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NONCOPRIME ACTION AND CHARACTER CORRESPONDENCES

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1. Introduction

In [7], Nagao extended the Glauberman Correspondence to the non-coprime case by restricting the attention to the S-invariant p-defect zero characters of a finite group G acted by a finite p-group S. Concretely, if G is a complemented normal subgroup of Γ and C is a set of representatives of G-conjugacy classes of complements of G in Γ, Nagao showed that there exists a natural bijection from the set of Γ-invariant p-defect zero characters of G onto $\cup_{S\in C}\{p$-defect zero characters of $C_G(S)\}$, whenever Γ/G is a p-group.

Now we want to make no assumptions on Γ/G (although we will end up making some assumptions on G) and still show that there exists a natural map from some subset of the Γ-invariant characters of G (those who have p-defect zero for the primes dividing $|\Gamma/G|$) into $\cup_{S\in C}\mathrm{Irr}(C_G(S))$.

As we mention, we pay for this extra generality: we impose some conditions on G (G must be π-separable for the set of primes π dividing $|\Gamma/G|$). Also, although defect zero characters of G will map into defect zero characters of $C_G(S)$ it will not be true, in general, that our map is onto (think on a π-group acted by another π-group with trivial fixed points subgroup). This will be the case, however, when the Hall π-subgroups of Γ are nilpotent (as it happens in Nagao's case). When Γ/G is a p-group (and G is p-solvable) we will certainly show that our map coincides with Nagao's.

The key point in this note is to consider an interesting subset of the irreducible characters of a finite group G acted by a finite group S whose order is non-necessarily coprime to $|G|$. If $\mathrm{Ind}_S(G)=\{\chi\in\mathrm{Irr}(G)$ such that $\chi=\mu^G$, where μ is an S-invariant character of an S-invariant subgroup H of G with order coprime to $S\}$, then there exists a natural one to one map from $\mathrm{Ind}_S(G)$ into $\mathrm{Irr}(C_G(S))$. We will show that the image of $\chi\in\mathrm{Ind}_S(G)$ is $\mu^{*C_G(S)}$, where $\mu^*\in\mathrm{Irr}(C_H(S))$ is the Glauberman-Isaacs correspondent of $\mu\in\mathrm{Irr}_S(H)$. Of course, one of the problems in this note will be to show that if μ induces irreducibly to G, then μ^* induces irreducibly to $C_G(S)$ (this was done in [6] when the

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$(|G|, |S|)=1$. Now, of course, we are not assuming that the orders of G and S are coprime).

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2. Preliminaries

While Nagao makes use of general block theory for proving his correspondence, the tools we use here to prove ours are basically our main result in [6] and Isaacs π-theory. Since modular theory for sets of primes is only available for π-separable groups we have to restrict ourselves from the very beginning to this class of groups.

If S acts on G coprimely, let us denote by $*: \mathrm{Irr}_S(G) \to \mathrm{Irr}(C_G(S))$ the Glauberman-Isaacs correspondence. Next is our main result in [6].

(2.1) **Theorem.** *Suppose that S acts on G coprimely and assume that H is an S-invariant subgroup of G. If $\mu \in \mathrm{Irr}_S(H)$ induces $\mu^G \in \mathrm{Irr}(G)$ then $(\mu^G)^* = \mu^{*C_G(S)}$.*

Proof. See Theorem A of [6].

If π is any set of primes, let us say that $\chi \in \mathrm{Irr}(G)$ has π-defect zero if $\chi(1)_\pi = |G|_\pi$ (i.e., χ has p-defect zero for any prime p in π).

The following are easy properties of π-defect zero characters.

(2.2) **Proposition.**

(a) *Let H be a subgroup of G and let $\mu \in \mathrm{Irr}(H)$ with $\mu^G = \chi \in \mathrm{Irr}(G)$. Then χ has π-defect zero if and only if μ has π-defect zero.*

(b) *If N is a normal subgroup of G and $\chi \in \mathrm{Irr}(G)$ has π-defect zero, then every irreducible constituent of χ_N has π-defect zero.*

Proof. See, for instance, (3.2) of [1].

