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ON TIME CHANGE OF SYMMETRIC MARKOV PROCESSES

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1. Introduction

Let X be a locally compact separable metric space and m be an everywhere dense positive Radon measure on X. Suppose that we are given an irreducible regular Dirichlet space $(\mathcal{E}, \mathcal{F})$ on $L^2(X; m)$. There then corresponds an m-symmetric Markov process $\boldsymbol{M}=(\Omega, \mathcal{B}, X_t, P_x)$. Let us fix a positiv Radon measure μ charging not a set of zero capacity. Let A be the positive continuous additive functional (PCAF) associated with μ and set $Y_t=X(A^{-1}(t))$. We shall denote by $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ the $L^2(X; \mu)$-Dirichlet space of $\boldsymbol{M}^\mu=(\Omega, \mathcal{B}, Y_t, P_x)$. The purpose of this paper is to characterize the extended Dirichlet space $(\mathcal{E}^\mu, \mathcal{F}^\mu_e)$ of $(\mathcal{E}^\mu, \mathcal{F}^\mu)$. The characterization given here is originally discussed by Silverstein [7], but his proof seems to be insufficient. In the transient case however, Fukushima [3] has established the characterization.

If $(\mathcal{E}, \mathcal{F})$ is transient, then its extended Dirichlet space $(\mathcal{E}, \mathcal{F}_e)$ becomes a Hilbert space continuously embedded in an $L^1(X; \text{gdm})$-space. If $\boldsymbol{M}$ is recurrent in the sense of Harris, then the quotient space of $\mathcal{F}_e$ by the family of constant functions becomes a Hilbert space continuously embedded in an $L^1(X; gdm)$-space. In the latter case, we shall identify $\mathcal{F}_e$ and the quotient space. Let Y be the support of A, γ be the restriction operator to Y and $\mathcal{F}_{X-Y}=\{u\in \mathcal{F}_e; u=0 \text{ q.e. on } Y\}$. Let

$$(1.1) \qquad \mathcal{F}_e = \mathcal{F}_{X-Y}+\mathcal{H}^Y$$

be an orthogonal decomposition. Then the main result is, for a suitable choice of the version, $\mathcal{F}^\mu_e=\mathcal{H}^Y$ and $\mathcal{E}^\mu(\gamma u, \gamma u)=\mathcal{E}(u, u)$ for all $u\in\mathcal{H}^Y$.

We shall prove this in section 3 by assuming that $\boldsymbol{M}$ is recurrent in the sense of Harris. See [3] for the same result in transient case. We believe that the present generalization to recurrent case is important because recurrent symmetric Markov processes appear in many applications and besides the present additional condition of the Harris property can be checked by a kind of coerciveness condition on the Dirichlet form ([4]).

In section 4, under the hypothesis that $Y=X$, we shall be concerned with the regularity property of $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ for general regular Dirichlet space $(\mathcal{E}, \mathcal{F})$. If C is a core of $(\mathcal{E}, \mathcal{F})$, then for general (not necessarily smooth) positive Radon measure μ such that $m \leqq \mu$, $(\mathcal{E}, C)$ is closable on $L^2(X; \mu)$. But, if μ is not a smooth measure, then the corresponding process cannot be obtained by time change. In addition to the conditions on μ in the first paragraph, if we assume that $\mu - m$ is a bounded measure or a measure with finite energy integral, then $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ becomes the Dirichlet space obtained by the smallest closed extension of $(\mathcal{E}, C)$ on $L^2(X; \mu)$.

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2. Preliminaries

In this section, we shall state some general results for the following sections. In the next section, we shall restrict the situation to the Harris recurrent case.

