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On Covering Property of Abstract Riemann Surfaces

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Let R be an abstract Riemann surface of finite genus belonging to the class O_{AB}, then it is well known that any covering surface on the w-plane, defined by a non-constant analytic function on R covers any point except at most a null-set, that is, the boundary of the surface of O_{AB} on the w-plane. In this paper we shall study Iversen's and Gross's property, but at present what we can prove is only that a subclass of O_{AB} has Iversen's property, thus the validity of Iversen's property of O_{AB} is an open problem.

1) We suppose a conformal metric is given on R, of which a line element is given by the local parameter $ds=\lambda(t)|dt|$, and let O be a fixed point of R. Denote by D_ρ the domain bounded by the point set having a distance $\rho:\rho<\infty$ from O, and suppose for $\rho<\infty$ that the domain D_ρ is compact, $\lim\limits_{\rho=\infty} D_\rho=R$, the boundary Γ_ρ of D_ρ is composed of $n(\rho)$ components, $r_1, r_2, \cdots, r_n$, and that $\Lambda(\rho)$ is the largest length of r_k $(k=1.2,\cdots n,)$:

$$l_k=\int\limits_{r_k} ds\,, \qquad \Lambda(\rho)=\max_k l_k\ .$$

Put
$$N(\rho)=\max_{\rho'\leqq\rho} n(\rho')\ .$$

Pfluger proved[1] that if

$$\limsup_{\rho=\infty}\left[4\pi\int\limits_{\rho_0}^{\rho}\frac{d\rho}{\Lambda(\rho)}-\log N(\rho)\right]=\infty\ ,$$

then $R\in O_{AB}$.

Theorem 1. *If*

$$\limsup_{\rho=\infty}\left[\pi\int\limits_{\rho_0}^{\rho}\frac{d\rho}{\Lambda(\rho)}-\log N(\rho)\right]=\infty \quad (\textit{genus of } R\leqq\infty)\,,$$

1) A. Pfluger: Sur l'existence de fonctions non constantes, analytiques, uniformes et bornées sur une surface de Riemann ouverte, C. R. Acad. Sci. Paris, 230, 1950, pp. 166–168,

then every connected piece of R over $|w-w_0|<S$, *covers every points except at most the null-set* [2]E_{AB}, *which is the boundary set of a domain of* O_{AB} *on the w-plane.*

Proof. If there exists a lacunary set E, which is being clearly closed, not contained in E_{AB} in $|w-w_0|<S$, we can construct a bounded analytic function $A(w)$ in the w-plane except E and regular on $|w-w_0|=S$. Define a harmonic function $U(w)$ on $|w-w_0|\leqq S$ such that $U(w)=$ real part of $A(w)$ on $|w-w_0|=S$, then it is clear that the conjugate function $V(w)$ of $U(w)$ is bounded on $|w-w_0|\leqq S$ and $A(w)-U(w)-iV(w)=B(w)$ is bounded on $|w-w_0|\leqq S$ and further $B(w)\neq$ constant. Consider the closed domain $\bar{G}$ such that[3] $Re\,(B(w))\geqq 0: |w-w_0|\leqq S$, and let V be the image of $\bar{G}$ on R, then V has relative boundaries $l_1, l_2, \cdots, l_p, \cdots$, on which the $Re\,B(w)$ vanishes.

Each l_i is non compact, since otherwise $\Im_m\ B(w)$ is not one valued on account of $\int_{l_i}\frac{\partial\ Re\ B(w)}{\partial n}\,ds>0$.

Every l_i converges to the boundary of R. Let $B(p)$ be the function $B(w)$ considered as the function on $R\frown V$, $p\in R\frown V$.

