^1, \cdots, x^{2m}\}$ be the geodesic polar coordinate system of N centered at e such that $x^1 = r_N$. Then on $N - \{e\}$ we see

$$\Delta f_R(r_N) = \frac{1}{\sqrt{G}} \sum_{A,B} \frac{\partial}{\partial x^A} \left(\sqrt{G} g_N^{AB} \frac{\partial}{\partial x^B} f_R(r_N) \right)$$

$$= (p(t)^{2(2m-1)})^{-1/2} \frac{d}{dt} \left\{ (p(t)^{2(2m-1)})^{1/2} \frac{d}{dt} f_R(t) \right\}$$

$$= 0 .$$

Since $-f_R(t)$ is a nondecreasing C^∞ function on $(0, \infty)$, Fact 2-4 implies $\Delta(-f_R(r))$

$(\leqq \Delta(-f_R(r_N)))=0$ on $M-\{o\}$.   q.e.d.

Lemma 2–3. *Let (M, o) be a Kähler manifold with a pole satisfying condition (1–1). Let $q(t)$ be the C^∞ function defined in Fact 2–1. Set $F(t)=$ $\exp\left(2\int_1^t \frac{dt}{q}\right)$. Then we have the following:*

(i)   *$F(r)$ is an C^∞ function on M,*

(ii) *$F(r)/q(r)^2$ is a positive monotone increasing C^∞ function on $[0, \infty)$*

(iii) *$\dfrac{2\mu F(r)}{q(r)^2}\,\Omega \leqq i\partial\bar{\partial}F(r)\leqq \dfrac{2(1+\eta)F}{q(r)^2}$,*

(iv) *$i\partial\bar{\partial}\log F(r)\geqq 0$ on $M-\{o\}$.*

Proof. As we obtain Fact 2–3, there exists a Riemannian metric g on R^2 which can be written as

$$g = dt^2+q(t)^2 d\theta^2$$

on $R^2-\{o\}$, where (t, θ) is the usual polar coordinates. We put a complex structure on R^2 so that g becomes a Kähler metric. Define a map $I: R^2-\{o\} \to R^2-\{o\}$ by

$$I(t, \theta) = \left(\exp\int_1^t \frac{dt}{q}, \theta\right),$$

where (t, θ) is the polar coordinates on $R^2-\{o\}$. Then I is a diffeomorphism. Since we have

(2–16) $I^*(dt^2+t^2 d\theta^2) = \left\{\dfrac{\exp\left(\int_1^t \frac{dt}{q}\right)}{q}\right\}^2(dt^2+q^2 d\theta^2)$,

I is a conformal map. Hence, if we consider I to be a C-valued function, I is a holomorphic function on $R^2-\{o\}$. Since I is bounded on a neighbourhood of o, I can be extended holomorphically to o and we have $I(o)=o$. Set $\widetilde{F}(s)= |I((s, o))|^2$ for any $s\in R$. Then $\widetilde{F}$ is an even C^∞ function on R and $F(t)=\widetilde{F}(t)$ for any $t>0$. Since r^2 is a C^∞ function on M, we know that $F(r)$ is a C^∞ function on M. Since I is holomorphic at o, I is a biholomorphic map. Consequently, (2–16) implies that $F(t)/q(t)^2$ is an even positive C^∞ function. Hence $F(r)/q(r)^2$ a positive C^∞ function on M. On the other hand, we have

$$\left(\frac{F(t)}{q(t)^2}\right)' = \frac{2F(t)}{q(t)^3}(1-q'(t)).$$

By (2–4), we see $F(t)/q(t)^2$ is an increasing function. Since we have

$$i\partial\bar{\partial}F(r) = F'(r)i\partial\bar{\partial}r+F''(r)i\partial r\wedge\bar{\partial}r$$

$$= \frac{2F(r)}{q(r)} i\partial\bar{\partial}r + \frac{2F(r)}{q(r)^2}(2-q'(r))i\partial r \wedge \bar{\partial}r \,,$$

by (2·–11) and (2–4), we see

$$i\partial\bar{\partial}F(r) \geqq \frac{2F(r)}{q(r)}\frac{\mu^*(r)}{r}(\Omega - i\partial r \wedge \bar{\partial}r) + \frac{2F(r)}{q(r)^2}(2-q'(r))i\partial r \wedge \bar{\partial}r$$

$$= \frac{2F(r)}{q(r)}\{q'\Omega + 2(1-q')i\partial r \wedge \bar{\partial}r\} \geqq \frac{2\mu F(r)}{q(r)^2}\Omega \,.$$