For two PCAFs (Φ_t) and (Ψ_t), we define the kernels $V^{pq}_{\Phi\Psi}$ and V^q_Ψ by

$$V^{pq}_{\Phi\Psi} f(x) = E_x \left[\int_0^\infty \exp(-p\Phi_t - q\Psi_t) f(X_t)\, d\Psi_t\right]$$

and $V^q_\Psi = V^{0q}_{\Phi\Psi}$, respectively. When $\Psi_t = t$, then we shall set $V^{pq}_{\Phi t} = V^{pq}_{\Phi\Psi}$, $V^{qp}_{t\Phi} = V^{qp}_{\Psi\Phi}$ and $V^q = V^q_\Psi$. If $V^{r0}_{\Phi\Psi}|f|$ is bounded a.e. for some $r \geqq 0$, then we have the following generalized resolvent equation ([5]).

$$V^{pq}_{\Phi\Psi} f - V^{rs}_{\Phi\Psi} f + (p-r)\, V^{qp}_{\Psi\Phi}\, V^{rs}_{\Phi\Psi} f + (q-s)\, V^{pq}_{\Phi\Psi}\, V^{rs}_{\Phi\Psi} f = 0\,. \tag{2.1}$$

Let Y_μ and Y be the support of μ and A, respectively, that is, Y_μ is the smallest closed set outside of which μ vanishes and Y is the fine closed set defined by $Y = \{x; P_x[A_t > 0 \text{ for all } t > 0] = 1\}$. Then $Y \subset Y_\mu$ and $\mu(Y_\mu - Y) = 0$ by [3; Lemma 5.5.1]. Let σ_Y be the hitting time of Y and set $H\phi(x) = E_x[\phi(X(\sigma_Y))]$, then $\sigma_Y = \inf\{t; A_t > 0\}$ *a.s.* P_x for q.e. x by [2; V. 3.5], and, in particular,

$$H\phi(x) = \lim_{p \to \infty} p V^p_A \phi(x) \qquad \text{for} \quad q.e.x. \tag{2.2}$$

Let $\mathcal{D}$ be the class of functions defined by

$$\mathcal{D} = \{V^{pq}_{tA} f; p, q > 0, f \in C_0(X)\} \cup \{V^{qp}_{At} f; p, q > 0, f \in C_0(X)\}\,. \tag{2.3}$$

In [5; §5], we proved the following results under the hypothesis that $Y = X$, but, by using (2.2), the same arguments are valid under the present situation.

Lemma 2.1. *M^μ is a μ-symmetric normal strong Markov process on Y.*

Lemma 2.2. (i) *$\mathcal{D} \subset \mathcal{F}$ and $\gamma\mathcal{D} \subset \mathcal{F}^\mu$*

(ii) *$\mathcal{D}$ is $\mathcal{E}_1$-densei in $\mathcal{F}$ and $\gamma\mathcal{D}$ is $\mathcal{E}^\mu_1$-dense in $\mathcal{F}^\mu$, where* $\mathcal{E}_1(\cdot, \cdot) = \mathcal{E}(\cdot, \cdot) +$

$(\cdot, \cdot)_{L^2(m)}$ *and* $\mathcal{E}_1^\mu(\cdot, \cdot)=\mathcal{E}(\cdot, \cdot)+(\cdot, \cdot)_{L^2(\mu)}$.

(iii) *If* $u\in\mathcal{D}$, *then* $Hu\in\mathcal{F}$ *and*

$$\mathcal{E}(Hu, Hu) = \mathcal{E}^\mu(\gamma u, \gamma u)\,. \tag{2.4}$$

Since $(\mathcal{E}, \mathcal{F})$ is irreducible, so is $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ by [7]. The *extended Dirichlet space* $(\mathcal{E}, \mathcal{F}_e)$ of $(\mathcal{E}, \mathcal{F})$ is defined as follows: A function u belongs to $\mathcal{F}_e$ if there exists an $\mathcal{E}$-Cauchy sequence $\{u_n\}\subset\mathcal{F}$ such that $\lim_{n\to\infty} u_n=u$ m-a.e. In this case $\mathcal{E}(u, u)$ is defined by $\mathcal{E}(u, u)=\lim_{n\to\infty}\mathcal{E}(u_n, u_n)$.