Since $B(p): p\in(R\frown V)$ is bounded, we can suppose that V is mapped on the semi-circle $|\xi|<1$ $Re\ \xi\geqq 0$ and every l_i is mapped on the imaginary axis. After Pfluger we introduce in $|\xi|<1$ the hyperbolic metric by the line element defined by $ds=\frac{|d\xi|}{1-|\xi|^2}$. Consider V in D_ρ and put $D_\rho'=D_\rho\frown V$. The boundary of D_ρ' is composed of l_i and $\sum_{i=1}^{n(\rho)}\sum_{j=1}^{i(j)} r_i^j$, where r_i^j is an arc contained in r_i. Let L_i^j be a segment on imaginary axis connecting two end-points of the image r_i^j lying on the imaginary axis, and $\tilde{L}_i^j$ be image of r_i^j. Then

$$\tilde{L}_i^j=\int_{r_i^j} ds\geqq \text{length of } L_i^j\,.$$

Let A_j^i be the area bounded by $\tilde{L}_i^j$ and L_i^j. Then by the isoperimetric problem

$$4A_i^j(A_i^j+\pi)\leqq(\tilde{L}_i^j+L_i^j)^2\leqq 4(\tilde{L}_i^j)^2\,,$$
$$4A_i(A_i+\pi)\leqq 4\tilde{L}_i^2\,,$$

2) In this article, we denote by E_{AB} the null set of O_{AB} on the plane.

3) Without loss of generality we may assume that there exists a point w_0 satisfying the real part of $B(w_0)$ is poritive.

where

$$A_i = \sum_j A_i^j, \quad \tilde{L}_i = \sum_j \tilde{L}_i^j.$$

If r_j has no common point with any l_i, then we have

$$4A_j(A_j+\pi) \leqq \tilde{L}_j^2.$$

Thus

$$4A_i(A_i+\pi) \leqq 4\tilde{L}_i^2, \quad \text{for every } i.$$

Denote by A_ρ the area of image of D_ρ'. Then $A_\rho \leqq \sum A_i$, and in the same manner as used by Pfluger, we have

$$4A_\rho\left(\pi+\frac{A_\rho}{n}\right) \leqq 4\pi(\sum A_i + \sum A_i^2) \leqq 4\sum \tilde{L}_i^2.$$

On the other hand

$$\tilde{L}_i^2 \leqq l_i \int_{r_i} \frac{\left|\frac{d\xi}{dz}\right|^2}{(1-|\xi|^2)^2}\, dz,$$

$$\sum^n \tilde{L}_i^2 \leqq \Lambda(\rho)\ \frac{dA_\rho}{d\rho},$$

$$A_\rho\left(\pi+\frac{A_\rho}{n}\right) \leqq \Lambda(\rho)\frac{dA_\rho}{d\rho},$$

hence

$$\frac{A_{\rho_0}}{\pi+A_{\rho_0}} \leqq N(\rho)\ \exp\left(-\pi\int_{\rho_0}^{\rho}\frac{d\rho}{\Lambda(\rho)}\right).$$

Thus by assumption A_{ρ_0} must be zero, from which the conclusion follows.

Denote by $n(w)$ the number of sheets of connected piece of R on $|w-w_0|<S$ over a point w.

Theorem 2. *Let R be a Riemann surface belonging to $O_{AB}(O_{AD})$ of finite genus and let V be a connected piece on $|w-w_0|<\rho$ such that $\underset{w\in V}{n(w)} \leqq N < \infty$. Then V covers every point except at most a null-set $E_{AB}(E_{AD})$.*

Denote by D_N set of points of projection of V such that $n(w)=N$. Then from the lower semi-continuity of $n(w)$, it is clear that D_N is an open set and the boundary B_N of D_N is closed.

1) B_N is a totally disconnected set. If it were not so, take a continuum-component B_N' of B_N and a point p such that $n(p)=\max n(w)$