On the other hand, by (2.11) and (2.3), we see

$$i\partial\bar{\partial}F(r) \leqq \frac{2F(r)}{q(r)}\frac{\eta^*(r)}{r}(\Omega - i\partial r \wedge \bar{\partial}r) + \frac{2F(r)}{q(r)^2}(2-q'(r))i\partial r \wedge \bar{\partial}r$$

$$= \frac{2F(r)}{q(r)}\frac{p'(r)}{p(r)}(\Omega - i\partial r \wedge \bar{\partial}r) + \frac{2F(r)}{q(r)^2}(2-q'(r))i\partial r \wedge \bar{\partial}r$$

$$\leqq \frac{2F(r)}{q(r)}\frac{p'(r)}{p(r)}\Omega + \frac{2F(r)}{q(r)^2}\cdot 2i\partial r \wedge \bar{\partial}r$$

$$\leqq \frac{2\eta F(r)}{q(r)^2}\Omega + \frac{2F(r)}{q(r)^2}\Omega \,.$$

Finally by (2–11), we have

$$i\partial\bar{\partial}\log F(r) = \frac{2}{q(r)}i\partial\bar{\partial}r - \frac{2q'(r)}{q(r)^2}i\partial r \wedge \bar{\partial}r$$

$$\geqq \frac{2}{q(r)}\frac{\mu^*(r)}{r}(\Omega - i\partial r \wedge \bar{\partial}r) - \frac{2q'(r)}{q(r)^2}i\partial r \wedge \bar{\partial}r$$

$$= \frac{2q'(r)}{q(r)^2}\Omega - \frac{4q'(r)}{q(r)^2}i\partial r \wedge \bar{\partial}r$$

$$\geqq 0 \,. \hspace{4cm} \text{q.e.d.}$$

3. A volume estimate for analytic subsets. Let V be a closed analytic subset in M of pure dimension n. For a positive number t, we set $B(t) = \{x \in M, r(x) < t\}$ and $\partial B(t) = \{x \in M, r(x) = t\}$. Then $\overline{B(t)}$ is compact and $\partial B(t)$ is a hypersurface in M. We write Vol $(V \cap B(t))$ for the volume of $V \cap B(t)$. Then we have

$$\text{Vol}\,(V \cap B(t)) = \frac{1}{2^n n!}\int_{V \cap B(t)}\Omega^n \,.$$

As usual we set $d^c = i(\bar{\partial} - \partial)/2$ so that $dd^c = i\partial\bar{\partial}$. In this section we shall prove

Proposition 3–1. *Let (M, o) be as in Theorem 1 in section 1. Then there exists a positive constant A depending only on (M, o) such that for any closed an-*

alytic subset V in M of pure dimension n, we have $\text{Vol}\,(V \cap B(t)) \geqq A l(V) t^{2n}$ for $t \geqq 0$, where $l(V)$ is the multiplicity of V at o.

Proof. Set $B(t,s) = B(s) - \overline{B(t)}$ for $0 < t < s$. Using Stokes Theorem for analytic subsets (cf. [3] or Theorem 1.28 in [2]), for $0 < t < s$, we have

$$\frac{1}{F(s)^n} \int_{V \cap B(s)} (dd^c F(r))^n - \frac{1}{F(t)^n} \int_{V \cap B(t)} (dd^c F(r))^n$$

$$= \frac{1}{F(s)^n} \int_{V \cap \partial B(s)} d^c F(r) \wedge (dd^c F(r))^{n-1}$$

$$- \frac{1}{F(t)^n} \int_{V \cap \partial B(t)} d^c F(r) \wedge (dd^c F(r))^{n-1}$$

$$= \int_{V \cap \partial B(t)} \frac{d^c F(r)}{F(s)} \wedge \left(\frac{dd^c F(r)}{F(s)} - \frac{dF(r) \wedge d^c F(r)}{F(s)^2} \right)^{n-1}$$

$$- \int_{V \cap \partial B(t)} \frac{d^c F(r)}{F(t)} \wedge \left(\frac{dd^c F(r)}{F(t)} - \frac{dF(r) \wedge d^c F(r)}{F(t)^2} \right)^{n-1}$$