If $(\mathcal{E}, \mathcal{F})$ is transient, then by [3; p. 67], $(\mathcal{E}, \mathcal{F}_e)$ is a Hilbert space and there exists a strictly positive function $g\in L^1(X; m)$ such that

$$\int |u(x)|\, g(x)\, dm(x)\leqq\mathcal{E}(u, u)^{1/2} \quad \text{for all} \quad u\in\mathcal{F}_e\,. \tag{2.5}$$

Also, each function $u\in\mathcal{F}_e$ has a quasi-continuous (q.c.) modification. If $(f, V^0 f)_{L^2(m)}<\infty$ then $V^0 f\in\mathcal{F}_e$ and

$$\mathcal{E}(u, V^0 f) = \int u(x) f(x)\, dm(x) \tag{2.6}$$

holds for all $u\in\mathcal{F}_e$.

In the remainder of this section, we shall assume that $(\mathcal{E}, \mathcal{F})$ is recurrent. Let C be a measurable subset of X such that $0<m(C)<\infty$ and set $\Phi_t=\int_0^t I_C(X_s)\, ds$. Then $(V_{\Phi t}^{1p})_{p\geq 0}$ is the resolvent of the m-symmetric Markov process $\boldsymbol{M}^C$ given by the $(\exp(-\Phi_t))$-subprocess of $\boldsymbol{M}$. The $L^2(X; m)$-Dirichlet space $(\mathcal{E}^C, \mathcal{F}^C)$ of $\boldsymbol{M}^C$ is given by

$$\mathcal{F}^C = \mathcal{F} \quad \text{and} \quad \mathcal{E}^C(u, v) = \mathcal{E}(u, v)+(u, v)_{L^2(m_C)}\,, \tag{2.7}$$

where $m_C(\cdot)=m(\cdot\cap C)$ ([5]). Since $1=V_{\Phi t}^{10}\, I_C$ is a potential relative to $\boldsymbol{M}^C$, we see, using (2.7) that 1 belongs to the extended Dirichlet space $\mathcal{F}_e^C$ of $\mathcal{F}^C$ and $\mathcal{E}(1, 1)=0$.

Lemma 2.3. *For all C such that $0<m(C)<\infty$, $\mathcal{F}_e^C$ is contained in $\mathcal{F}_e$ and, conversely, if $u\in\mathcal{F}_e$ then there exists a measurable set C with $0<m(C)<\infty$ satisfying $u\in\mathcal{F}_e^C$.*

Proof. Since $\mathcal{E}(u, u)\leqq\mathcal{E}^C(u, u)$ for all $u\in\mathcal{F}$ by (2.7), the first statement is obvious. Conversely, if $u\in\mathcal{F}_e$, then there exists an $\mathcal{E}$-Cauchy sequence $\{u_n\}$ of functions of $\mathcal{F}$ such that $\lim_{n\to\infty} u_n=u$ m-a.e. Take the set C so that $0<m(C)<\infty$ and $\{u_n\}$ and u are bounded on C. Then, by (2.7), $\{u_n\}$ forms an $\mathcal{E}^C$-Cauchy sequence of functions of $\mathcal{F}^C$. This implies that $u\in\mathcal{F}_e^C$.