The next result is less trivial. The referee has found a shorter proof of it by using projective representations.

(2.3) **Theorem.** *Suppose that χ is a π-defect zero character of a π-separable group G. If $\chi_{O_{\pi'}(G)}$ is homogeneous, then G is a π'-group.*

Proof. Let (U, θ) be a maximal π-factorable subnormal pair of G below χ(see(3.1) and (3.2) of [4]). Now, since U is subnormal in G and χ has π-defect zero, by (2.2.b) it follows that θ has π-defect zero. Because θ is π-factorable, by definition, we can write $\theta = \alpha\beta$, where $\alpha \in \mathrm{Irr}(U)$ is π-special and $\beta \in \mathrm{Irr}(U)$

is π'-special. Let H be a Hall π-subgroup of U. Then $|H|=\theta(1)_\pi=\alpha(1)$. Now, since α is π-special, by Proposition (6.1) of [2], α_H is irreducible. By degrees, necessarily $H=1$ and thus $U\subseteq O_{\pi'}(G)$. Since the irreducible characters of $O_{\pi'}(G)$ are obviously π-factorable, by maximality $U=O_{\pi'}(G)$. (This shows that the maximal π-factorable subnormal pairs below a π-defect zero character are of the form $(O_{\pi'}(G), \theta)$. By (4.5) of [4], θ is G-invariant if and only if G is a π'-group. This proves the theorem.

In [4], for π-separable groups, G, Isaacs constructed a canonical set of irreducible complex characters, $B_\pi(G)$, whose restrictions to the classes of the π-elements of G behave like the irreducible Brauer characters (this set of "irreducible" restrictions is denoted by $I_\pi(G)$ ([5]) and, of course, when $\pi=p'$, $I_\pi(G)=IBr(G)$.

The way of defining $B_\pi(G)$ is complicated. Basically, for each $\chi\in\mathrm{Irr}(G)$ (where G is a π-separable group), Isaacs associates to χ, in a canonical way, a pair (W,γ), where $W\subseteq G$, $\gamma\in\mathrm{Irr}(W)$ is π-factorable and $\gamma^G=\chi$ (see (4.6) of [4]). The pair (W,γ) is uniquely determined up to G-conjugacy and the pairs (W,γ) in the G-class are called the nuclei for χ. $B_\pi(G)$ are those $\chi\in\mathrm{Irr}(G)$ such that γ is π-special.

It is well known that p-defect zero characters restricted to the p-regular classes are irreducible Brauer characters. The same happens for π-defect zero characters.

(2.4) **Theorem.** *If $\chi\in\mathrm{Irr}(G)$ has π-defect zero, where G is a π-separable group, then $\chi\in B_{\pi'}(G)$.*

Proof. Let (W,γ) be a nucleus for χ. Since $\gamma^G=\chi$, by (2.2a), γ has π-defect zero. Since γ is π-factorable, the same argument used in (2.3) tells us that W is a π'-group. Therefore γ is π'-special and thus $\chi\in B_{\pi'}(G)$.

3. The set $\mathrm{Ind}_S(G)$

For convenience let us write our hypothesis.

(3.1) **Hypothesis.** Suppose that S acts on G and let $\Gamma=GS$ be the semidirect product. If π is the set of primes dividing $|S|$, we will assume that G, and therefore Γ, is π-separable.

We will denote by $\mathrm{Ind}_S(G)=\{\chi\in\mathrm{Irr}(G)$ such that $\chi=\mu^G$, where μ is an S-invariant character of an S-invariant subgroup H of G with $(|H|, |S|)=1\}$.

If $\chi\in\mathrm{Ind}_S(G)$, then $\chi(1)_\pi=|G|_\pi$ and thus χ has π-defect zero. Therefore, by (2.4), $\chi\in B_{\pi'}(G)$. Since Γ/G is a π-group and χ is Γ-invariant, by (6.3) of [4], χ has a unique extension $\hat{\chi}\in B_{\pi'}(\Gamma)$.