($dF(r)$ being zero on $B(t)$ and $B(s)$)

$$= \int_{V \cap \partial B(t)} d^c \log F(r) \wedge (dd^c \log F(r))^{n-1}$$

$$- \int_{V \cap \partial B(t)} d^c \log F(r) \wedge (dd^c \log F(r))^{n-1}$$

$$= \int_{V \cap B(s)} (dd^c \log F(r))^n - \int_{V \cap B(t)} (dd^c \log F(r))^n$$

$$= \int_{V \cap B(t,s)} (dd^c \log F(r))^n$$

$$\geqq 0$$

(cf. (ii) of Lemma 2–3). Therefore we know

$$\frac{1}{F(t)^n} \int_{V \cap B(t)} (dd^c F(r))^n$$

is a non-negative increasing function for $t > 0$. In particular there exists

$$\lim_{t \downarrow 0} \frac{1}{F(t)^n} \int_{V \cap B(t)} (dd^c F(r))^n$$

which is denoted by $n^*(V, o)$. Now by (iii) and (ii) of Lemma 2–3, we have

$$(3\text{-}1) \qquad \frac{\text{Vol}\,(V \cap B(t))}{t^{2n}} = \frac{1}{2^n n! t^{2n}} \int_{V \cap B(t)} \Omega^n$$

$$\geqq \frac{1}{2^{2n} n! (1+\eta)^n t^{2n}} \cdot \int_{V \cap B(t)} \left(\frac{q(r)^2}{F(r)} \right)^n (dd^c F(r))^n$$

$$\geq \frac{1}{2^{2n}n!(1+\eta)^n t^{2n}} \cdot \frac{q(t)^{2n}}{F(t)^n} \int_{V \cap B(t)} (dd^c F(r))^n$$

$$\geq \frac{\mu^{2n}}{2^{2n}n!(1+\eta)^n} \frac{1}{F(t)^n} \int_{V \cap B(t)} (dd^c F(r))^n .$$

Hence we have

(3–2) $$\mathrm{Vol}\,(V \cap B(t)) \geq \frac{\mu^{2n}}{2^{2n}n!(1+\eta)^n} n^*(V, o) t^{2n}$$

for $t \geq 0$. By (2–4), we see

(3–3) $$\frac{1}{t^2} \leq \frac{1}{F(t)} \leq \frac{1}{t^{2\mu}}$$

for $0 < t \leq 1$. By (ii) and (iii) of Lemma 2–3, there exists a positive constant B_1 such that

(3–4) $$i\partial\bar{\partial}F(r) \geq B_1\Omega$$

on $B(1)$. Then (3–3) and (3–4) imply

(3–5) $$\frac{1}{F(t)^n} \int_{V \subset B(t)} (dd^c F(r))^n \geq \frac{B_1{}^n}{t^{2n}} \int_{V \cap B(t)} \Omega^n$$

for $0 < t \leq 1$. Now take sufficiently small $\varepsilon > 0$ so that $B(\varepsilon)$ is a holomorphic local coordinate neighbourhood with a holomorphic local coordinate system $\{z^1, \cdots, z^m\}$ $(z^i(o)=0, 1 \leq i \leq m)$. Let $g_0 = \sum dz^j d\bar{z}^j$ be the usual flat Kähler metric on $B(\varepsilon)$. Then $\alpha_0 = \{i\partial\bar{\partial}(\sum |z^j|^2)\}/2$ is the associated Kähler form of g_0. By using $i\partial\bar{\partial} \log (\sum |z^j|^2) \geq 0$ on $B(\varepsilon) - \{o\}$, the same argument to have obtained $n^*(V, o)$ implies that if we set