$=S: w \in B_N$, and let $v(p)$ be a neighbourhood of p with boundary l such that l has at least one component $l'(\in D_N)$ of $(l-B_N')$ and $v(\rho) \bigcap B_N') \not\subset D_{S'+1}$. Since p is covered S times by V, there exists at most S discs $k_{S'}, \cdots, k_{S'}$ $(S' \leqq S)$ on v and at least another disc k_0 on v, and V on k_0 has at least a connected piece with lacunary of a continuum, larger than $v(p) \bigcap B'_N$, and at most $(N-S)$ number of relative boundary components $L_1, L_2, \cdots, L_{N'-S'}$ lying on l' $(N'-S' \leqq N-S)$. We denote such a connected piece by $\tilde{V}$. Since the genus of R is finite, it can be mapped by $w=f(p)$ onto a sub-Riemann surface R in the other closed surface R^*. R^*-R is a totally disconnected set. Consider the image of $\tilde{V}$ in R^*. Then we can see easily that every image of L_i $(i=1,2,\cdots,N'-S')$ converge to a point of R^*, because R^*-R is totally disconnected and $p=f^{-1}(w): p \in R^*$ is continuous. Denote by $\tilde{\tilde{V}}$ the domain on R^* bounded by the image L_i and by a finite number of points of a subset of R^*-R. On the other hand by assumption $v(p)$ has a continuum boundary except the projection of L_i, thus we can define a bounded (Dirichlet bounded) analytic function $\varphi(w(p))$ on $v(p)$ with vanishing real part on L_i. If $\varphi(w(p))$ is analytic in $\tilde{\tilde{V}}$, it must be a constant, therefore there exists in $\tilde{\tilde{V}}$, a closed set E where $\varphi(p)$ is not regular. Therefore by Neumann's[4] method and by Abel's integral, we can construct a bounded analytic (Dirichlet bounded) function on R, which contradicts the fact that $R \in O_{AB}(O_{AD})$.

2) Since B_N is a totally disconnected closed set, we can take a neighbourhood $V'(p)$ such that the boundary[5] of $V'(p)$ is completely contained in D_N and enclosing a lacunary set E of the connected piece. Thus by the same method as above, we can conclude that $R \overline{\in} O_{AB}(O_{AD})$.

Remark 1) If $R \in O_{AB}(O_{AD})$ covers the w-plane a bounded number of times, then we can see easily that the mapping function is regular throughout R^*, and the function must be an algebraic function.

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4) Since $E(\subset(R^*-R))$ is a closed and totally disconnected set, we can find a domain D, with relative boundary ∂D, in $\tilde{\tilde{V}}$ such that $D \supset E'$ $(E \supset E')$, distance $(\partial D$. relative boundary of $\tilde{\tilde{V}}) > 0$, and distance $(E'. \partial D) > 0$. Then by Neumann's method, we can construct a non constant harmonic function $U_1(p)$ such that $(Re\ \varphi(p) - U_1(p)$ is harmonic in $\tilde{\tilde{V}}$, $U_1(p)$ is harmonic in R^*-D, and the conjugate of $U_1(p)$ is single valued in D, therefore we can construct a bounded (Diriclet bounded) function with a linear form of Abel's first kind of integral.

5) $\tilde{\tilde{V}}$ in R^*, above defined, of every connected piece on $V'(p)$ has at most N number of analytic curves as its relative boundary.

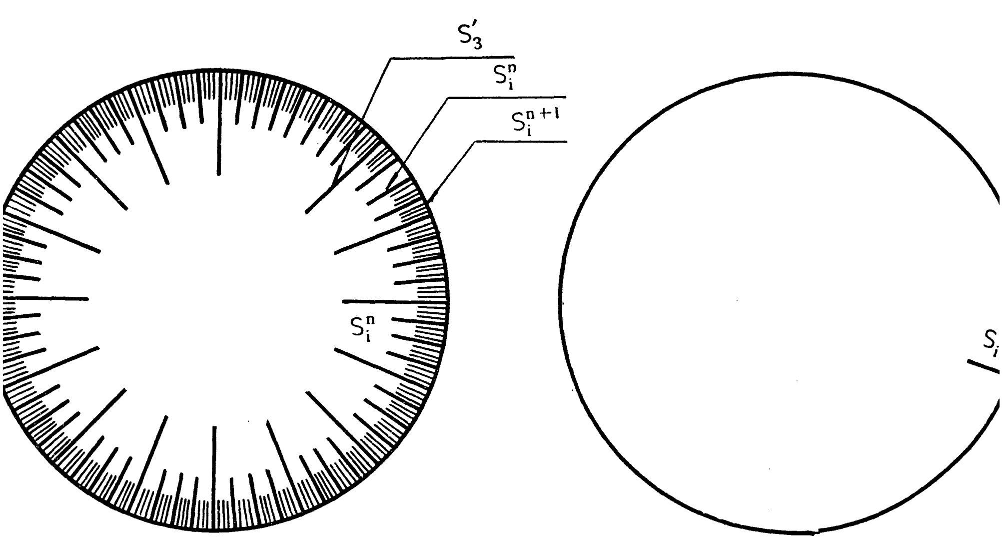
Remark 2) We conjecture that every Riemann surface belonging to O_{AB} of finite genus has Iversen's property but the present auther did not succeed to prove it.