Our first (easy) objective is to show that if $\chi\in\mathrm{Ind}_S(G)$ then χ has some

S-invariant constituent upon restriction to a normal subgroup. The following will be widely generalized in Section 5.

(3.2) **Theorem.** *If $\chi \in \mathrm{Ind}_S(G)$ and Y is a normal S-invariant π'-subgroup of G, then χ_Y has some S-invariant irreducible constituent.*

Proof. Write $\chi=\mu^G$, where $\mu \in \mathrm{Irr}_S(H)$, H is S-invariant and $(|H|, |S|) =1$. Then HY is also S-invariant and has order coprime with $|S|$. Now $\mu^{HY} \in \mathrm{Irr}_S(HY)$ and by (13.27) of [3], $(\mu^{HY})_Y$, and hence χ_Y, has an S-invariant irreducible constituent.

Now we want to distinguish some of the S-invariant irreducible constituents of χ_Y, where $\chi \in \mathrm{Ind}_S(G)$ and Y is as in (3.2). We will say that $\alpha \in \mathrm{Irr}_S(Y)$ is good for $\chi \in \mathrm{Ind}_S(G)$ if there exists an S-invariant π'-subgroup H of G containing Y with some $\mu \in \mathrm{Irr}_S(H|\alpha)$ such that $\mu^G=\chi$. Observe that in Theorem (3.2) it is shown that there exists a good constituent for any $\chi \in \mathrm{Ind}_S(G)$.

We need an immediate fact about good constituents.

(3.3) **Proposition.** *Let $\chi \in \mathrm{Ind}_S(G)$, let Y be a normal S-invariant π'-subgroup of G and let $\alpha \in \mathrm{Irr}_S(Y)$ be an irreducible constituent of χ_Y. Then α is good for χ if and only if the Clifford correspondent of χ over α lies in* $\mathrm{Ind}_S(T)$ *where $T=I_G(\alpha)$ is the stabilizer of α in G.*

Proof. Let $\eta \in \mathrm{Irr}(T|\alpha)$ be the Clifford correspondent of χ over α (i.e., $\eta^G=\chi$). If α is good for χ we may choose an S-invariant π'-subgroup H of G with $\mu \in \mathrm{Irr}_S(H)$ over α and with $\mu^G=\chi$. Since $T \cap H$ is the inertia subgroup of α in H, we pick $\tau \in \mathrm{Irr}_S(T \cap H|\alpha)$ with $\tau^H=\mu$. Then $\tau^G=\chi$ and by the uniqueness of the Clifford correspondent, $\tau^T=\eta$. This shows that $\eta \in \mathrm{Ind}_S(T)$. On the other hand, if $\eta=\delta^T$, where $\delta \in \mathrm{Irr}_S(J)$ and J is a π'-subgroup of T, then $(\delta^{JY})^G=\chi$ and since δ^{JY} lies over α, α is good for χ.

A key result in this paper will be to show that good constituents for $\chi \in \mathrm{Ind}_S(G)$ are $C_G(S)$-conjugate. This is something which requires, we believe, a nontrivial amount of π-theory.

First of all we need the following application of Glauberman's Lemma (13.8 and 13.9 of [3]).

(3.4) **Lemma.** *Suppose that S acts on G coprimely. Let $N \subseteq M \subseteq G$ be normal S-invariant subgroups of G, and let $\chi \in \mathrm{Irr}_S(G)$ lying over $\theta \in \mathrm{Irr}_S(N)$. Then there exists $\eta \in \mathrm{Irr}_S(M)$ lying under χ and over θ.*

Proof. See Lemma (2.3) of [8].