$$n(t, o) = \frac{1}{t^{2n}} \int_{V \cap \{\sum |z^j|^2 < t^2\}} \alpha_0{}^n ,$$

then $n(t, o)$ is an increasing function of t and $\lim_{t \to 0} n(t, o) = B_2 l(V)$ where B_2 is a universal constant (cf. Corollary 1.29 in [2]). Since there exists a positive constant B_3 such that

$$\frac{1}{B_3} \alpha_0 \leq \Omega \leq B_3 \alpha_0$$

on $B\left(\frac{\varepsilon}{2}\right)$, we know there exists a positive constant B_4 such that

(3–6) $$\frac{1}{t^{2n}} \int_{V \cap B(t)} \Omega^n \geq B_4 l(V)$$

for sufficiently small t. By (3–5) and (3–6) we have

$$(3\text{–}7) \qquad\qquad n^*(V, o) \geqq B_1{}^n B_4 l(V) .$$

Then (3–7) and (3–2) imply Proposition 3–1. q.e.d.

Corollary. *For a positive number R, there exists a positive constant $B^*(R)$ such that for $t \geqq R$ we have*

$$\mathrm{Vol}\,(V \cap B(t)) \geqq B^*(R) V^*(R) t^{2n} ,$$

where $V^*(R) = \displaystyle\int_{V \cap B(R)} (dd^c F(r))^n .$

Proof. By (3–1) we see

$$\frac{\mathrm{Vol}\,(V \cap B(t)}{t^{2n}} \geqq \frac{2^{2n} n! (1+\eta)^n}{2n} \frac{1}{F(R)^n} V^*(R) . \qquad\qquad \text{q.e.d.}$$

4. Proof of Theorem 1. We keep the notation of the previous sections. Let $p(t)$ be the function defined in Fact 2–1. For any positive number R, define a C^∞ function F_R on $M-\{o\}$ by $F_R(x)=f_R(r(x))$ where f_R is the function defined in Lemma 2–2.

Let s by any nonzero holomorphic section of L such that

$$(4\text{–}1) \qquad\qquad \{x \in M; s(x) = 0\} \ \text{ is not empty,}$$

$$(4\text{–}2) \qquad\qquad \|s(x)\| \leqq C(1+r(x))^\nu$$

for some positive constants C and ν. Let V be the divisor defined by the zeros of s. For $R>1$, we set $B(R, 1)=B(R)-B(1)$. Then Green's formula implies

$$\int_{B(R,1)} \Delta F_R \cdot \log\|s\|^2 - \int_{B(R,1)} F_R \cdot \Delta \log\|s\|^2$$

$$= \int_{\partial B(R)} \log\|s\|^2 *dF_R - \int_{\partial B(1)} \log\|s\|^2 *dF_R$$

$$- \int_{\partial B(r)} F_R *d \log\|s\|^2 + \int_{\partial B(1)} F_R *d \log\|s\|^2 .$$

By (2–3) we have

$$\frac{1}{\eta^{2m-1}} \int_t^R \frac{1}{t^{2m-1}} \leqq f_R(t) \leqq \int_t^R \frac{dt}{t^{2m-1}} .$$

Hence we obtain

$$(4\text{–}1) \qquad \frac{1}{(2m-1)\eta^{2m-1}} \left(\frac{1}{t^{2m-2}} - \frac{1}{R^{2m-2}} \right) \leqq f_R(t) \leqq \frac{1}{2m-2} \left(\frac{1}{t^{2m-2}} - \frac{1}{R^{2m-2}} \right) .$$