Since $(\mathcal{E}^C, \mathcal{F}^C)$ is a regular transient Dirichlet space, we have the following

Corollary 2.4. *Each function of $\mathcal{F}_e$ has a q.c. modification.*

In the followings, all functions of $\mathcal{F}_e$ are assumed to be quasi-continuous. In the next section, we will assume that $\boldsymbol{M}$ is recurrent in the sense of Harris relative to m, that is,

$$(2.8) \qquad \int_0^\infty f(X_t)\, dt = \infty \quad P_x\text{-a.s.}$$

for any $f \geqq 0$ such that $\int f(x)\, dm(x) > 0$, for each $x \in X$. For the sufficient conditions of this, we shall refer Fukushima [4]. In this case, if we consider $\mathcal{F}_e$ to be the quotient space of $\mathcal{F}_e$ by the family of constant functions and $\mathcal{E}$ defined naturally on it, then $(\mathcal{E}, \mathcal{F}_e)$ becomes a Hilbert space. Moreover, there exists a strictly positive function $g \in L^1(X; m)$ and a linear functional $I(\cdot)$ such that

$$(2.9) \qquad \int |u(x) - I(u)|\, g(x)\, dm(x) \leqq \mathcal{E}(u, u)^{1/2},$$

for all $u \in \mathcal{F}_e$ ([5]). If $\boldsymbol{M}$ is recurrent in the sense of Harris relative to m, then so is $\boldsymbol{M}^\mu$ relative to μ by Azéma, Duflo and Revuz [1]. Hence $(\mathcal{E}^\mu, \mathcal{F}_e^\mu)$ becomes also a Hilbert space.

REMARK. In view of Lemma 2.3 and the paragraph preceding it, the condition

$$(2.10) \qquad 1 \in \mathcal{F}_e \quad \text{and} \quad \mathcal{E}(1, 1) = 0$$

is a necessary condition for an irreducible regular Dirichlet space to be recurrent. This is also a sufficient condition ([4]).

3. Characterization of $(\mathcal{E}^\mu, \mathcal{F}_e^\mu)$

In this section, we shall assume that $\boldsymbol{M}$ is recurrent in the sense of Harris relative to m. Then the decomposition (1.1) is well defined. For each measurable set C such that $0 < m(C) < \infty$, let $\mathcal{F}_{X-Y}^C = \{u \in \mathcal{F}_e^C;\, u = 0 \text{ q.e. on } Y\}$ and

$$(3.1) \qquad \mathcal{F}_e^C = \mathcal{F}_{X-Y}^C + \mathcal{H}_C^Y$$

be the orthogonal decomposition of $(\mathcal{E}^C, \mathcal{F}_e^C)$. Define H^C by

$$(3.2) \qquad H^C u(x) = E_x\left[\exp(-\Phi(\sigma_Y))\, u(X(\sigma_Y))\right].$$

Then, by the result on transient Dirichlet space, the orthogonal projection of $u \in \mathcal{F}_e^C$ on $\mathcal{H}_C^Y$ is given by $H^C u$ ([3; (5.5.6)]). In the next lemma, we shall show that the projection of $u \in \mathcal{F}_e$ on $\mathcal{H}^Y$ is given by Hu.

Lemma 3.1. *If $u \in \mathcal{F}_e$, then Hu is its orthogonal projection on $\mathcal{H}^Y$.*

Proof. We may suppose that u is bounded. Let R^{X-Y} be the potential operator of the process X killed on Y, that is,

$$R^{X-Y}f(x) = E_x[\int_0^{\sigma_Y} f(X_t)\,dt]\,,$$

then $R^{X-Y}(I_C H^C u)(x)$ belongs to $\mathcal{F}_{X-Y}$ and satisfies

$$\mathcal{E}(v, R^{X-Y}(I_C H^C u)) = (v, I_C H^C u)_{L^2(m)}\,, \tag{3.3}$$

for all $v \in \mathcal{F}^C_{X-Y}$. Since

$$\begin{aligned} R^{X-Y}(I_C H^C u)(x) &= E_x[\int_0^{\sigma_Y} H^C u(X_t)\,d\Phi_t] \\ &= E_x[\int_0^{\sigma_Y} E_{X_t}[\exp(-\Phi(\sigma_Y))\,u(X(\sigma_Y))]\,d\Phi_t] \\ &= E_x[\int_0^{\sigma_Y} \exp(-\Phi(\sigma_Y)+\Phi_t)\,u(X(\sigma_Y))\,d\Phi_t] \\ &= Hu(x) - H^C u(x)\,, \end{aligned}$$