Theorem 3. *A Riemann surface belonging to O_{AB} of finite genus has not necessarily the Gross's property.*

Example. Let F_0 be the unit-circle $|z|<1$ with slits $S_i^n : n=1,2,3,\cdots ; i=1,2,\cdots, q_n$ such that (Fig. 1)

$$S_i^n : 1-\frac{1}{p_n} \leq |z| < 1, \quad \arg z = \frac{2\pi i}{q_n},$$
$$i=1,2,3,\cdots, q_n, \quad p_n = \tilde{a}^n : \tilde{a} > 4.$$

Lemma. Let F_i^n be the unit-circle with slits S_i^n, and connect F_0 with every F_i^n on corresponding slits S_i^n crosswise, then we have infinitely many sheeted covering surface on the unit-circle. If we take q_n sufficiently large, then we have $\omega(p) \equiv 0$, where $\omega(p)$ is the harmonic measure of the boundary of F_0 on $|z|=1$.



F_0 F_i^n

$F_i^n : i=1,2,\cdots, i_0(n). \quad n=1,2,3,\cdots\cdots.$

Fig. 1

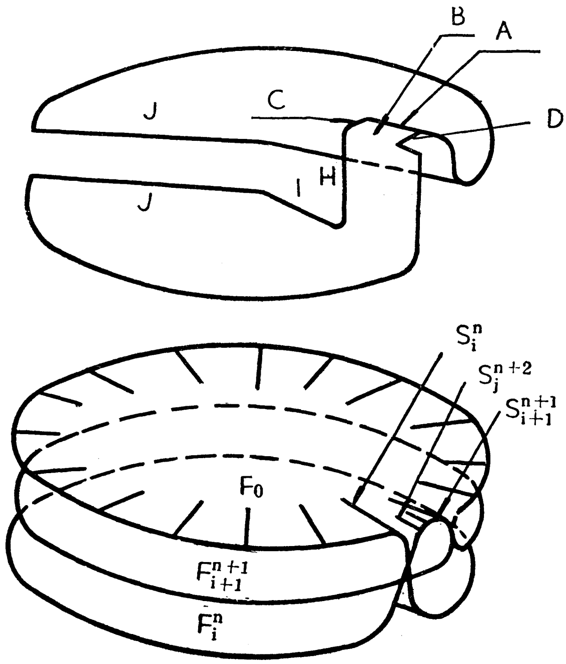


Fig. 2

We denote by $G^{m,n}_{i,i+1}$ $(n \geqq m)$ the domain of F_0 enclosed by straight lines A, B and circular arcs C, D such that (Fig. 2)

$$A: \ 1-\frac{1}{p_n} \leqq |z| \leqq 1-\frac{1}{p_{n+1}}; \ \arg A = \arg S^m_i = \arg S^n_i ,$$

$$B: \ 1-\frac{1}{p_n} \leqq |z| \leqq 1-\frac{1}{p_{n+1}}; \ \arg B = \arg S^n_{i+1} ,$$

$$C: \ |z| = 1-\frac{1}{p_{n+1}}; \ \arg S^n_{i+1} \leqq \arg z \leqq \arg S^n_i ,$$

$$D: \ |z| = 1-\frac{1}{p_n}; \ \arg S^n_{i+1} \leqq \arg z \leqq \arg S^n_{i+1} .$$