Therefore we have an estimate:

$$\left| -\int_{\partial B(1)} \log ||s||^2 * dF_R + \int_{\partial B(1)} F_R * d \log ||s||^2 \right|$$

$$= \left| -f_R'(1) \int_{\partial B(1)} \log ||s||^2 * dr + f_R(1) \int_{\partial B(1)} * d \log ||s||^2 \right|$$

$$\leq \frac{1}{p(1)^{2m-1}} \left| \int_{\partial B(1)} \log ||s||^2 * dr \right| + \frac{1}{2m-2} \left(1 - \frac{1}{R^{2m-2}} \right) \left| \int_{\partial B(1)} * d \log ||s||^2 \right|$$

$$= 0(1),$$

where $0(1)$ stands for a bounded term as $R \to \infty$. Since

$F_R \equiv 0$ on $\partial B(R)$, we have

$$(4\text{--}2) \qquad \int_{B(R,1)} \Delta F_R \cdot \log ||s||^2 - \int_{B(R,1)} F_R \cdot \Delta \log ||s||^2$$

$$= \int_{\partial B(R)} \log ||s||^2 f_R'(R) * dr + O(1).$$

On the other hand, Poincaré-Lelong's formula implies

$$(4\text{--}3) \qquad \frac{2^m m!}{2\pi} \int_{B(R,1)} F_R \Delta \log ||s||^2 = \int_{B(R,1) \cap V} F_R \Omega^{m-1} - \int_{B(R,1)} F_R \omega \wedge \Omega^{m-1}$$

(cf. [3] or Theorem 1.11 in [2]). Since $\Delta F_R \geq 0$ by Lemma 2–2, we see by (4–2)

$$\int_{B(R,1)} \Delta F_R \cdot \log ||s||^2 \leq \int_{B(R,1)} \Delta F_R \cdot \log \{C(1+r)^\nu\}.$$

Now by Green's formula, we have

$$\int_{B(R,1)} \Delta F_R \cdot \log \{C(1+r)^\nu\}$$

$$= \int_{B(R,1)} F_R \cdot \Delta \log \{C(1+r)^\nu\} + \int_{\partial B(R)} \log \{C(1+r)^\nu\} * dF_R$$

$$- \int_{\partial B(1)} \log \{C(1+r)^\nu\} * dF_R - \int_{\partial B(R)} F_R * d \log \{C(1+r)^\nu\}$$

$$+ \int_{\partial B(1)} F_R * d \log \{C(1+r)^\nu\}$$

$$= \int_{B(R,1)} F_R \cdot \Delta \log C(1+r)^\nu\} + \int_{\partial B(R)} \log \{C(1+r)^\nu\} * dF_R + O(1).$$

Therefore we have

$$(4\text{--}4) \qquad \int_{B(R,1)} \Delta F_R \cdot \log ||s||^2 \leq \int_{B(R,1)} F_R \cdot \Delta \log \{C(1+r)^\nu\}$$

$$+ \int_{\partial B(R)} \log \{C(1+r)^{\nu}\} f_R'(R) * dr + O(1) \, .$$

From (4–4), (4–2) and (4–3), we obtain

$$\int_{B(R,1)} F_R \cdot \Delta \log \{C(1+r)^{\nu}\} + \int_{\partial B(R)} \log \{C(1+r)^{\nu}\} f_R'(R) * dr + O(1)$$

$$\geqq \int_{B(R,1)} \Delta F_R \cdot \log \|s\|^2$$

$$= \int_{B(R,1)} F_R \cdot \Delta \log \|s\|^2 + \int_{\partial B(R)} \log \|s\|^2 f_R'(R) * dr + O(1)$$

$$= \frac{2\pi}{2^m m!} \int_{B(R,1) \cap V} F_R \Omega^{m-1} - \frac{2\pi}{2^m m!} \int_{B(R,1)} F_R \omega \wedge \Omega^{m-1}$$

$$+ \int_{\partial B(R)} \log \|s\|^2 f_R'(R) * dr + O(1) \, .$$

Since $f_R'(R) < 0$, we see

$$\int_{\partial B(R)} \log \{C(1+r)^{\nu}\} f_R'(R) * dr \leqq \int_{\partial B(R)} \log \|s\|^2 f_R'(R) * dr \, .$$

Therefore we obtain

$$(4\text{--}5) \qquad \frac{2^m m!}{2\pi} \int_{B(R,1)} F_R \cdot \Delta \log \{C(1+r)^{\nu}\}$$

$$\geqq \int_{B(R,1) \cap V} F_R \Omega^{m-1} - \int_{B(R,1)} F_R \omega \wedge \Omega^{m-1} + O(1) \, .$$