using Lemma 2.3, (2.7), (3.3) and the remark after (3.2), we have

$$\begin{aligned} \mathcal{E}(v, Hu) &= \mathcal{E}(v, R^{X-Y}(I_C H^C u) + H^C u) = (v, I_C H^C u) + \mathcal{E}(v, H^C u) \\ &= (v, I_C H^C u) + \mathcal{E}^C(v, H^C u) - (v, H^C u)_{m_C} = 0 \end{aligned}$$

for all $v \in \mathcal{F}_{X-Y}$ under a suitable choice of C. This implies that $Hu \in \mathcal{H}^Y$.

Theorem 3.2. *$\mathcal{F}^\mu_e = \gamma\mathcal{H}^Y$ in the sense $\gamma\mathcal{H}^Y \subset \mathcal{F}^\mu_e$ and, conversely, for each $\phi \in \mathcal{F}^\mu_e$ there exists $u \in \mathcal{H}^Y$ such that $\gamma u = \phi$ μ–a.e. In this case,*

$$\mathcal{E}(u, u) = \mathcal{E}^\mu(\phi, \phi)\,. \tag{3.4}$$

Proof. Suppose that $u \in \mathcal{H}^Y$, then there exists an $\mathcal{E}$-Cauchy sequence $\{u_n\}$ of functions of $\mathcal{D}$ such that $\lim_{n\to\infty} u_n = u$ m-a.e. by Lemma 2.2. Take the set C so that u_n and u are uniformly bounded on C. Then $\{u_n\}$ is an $\mathcal{E}^C$-Cauchy sequence which converges m-a.e. to u. Since $(\mathcal{E}^C, \mathcal{F}^C)$ is transient and regular, by the 0-th order version of [3; Theorem 3.1.4], it contains a subsequence $\{u_{n_k}\}$ which converges to u q.e. Hence we have

$$\gamma u = \lim_{k\to\infty} \gamma u_{n_k} \quad \text{q.e.}\,, \tag{3.5}$$

and, in particular, the convergence is μ-a.e. By Lemma 2.2, $\{\gamma u_n\}$ forms an $\mathcal{E}^\mu$-Cauchy sequence of functions of $\mathcal{F}^\mu$. Thus we have $\gamma u \in \mathcal{F}^\mu_e$.

Suppose, conversely, that $\phi \in \mathcal{F}^\mu_e$. Then there exists a sequence $\{u_n\}$ of

functions of $\mathcal{D}$ such that $\{\gamma u_n\}$ is $\mathcal{E}^\mu$-Cauchy and $\lim_{n\to\infty} u_n=\phi$ μ-a.e. by Lemma 2.2. Since $\{Hu_n\}$ forms an $\mathcal{E}$-Cauchy sequence of functions of $\mathcal{H}^Y$ by (2.4), it converges to some $u\in\mathcal{H}^Y$ in $(\mathcal{E}, \mathcal{F}_e)$. We can see using (2.9) that Hu_n converges to u in $L^1(X; gdm)$ up to an additive constant. By taking a subsequence, we may assume that Hu_n converges to u m-a.e. As in the first part of the proof, there then exists a subsequence $\{Hu_{n_k}\}$ which converges to u q.e.
Hence we have

$$\gamma u=\lim_{k\to\infty}\gamma Hu_{n_k}=\lim_{k\to\infty}\gamma u_{n_k}=\phi \quad \mu\text{-a.e.}$$

4. Minimality of the time changed process

In the preceding section, we put no assumption on the support Y of the CAF A but assumed that $\boldsymbol{M}$ is recurrent in the sense of Harris. Under the assumption, since Y is not open in general, the regularity of $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ does not make sense.