$F^m_i(F^n_{i+1})$ has a slit $S^m_i(S^n_{i+1})$ with edges ${}^+S^m_i$, ${}^-S^m_i({}^+S^n_{i+1}, {}^-S^n_{i+1})$ (S^n_i has two edges). We consider $\omega(p)$ in the surface $F^m_i + G^{m,n}_{i,i+1} + F^n_{i+1}$, where $G^{m,n}_{i,i+1}$ is connected with F^m_i on A by ${}^+S^m_i$, with F^n_{i+1} on B by ${}^-S^n_{i+1}$. $F^n_i + G^{m,n}_{i,i+1} + F^n_{i+1}$ has boundaries C, D, ${}^-S^m_i$, ${}^+S^n_{i+1}$, $({}^+S^m_i - A)$, $({}^-S^m_{i+1} - B)$ and the boundary on $|z|=1$.

Let $\omega^*(p)$ be harmonic measure of ${}^-S^m_i + {}^+S^n_{i+1} + ({}^+S^m_i - A) + ({}^-S^n_{i+1} - B)$ with respect to $F^n_i + G^{m,n}_{i,i+1} + F^n_{i+1}$. Then it is clear

$$\omega^*(p) \geqq \omega(p) .$$

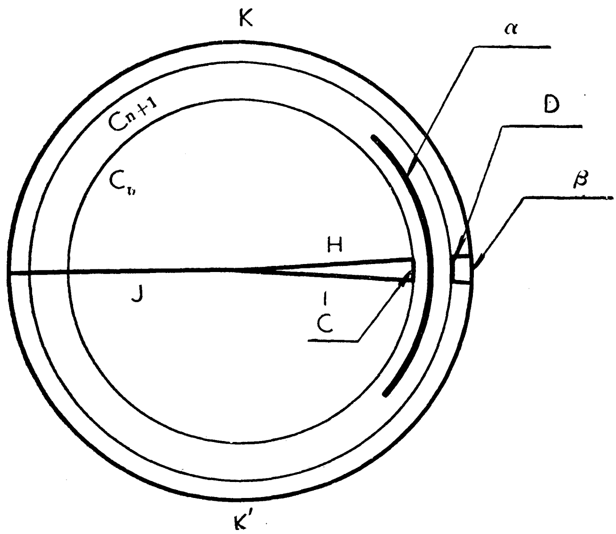


Fig. 3

Denote by $(F_i^n+G_{i,i+1}^{m,n}+F_{i+1}^m)^*$ the simply connected domain with boundaries such that (Fig. 3)

$$
\begin{aligned}
H &: \quad 0 \leqq |z| \leqq 1-\frac{1}{p_n}, \quad \arg z = \arg S_i\,, \\
E &: \quad 1-\frac{1}{p_{n+1}} \leqq |z| \leqq 1, \quad \arg z = \arg S_i\,, \\
I &: \quad 0 \leqq |z| \leqq 1-\frac{1}{p_n}, \quad \arg z = \arg S_{i+1}\,, \\
F &: \quad 1-\frac{1}{p_{n+1}} \leqq |z| \leqq 1, \quad \arg z = \arg S_{i+1}\,, \\
J &: \quad 0 \leqq |z| \leqq 1, \quad \arg z = \pi + \frac{\arg S_i + \arg S_{i+1}}{2}\,, \\
K &: \quad |z| = 1, \quad 0 \leqq \arg z \leqq \pi\,, \\
K' &: \quad |z| = 1, \quad \pi \leqq \arg z \leqq 2\pi\,,
\end{aligned}
$$

and $C+D$.