(3.5) **Theorem.** *Assume* (3.1). *Suppose that $\chi \in \mathrm{Ind}_S(G)$ and let $\theta \in \mathrm{Irr}_S(Y)$ be a good constituent for χ, where Y is a normal S-invariant π'-subgroup* *of G. Then there exists a nucleus (V, γ) of $\hat{\chi}$ with $YS \subseteq V$ and with γ_Y containing θ. Also, $(V \cap G, \gamma_{V \cap G})$ is a nucleus for χ and $V \cap G$ is a π'-group.*

Proof. We argue by induction on $|G|$. First of all we claim that there exists an S-invariant pair (U, α), where $U = O_{\pi'}(G)$, with $(Y, \theta) \leq (U, \alpha) \leq (G, \chi)$ and with α good for χ. To prove the claim, suppose that $\chi = \mu^G$, where $\mu \in \mathrm{Irr}_S(H)$, H is an S-invariant π'-subgroup of G and μ_Y contains θ. Now consider $\mu^{HU} \in \mathrm{Irr}_S(HU)$. By the previous Lemma we may choose $\alpha \in \mathrm{Irr}_S(U)$ over θ and under μ^{HU}. Certainly α is good for χ and this proves the claim.

Now (U, α) is a π-factorable subnormal pair of Γ below $\hat{\chi} \in B_{\pi'}(\Gamma)$. By (3.2) of [4], we may choose (X, η) a maximal π-factorable subnormal pair of Γ such that $(U, \alpha) \leq (X, \eta) \leq (\Gamma, \hat{\chi})$. By (5.2) of [4], observe that η is π'-special. Since $|X : X \cap G|$ is a π-number and η has π'-degree, we have that $\eta_{X \cap G}$ is irreducible. Since $X \cap G \triangleleft X$, by (4.1) of [2], $\eta_{X \cap G}$ is also π'-special and, in particular, π-factorable. As it was said in the proof of (2.3), since χ has π-defect zero, we know that (U, α) is a maximal π-factorable subnormal pair below χ. Therefore $U = X \cap G$ and hence X/U is a π-group. By Lemma (6.1) of [4], S fixes X. Since $\eta_{X \cap G} = \alpha$ and X/U is a π-group, η is the unique π'-special character of X over α ((6.1) of [2]). Therefore η is S-invariant and by the same reasons, $T \cap G = I_G(\alpha)$, where $T = I_\Gamma(X, \eta)$ (see (4.4) of [4]). Observe that $S \subseteq T$.

Now, by (4.4) of [4], we can find $\psi \in \mathrm{Irr}(T | \eta)$ such that $\psi^\Gamma = \hat{\chi}$ and notice that $(\psi_{T \cap G})^G = \chi$ and that $\psi_{T \cap G}$ is the Clifford correspondent of χ over α. Since α is good for χ, by (3.3), then $\psi_{T \cap G} \in \mathrm{Ind}_S(T \cap G)$.

We want now to apply an inductive hypothesis, so we must check that θ is good for $\psi_{T \cap G}$. But this is easy: since by (3.3) α is good for $\psi_{T \cap G}$ and θ lies under α, certainly θ is good for $\psi_{T \cap G}$. Now, since $\psi \in B_{\pi'}(T)$ (because, by definition, the nuclei for ψ are nuclei for $\hat{\chi}$), if follows that $\widehat{\psi_{T \cap G}} = \psi$. If $T < G$, the theorem follows by induction.

If α is G-invariant, by (2.3), G is a π'-group, $\hat{\chi}$ is π'-special (because $\hat{\chi}$ has π'-degree and lies in $B_{\pi'}(\Gamma)$, (5.4) of [4]), and hence $\hat{\chi}$ is π-factorable. Then, $V = \Gamma$ and this proves the theorem.

We will give a more general result of the following in Section 5. Now we prove what we really need to show the existence of our correspondence.

(3.6) **Corollary.** *Assume* (3.1). *Let $\chi \in \mathrm{Ind}_S(G)$ and let α and $\beta \in \mathrm{Irr}_S(O_{\pi'}(G))$ be good for χ. Then α and β are conjugate in $C_G(S)$.*

Proof. By Theorem (3.5), there exist nuclei (V, γ) and (W, η) for $\hat{\chi}$ such that $S \subseteq V \cap W$ and with $\gamma_{O_{\pi'}(G)}$ and $\eta_{O_{\pi'}(G)}$ containing α and β, respectively. Since $(O_{\pi'}(G), \alpha)$ and $(O_{\pi'}(G), \beta)$ are maximal π-factorable pairs below χ, and