Now by (2–9) and (2–7), we have

$$(4\text{--}6) \qquad \qquad \Delta r \leqq \frac{(2m-1)\eta}{r} \, .$$

Then (4–6) implies

$$(4\text{--}7) \qquad \qquad \Delta \log C(1+r) = \frac{\Delta r}{1-r} - \frac{1}{(1+r)^2}$$

$$\leqq \frac{(2m-1)\eta}{(1+r)r} \, .$$

Now (4–7) and (4–1) imply

$$(4\text{--}8) \quad \int_{B(R,1)} F_R \Delta \log \{C(1+r)^{\nu}\} \leqq \nu \int_{B(R,1)} \frac{1}{(2m-2)r^{2m-2}} \cdot \frac{(2m-1)\eta}{(1+r)r}$$

$$\leqq \frac{(2m-1)S(2m-1)\eta^{2m}\nu}{2m-2} \log (1+R) \, ,$$

where $S(2m-1)$ denotes the Euclidian volume of $(2m-1)$ dimensional unit sphere. By the same way we have

$$
\begin{aligned}
\int_{B(R,1)\cap V} F_R \Omega^{m-1} &= \int_1^R dt \int_{\partial B(t)\cap V} F_R \iota\left(\frac{\partial}{\partial r}\right)\Omega^{m-1} \\
&= \int_1^R f_R(t)dt \int_{\partial B(t)\cap V} \iota\left(\frac{\partial}{\partial r}\right)\Omega^{m-1} \\
&= \int_1^R \left[\frac{d}{dt}\left\{f_R(t)\int_1^t dt \int_{\partial B(t)\cap V} \iota\left(\frac{\partial}{\partial r}\right)\Omega^{m-1}\right\}\right]dt \\
&\quad -\int_1^R \left\{f_R'(t)\int_1^t dt \int_{\partial B(t)\cap V} \iota\left(\frac{\partial}{\partial r}\right)\Omega^{m-1}\right\}dt \\
&= 2^{m-1}(m-1)![f_R(t)\,\mathrm{Vol}(B(t)\cap V)]_1^R \\
&\quad +2^{m-1}(m-1)!\int_1^R \frac{\mathrm{Vol}(B(t,1)\cap V)}{p(t)^{2m-1}}.
\end{aligned}
$$

Then by (2–3) and $f_R(R)=0$, we have

$$
(4\text{–}9) \qquad \int_{B(R,1)\cap V} F_R \Omega^{m-1} \geqq \frac{2^{m-1}(m-1)!}{\eta^{2m-1}}\int_1^R \frac{\mathrm{Vol}(B(t)\cap V)}{t^{2m-1}}\,dt+O(1).
$$

From (4–5), (4–8) and (4–9) we obtain

$$
(4\text{–}10) \qquad E_1\eta\nu \log(1+R) \geqq \frac{2^{m-1}(m-1)!}{\eta^{2m-1}}\int_1^R \frac{\mathrm{Vol}(B(t)\cap V)}{t^{2m-1}}\,dt
$$
$$
-\int_{B(R,1)} F_R\omega\wedge\Omega^{m-1}+O(1),
$$

where E_1 is a positive constant depending only on m.

If the Chern form ω satisfies (1–3), we have

$$
(4\text{–}11) \qquad\qquad -\int_{B(R,1)} F_R\omega\wedge\Omega^{m-1} \geqq 0.
$$

If ω satisfies (1–4), we see by (4–1)

$$
\begin{aligned}
(4\text{–}12) \qquad \left|\int_{B(R,1)} F_R\omega\wedge\Omega^{m-1}\right| &\leqq D\int_{B(R,1)} F_R\|\omega\|\Omega^m \\
&\leqq D'S(2m-1)\int_1^R f_R(t)v(t)t^{2m-1}dt \\
&\leqq D''\int_1^R tv(t)dt = O(1),
\end{aligned}
$$

where D, D', D'' are constants independent of s. Therefore by (4–10), (4–11) and (3–12), we have

$$(4\text{-}13) \qquad E_1 \eta \nu \log(1+R) \geqq \frac{2^{m-1}(m-1)!}{\eta^{2m-1}} \int_1^R \frac{\mathrm{Vol}(B(t) \cap V)}{t^{2m-1}} \, dt + O(1) \,.$$