Let $\omega^{**}(p)$ be the harmonic measure of $E+D+H+I+J+F+C$. Then

$0 \leqq \omega(p) \leqq \omega^*(p) \leqq \omega^{**}(p)$. Let α be a half of the semi-circle passing through the point $|z| = \frac{1}{2}\left(2-\frac{1}{p_n}-\frac{1}{p_{n+1}}\right)$, $\arg z = \frac{1}{2}(\arg S_i + \arg S_{i+1})$.

1°) *The value of* $\omega^{**}(p) : p \in \alpha$.

Suppose $(\arg S_i + \arg S_{i+1}) = 0$, and let S be a point on $\alpha : S = \left(1-\frac{p_{n+1}-p_n}{2p_n p_{n+1}}\right)e^{i\theta}$, $|\theta| \leqq \frac{\pi}{4}$. To investigate the behaviour of $\omega^{**}(p)$,

we transform these figures by a linear function $w=\dfrac{z-S}{1-\bar{S}z}$.
Let C_n, C_{n+1} be circles such that

$$C_n : |z| = 1-\frac{1}{p_n},\quad C_{n+1} : |z| = 1-\frac{1}{p_{n+1}}.$$

Then C_n, C_{n+1} will be mapped on to circles C'_n, C'_{n+1} such that

$$C'_n : \left| w - \frac{a^2r-r}{1-a^2r}e^{i\theta}\right| = \frac{a-r^2a}{1-a^2r^2},\quad C'_{n+1} : \left| w-\frac{r^2-rb^2}{1-b^2r}e^{i\theta}\right| = \frac{b-r^2b}{1-b^2r},$$

where $$a=1-\frac{1}{p_n},\quad b=1-\frac{1}{p_{n+1}},\quad r=1-\frac{p_{n+1}-p_n}{2p_np_{n+1}}. \cdots \text{(A)}$$

1) distance $(C'_n, 0) = \dfrac{p_n(2p_np_{n+1}-p_{n+1}-p_n)}{4p_n^2p_{n+1}-3p_np_{n+1}+p_n^2+p_{n+1}-p_n}$

$$= \frac{2-\dfrac{1}{\breve{a}^n}-\dfrac{1}{\breve{a}^{n+1}}}{4-\dfrac{3}{\breve{a}^n}+\dfrac{1}{\breve{a}^{n+1}}+\dfrac{1}{\breve{a}^{2n}}+\dfrac{1}{\breve{a}^{2n+1}}}.$$

2) distance $(C'_{n+1}, 0) = \dfrac{(p_{n+1}+3p_n)p_{n+1}}{p_{n+1}^2+3p_np_{n+1}+p_{n+1}-p_n} = \dfrac{1-\dfrac{3}{\breve{a}}}{1+\dfrac{1}{\breve{a}}+\dfrac{1}{\breve{a}^{n+1}}-\dfrac{1}{\breve{a}^{n+2}}}.$

3) $z=S \longrightarrow w=0$.

4) $z=0 \longrightarrow w=-S$.

5) $z=e \longrightarrow w=e^{i\theta_1'}$: $\theta_1'=\arg S_i$.

6) $z=e \longrightarrow w=e^{i\theta_2'}$: $\theta_2'=\arg S_{i+1}$.

7) the radius $\overleftrightarrow{z=0,\ z}=e^{i\theta_1} \longrightarrow$ orthogonal circle $\widehat{e^{i\theta_1'},\ -S}$.

8) the radius $0, e^{i\theta_2} \longrightarrow$ orthogonal circle $\widehat{e^{i\theta'_2},\ -S}$.

9) the radius $0, -1 \longrightarrow$ orthogonal circle $\widehat{-re^{i\theta},\ \dfrac{-1-re^{i\theta}}{1+re^{i\theta}}}$,
(this circle tends to $e^{-i\theta}$ when $r\to 1$.)

$$z=1 \longrightarrow w=\frac{1-re^{i\theta}}{1-re^{-i\theta}} :\quad \theta \leqq \frac{\pi}{2^2}.$$