$(V\cap G, \gamma_{V\cap G})$ and $(W\cap G, \eta_{W\cap G})$ are nuclei for χ, it follows that $\gamma_{O_{\pi'}(G)}$ and $\eta_{O_{\pi'}(G)}$ are multiples of α and β, respectively. Now by (3.2) of [4], $(V, \gamma)^g=(W, \eta)$, for some $g\in G$. Since S^g and S are Hall π-subgroups of $W=(W\cap G)S$, it follows that $S^{gw}=S$, for some $w\in W\cap G$. Then $gw\in C_G(S)$ and $\gamma^{gw}=\eta^w=\eta$. Therefore, $\alpha^{gw}=\beta$, as wanted.

4. A correspondence of characters

We need an easy Lemma.

(4.1) **Lemma.** *Suppose that S acts on G and let Y be a normal S-invariant subgroup of G with $(|Y|, |S|)=1$. If $\theta\in \mathrm{Irr}_S(Y)$ then $I_G(\theta)\cap C_G(S)=I_{C_G(S)}(\theta^*)$.*

Proof. By naturality, if x is any automorphism of YS fixing S, we have that $(\theta^x)^*=(\theta^*)^x$.

(4.2) **Theorem.** *Assume* (3.1) *and suppose that H is an S-invariant subgroup of G with $(|H|, |S|)=1$. Let $\alpha\in\mathrm{Irr}_S(H)$ with $\alpha^G\in\mathrm{Irr}(G)$. Then $(\alpha^*)^{C_G(S)}\in\mathrm{Irr}(C_G(S))$. Also, if J is another S-invariant subgroup of G with $(|J|, |S|)=1$ and $\beta\in\mathrm{Irr}_S(J)$ is such that $\beta^G\in\mathrm{Irr}(G)$, then $\alpha^G=\beta^G$ if and only if $(\alpha^*)^{C_G(S)}=(\beta^*)^{C(_GS)}$.*

Proof. We argue by induction on $|G|$. Let $U=O_{\pi'}(G)$, $K=HU$ and $\mu=\alpha^K\in\mathrm{Irr}_S(K)$. By Theorem A of [6], we have that $\mu^*=\alpha^{*C_K(S)}\in\mathrm{Irr}(C_K(S))$.

Now let $\theta\in\mathrm{Irr}_S(U)$ be an irreducible constituent of μ_U. Since α^G has π-defect zero and θ is a constituent of $(\alpha^G)_U$, by (2.3), it follows that $T=I_G(\theta)<G$ or G is a π'-group. In the latter case, $K=G$ and $\alpha^{*C_G(S)}=\alpha^{*C_K(S)}$ is irreducible. So we may assume that $T<G$.

Since $T\cap K=I_K(\theta)$, let $\delta\in\mathrm{Irr}(T\cap K|\theta)$ with $\delta^K=\mu$. By uniqueness, notice that δ is S-invariant. Again, by Theorem A of [6], $\delta^{*C_K(S)}=\mu^*$ is irreducible. Now, $\delta^T\in\mathrm{Irr}(T)$, $T\cap K$ is an S-invariant subgroup of T with $(|T\cap K|, |S|)=1$ and by induction, $\delta^{*C_T(S)}=(\delta^T)^*$ is irreducible. Since δ lies over θ, by (5.3) of [9], δ^* lies over θ^*. By (4.1), $C_T(S)=I_{C_G(S)}(\theta^*)$ and hence $\delta^{*C_G(S)}\in\mathrm{Irr}(C_G(S))$. Now, $\alpha^{*C_G(S)}=\mu^{*C_G(S)}=\delta^{*C_G(S)}$ is irreducible.