Therefore from Proposition 3–1, we obtain

$$E_1 \nu \log(1+R) \geqq \frac{2^{m-1}(m-1)!}{\eta^{2m}} \int_1^R \frac{Al(V) t^{2m-2}}{t^{2m-1}} \, dt + O(1)$$

$$\geqq E_2 l(V) \log R + O(1) \,,$$

where E_2 is a positive constant depending only on (M, o). Hence, taking the limit, we have

$$(4\text{-}14) \qquad\qquad\qquad \nu \geqq \frac{E_2}{E_1} l(V) \,,$$

where E_1 and E_2 are positive constants depending only on (M, o).

Lemma 4–1. *Let (M, o) and $\{L, h\}$ be as in* Theorem 1. *For a positive number ν, denote by $\Gamma(M, L; \nu)$ the complex vector space of holomorphic sections s over M which satisfy*

$$(4\text{-}15) \qquad\qquad\qquad \|s(x)\| \leqq C(1+r(x))^\nu$$

for some positive C. Then there exists a positive number ν^* depending only on (M, o) such that the dimension of $\Gamma(M, L; \nu^*)$ is at most one.

Proof. Take $E_2/2E_1$ as ν^*, where E_1, E_2 are as in (4–14). Take any holomorphic section s in $\Gamma(M, L; \nu^*)$. Then by (4–15), we see $l(V) = 0$, i.e., $s(o) \neq 0$. Suppose there were two elements s_1 and s_2 in $\Gamma(M, L; \nu^*)$ which are linearly independent. Since $s_1(o) \neq 0$ and $s_2(o) \neq 0$, there would exist a number a such that $(as_1 + s_2)(o) = 0$. Then $as_1 + s_2$ should be zero. This is a contradiction.

Proof of Theorem 1. Let ν^* be as in Lemma 4–1. It is enough to check the case when $\Gamma(M, L; \nu^*)$ contains an element s_0 such that $\{x \in M; s_0(x) = 0\}$ is non empty. Fix a sufficiently large number R_0 so that

$$(4\text{-}16) \qquad\qquad B(R) \cap \{x \in M; s_0(x) = 0\} \neq \emptyset \,.$$

Then by (4–13) and Corollary to Proposition 3–1, for $R > R_0$ we have

$$E_1 \nu \log(1+R) \geqq \frac{2^{m-1}(m-1)!}{\eta^{2m}} \int_{R_0}^R \frac{\mathrm{Vol}(B(t) \cap V)}{t^{2m-1}} \, dt + O(1)$$

$$\geqq \frac{2^{m-1}(m-1)! \, B^*(R_0) V^*(R_0)}{\eta^{2m}} \log R + O(1) \,.$$

Hence, taking the limit, we see

$$(4\text{-}17) \qquad\qquad \nu \geqq \frac{E_3}{E_1} B^*(R_0) V^*(R_0),$$

where E_1, E_3 are positive constants depending only on (M, o), and $B^*(R_0)$ is positive. By (4-16), we see $V^*(R_0)$ is positive. Now set

$$\nu_0 = \min\left\{\nu^*, \frac{E_3 B^*(R_0) V^*(R_0)}{2E_1}\right\}.$$

Then we have $\Gamma(M, L; \nu_0)=0$. In fact take any element s in $\Gamma(M, L; \nu_0)$. Then by Lemma 4-1, there exists a real number a such that $s=as_0$. Suppose $a \neq 0$. Then since $\{x \in M;\ s(x)=0\} = \{x \in M;\ s_0(x)=0\}$, (4-17) implies $\nu_0 \geqq E_3 B^*(R_0)$ $V^*(R_0)/E_1$. This is a constradiction.   q.e.d.

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