Let $\omega^\beta(z)$ be the harmonic measure of β (β lies on $|z|=1$ and $\arg S_i \leqq \arg z \leqq \arg S_{i+1}$) with respect to the unit-circle. Then $\omega^\beta(z) : z\in\alpha$ attains its maximum when $\arg z=\dfrac{1}{2}(\arg S_i+\arg S_{i+1})$, which implies that the length of the image of β is largest when $\arg S=0$, in which case the mapping function is reduced to $w=\dfrac{z-r}{1-rz}$. If we denote by β_1^* and β_2^* the end-points of the image β^* of β, then we have

$$\arg \beta_1^* = \tan^{-1}\left[\frac{\sin\theta\,(1-r^2)}{(-2r)+\cos\,(1+r^2)}\right].$$

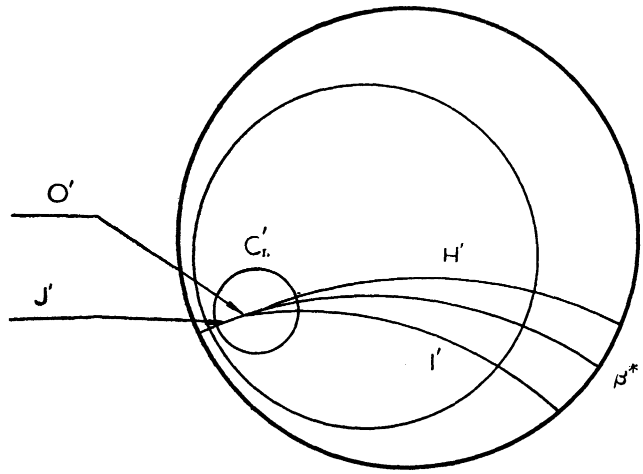


Fig. 4

If we take θ so small that

(B) $\cos\theta > \frac{2r}{1+r^2}$,

then the length of β^* is smaller than π.

Elementary calculation yields from (A) and (B)

$$\theta < \sqrt{\frac{\tilde{a}-1}{2\tilde{a}^{n+1}}},$$

i.e.,

$$q_n > 4\pi\sqrt{\frac{2\tilde{a}^{n+1}}{\tilde{a}-1}}.$$

If we consider in 9) the radius $\overleftrightarrow{0, -1}$, then the argument of its image $\arg\left(\left(\frac{-1-re^{i\theta}}{1+re^{-i\theta}}\right)\right)$ is smallest when $\theta = \frac{\pi}{4}$: thus

$$\arg\frac{-1-re^{i\theta}}{1+re^{-i\theta}} \geqq 2\left(\tan^{-1}\frac{1+\sqrt{2}}{\sqrt{2}}\right) > \frac{\pi}{2}+\varepsilon_0 \quad \left(\frac{\pi}{4} \geqq \theta \geqq 0\right),$$

and the argument of $w = re^{\theta+\pi}$ is $\theta+\pi$. Therefore the distance from $w=0$ to the image J is larger than a positive number δ_1. The same fact holds true when $-\frac{\pi}{4} \leqq \theta \leqq 0$.

Since $\omega^{**}(p)$ attains 1 only on D, E, F, H, C, I, and J, and since the distance from $w=0$ to the images of E, D, F, H, C, I, J has a positive distance larger than δ_3, and further since the length of β^* is less than π, we see that $\omega^{**}(z) \leqq \delta_4 < 1$, where δ_4 is a positive number whenever S is on α.

2°) F_i^n has a slit S_i^n. We denote by α' the part of the circle such that $|z| = 1-\frac{p_{n+1}-p_n}{2p_n p_{n+1}}$, $\frac{1}{2}(\arg S_i+\arg S_{i+1})+\frac{\pi}{4} < \arg z < -\frac{1}{2}(\arg S_i+\arg S_{i+1})+\frac{7\pi}{4}$, and denote by $\omega^{***}(z)$ the harmonic measure of S_i with respect to the domain (unit-circle$-S_i$). Then clearly $\omega^{***}(z) \leqq \delta_5 < 1: z\in\alpha'$ and $\omega(p) \leqq \omega^{***}(z)$.