In this section, for general regular Dirichlet space $(\mathcal{E}, \mathcal{F})$, under the assumptions that $Y=X$ and an additional finiteness condition of μ, we shall show that $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ becomes regular. Let C be a core of $(\mathcal{E}, \mathcal{F})$, that is, C is a family of functions of $C_0(X)$ which is uniformly dense in $C_0(X)$ and $\mathcal{E}_1$-dense in $\mathcal{F}$.

Theorem 4.1. *Suppose that $Y=X$ q.e. and μ satisfies one of the following conditions:*
(i) $|\mu-m|$ *is a finite measure,*
(ii) $|\mu-m|$ *is a measure with finite energy integral.*
Then $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ is a regular Dirichlet space with C as core.

Proof. Since μ is a smooth measure, there exists a strictly positive function $h\in L^1(X; \mu)$ such that

$$\int |v(x)|\,h(x)\,d\mu(x)\leqq K\,\mathcal{E}_1(v, v)^{1/2} \tag{4.1}$$

holds for some constant K and all quasi-continuous function $u\in\mathcal{F}$ (see [3; Theorem 3.2.3]).

Suppose that $u\in C$, there then exists a sequence $\{u_n\}$ of functions of $\mathcal{D}$ such that $\mathcal{E}_1(u_n-u, u_n-u)\to 0$ as $n\to\infty$ by Lemma 2.2. Hence $\lim_{n\to\infty} u_n=u$ in $L^1(X; hd\mu)$ by (4.1) and, in particular, by choosing a subsequence, $\lim_{n\to\infty} u_n=u$ μ-a.e. This implies that $u\in\mathcal{F}^\mu_e$ and $\mathcal{E}^\mu(u, u)=\mathcal{E}(u, u)$. Since $u\in L^2(X; \mu)$, we have $u\in\mathcal{F}^\mu$ (see [7; Remark 1 after Lemma 1.7]). It thus enough to show that C is $\mathcal{E}^\mu_1$-dense in $\mathcal{D}$ by Lemma 2.2.

Let $u\in\mathcal{D}$, then there exists a sequence $\{u_n\}$ of functions of C such that $\lim_{n\to\infty}\mathcal{E}_1(u_n-u, u_n-u)=0$, by the regularity of $(\mathcal{E}, \mathcal{F})$. It then follows that $\lim_{n\to\infty} u_n=u$ μ-a.e. and $\lim_{n\to\infty}\mathcal{E}^\mu(u_n-u, u_n-u)=0$, as above. It is hence enough to show that $\lim_{n\to\infty} u_n=u$ in $L^2(X; \mu)$. Since u is bounded, by truncating u_n by an upper bound of $|u(x)|$, we may assume that $\{u_n\}$ are uniformly bounded. Then, under the hypothesis (i), the $L^2(X; \mu)$-convergence of $\{u_n\}$ to u is obvious. On the other hand, if (ii) is satisfied, then

$$\int|u_n-u|(x)d|\mu-m|(x)\leqq K\mathcal{E}_1(u_n-u, u_n-u)^{1/2}, \tag{4.2}$$

for some constant K. Hence $\lim_{n\to\infty} u_n=u$ in $L^2(X; |\mu-m|)$. This shows that $\{u_n\}$ converges to u in $L^2(X; \mu)$, since it converges in $L^2(X; m)$.

Corollary 1. *Let μ be a smooth measure such that $K\mu\geqq m$ for some constant K. If μ satisfies (i) or (ii) in Theorem 4.1, then $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ is regular.*

Proof. If $K\mu\geqq m$ then, the CAF associated with $K\mu$ is strictly increasing and so is A. Hence $Y=X$ and the result follows from the Theorem.

Corollary 2. *Let μ be a positive Radon measure such that $Km\geqq\mu$ for some constant K. If $Y=X$, then $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ is regular.*

The proof is obvious, since $\mathcal{E}^\mu\leqq\mathcal{E}_K$ holds under the present condition.

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