Let $\tilde{\omega}_n(p)$ be a harmonic function $0 \leqq \tilde{\omega}_n(p) \leqq 1$ such that $\tilde{\omega}_n(p)=0$ on the boundary of $F_{i_1}^1, F_{i_2}^2, \cdots, F^n$ and $=1$ on the circle on F_0 with radius $=1-\frac{1}{p_{n+1}}$ and on the part of slits S_{n+1}^i contained in the part $|z| \geqq 1-\frac{1}{p_{n+1}}$. On the other hand F_0 has no common point with $F_i^{n+1}, F_j^{n+2}, \ldots$. Thus we see from 1°) and 2°)

$$\tilde{\omega}_n(p) \leqq \max(\delta_4, \delta_5),$$

where the projection of p is on the circle $|z| = 1-\frac{p_{n+1}-p_n}{2p_n p_{n+1}}$.

Let $\{V_i(p)\}$ be non negative continuous super-harmonic functions on F such that $V_i(p) \leqq 1$ and $\lim\limits_{\substack{|p|=1\\ p\in F_0}} V_i(p) = 1$, and denote $V(p)$ its lower envelope. Then

$$V(p) \leqq \max(\delta_4, \delta_5) \quad \text{on} \quad |z| = r_n \quad n=1,2,\cdots,$$

thus

$$V(p) \leqq \max(\delta_4, \delta_5)V(p), \quad \text{and} \quad V(p) \equiv 0.$$

We denote by $\hat{F}$ the symmetric surface with respect to the unit circle and identify the boundary of $F_i^n (n \geqq 1)$ with that of $\hat{F}$, then we have a planer Riemann surface $\widetilde{F}$ over the z-plane.

Proposition. $\widetilde{F}$ is contained in the class O_{AB}.[6)]

If there were a non-constant bounded analytic function $A(p) = U(p)+iV(p)$ on $\widetilde{F}$, where $\tilde{p}$ is the symmetric point of p with respect to the unit circle, then we have

$$U(p)-U(\tilde{p}) = 0, \quad V(p)-V(\tilde{p}) = 0,$$

which implies the constancy of $A(p)$. It is clear that $\widetilde{F}$ has not the Gross's property.

Theorem (*W. Gross*). *Let $z = z(p) : p\in R$ be a meromorphic function and let R be a Riemann surface belonging to O_G. If we denote by $p = p(z)$ its inverse function, if $p = p(z)$ is regular at z_0, then we can continuate $p(z)$ analytically on half lines $z = z_0+re^{i\theta}$ $(0 \leqq r < \infty)$ except a set of θ of angular measure zero.*

Thus our example is not contained in O_G.

When the genus of an abstract Riemann surface is finite, it is known

$$O_G = O_{HB} = O_{HD} \subset O_{AB} \subset O_{AD} = O_{ABD}.$$

Since there is a Riemann surface of finite genus of O_{AD} on which a non-constant bounded analytic function exists, O_{AD} has not necessarily Iversen's property. In the previous[7] paper we proved that O_G is the only class in which any Riemann surface always has Gross's property. Now even when we confine ourselves to Riemann surfaces of finite genus, we know that O_G is the maximal class in which Gross's theorem holds.

Denote by P_I, P_G the class of Riemann surfaces having Iversen's or Gross's property respectively. Then

1) Case of infinite genus

$$P_I \supset O_{HB} \supset O_{HP} \supset O_G, \quad P_I \not\supset O_{AB}, \quad P_I \not\supset O_{HD}, P_G \not\supset O_{HP}.$$

2) Case of finite genus

$$P_I \overset{?}{\supset} O_{AB}, \quad P_I \supset O_G, \quad P_I \not\supset O_{AD}, \quad P_G \supset O_G, \quad P_G \not\supset O_{AB}.$$

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6) T. Kuroda: A property of some open Riemann surfaces and its applications, Nagoya Math. Journ., Vol. 6 (1953). pp. 77–84.

7) Z. Kuramochi: On covering surfaces, Osaka Math. Journ., vol. 5 (1953). pp. 155–201.