Now, suppose that J is another S-invariant subgroup of G with $(|J|, |S|)=1$ and that $\beta\in\mathrm{Irr}_S(J)$ is such that $\beta^G\in\mathrm{Irr}(G)$. Let $L=JU$ and let $\eta=\beta^L\in\mathrm{Irr}_S(L)$. Let $\nu\in\mathrm{Irr}_S(U)$ be an irreducible constituent of η_U and let $I=I_G(\nu)$. Since $I\cap L=I_L(\nu)$, we may choose $\tau\in\mathrm{Irr}(I\cap L|\nu)$ with $\tau^L=\eta$. By Theorem A of [6], we have that $\beta^{*C_L(S)}=\eta^*=\tau^{*C_L(S)}$.

Suppose first that $\alpha^G=\beta^G=\chi$. We want to show that $\alpha^{*C_G(S)}=\beta^{*C_G(S)}$, and certainly, we may replace (L, η) and (K, μ) by $C_G(S)$-conjugates. Now $\chi\in\mathrm{Ind}_S(G)$ and ν and θ are good constituents for χ. By (3.6), we know that ν and θ are $C_G(S)$-conjugate. So we may assume in fact that $\nu=\theta$ and hence $I=T$.

Also, $\delta^T=\tau^T$, because both are the Clifford correspondents of χ over $\theta=\nu$.

If $T=G$, then G is a π'-group, and then $\alpha^{*C_G(S)}=\chi^*=\beta^{*C_G(S)}$, by Theorem A of [6]. If $T<G$, by induction, $\delta^{*C_T(S)}=\tau^{*C_T(S)}$, and then $\alpha^{*C_G(S)}=\delta^{*C_G(S)}=\tau^{*C_G(S)}=\beta^{*C_G(S)}$.

Suppose now that $\alpha^{*C_G(S)}=\beta^{*C_G(S)}=\varepsilon$. Since both θ^* and ν^* lie under ε, it follows that $\theta^{*c}=\nu^*$ for some $c\in C_G(S)$. Then $\theta^c=\nu$ and certainly we may assume that $\theta=\nu$. In this case, $\delta^{*C_T(S)}=\tau^{*C_T(S)}$, because $C_T(S)=I_{C_G(S)}(\theta^*)$ and both are the Clifford correspondents of ε over θ^*. If G is a π'-group, by Theorem A of [6], we have that $(\alpha^G)^*=(\beta^G)^*$ and then $\alpha^G=\beta^G$. Otherwise, $T<G$ and by induction, $\delta^T=\tau^T$ and hence $\alpha^G=\delta^G=\tau^G=\beta^G$.

By Theorem (4.2), we have defined an injective map (which we will continue denoting by *) from $\mathrm{Ind}_S(G)$ into $\mathrm{Irr}(C_G(S))$. The image of this map is in the set of π-defect zero characters of $C_G(S)$, but we do not know exactly what it is in general. We will have control on it, however, when the Hall π-subgroups of Γ are nilpotent. Another observation is that we have assumed π-separability on G. Is this really necessary? Since the relationship between Glauberman-Isaacs correspondents is so tight, perhaps Theorem (4.2) is true with complete generality.

5. Clifford theory and the correspondence

Suppose that $\chi\in\mathrm{Ind}_S(G)$ and let N be a normal S-invariant subgroup of G. When N is a π'-group, we distinguished in $\mathrm{Irr}_S(N)$ the good constituents of χ_N. Now, in more genrality, we say that $\theta\in\mathrm{Irr}_S(N)$ is *good* for $\chi\in\mathrm{Ind}_S(G)$ if θ lies under χ and the Clifford correspondent of χ over θ lies in $\mathrm{Ind}_S(I_G(\theta))$. By (3.3), observe that when N is a π'-group the new definition agrees with that in Section 3.

Now we give a Clifford type theorem for Ind_S-characters. It also extends Corollary (3.6).

(5.1) **Theorem.** *Assume* (3.1). *Let* $\chi\in\mathrm{Ind}_S(G)$ *and let N be a normal S-invariant subgroup of G. Then there exists a good* $\theta\in\mathrm{Irr}_S(N)$ *for* χ *and all of them are conjugate in* $C_G(S)$. *Also, good constituents are* Ind_S*-characters.*

Proof. We argue by induction on $|G|$. Let $Y=O_{\pi'}(N)$ and let $\alpha\in\mathrm{Irr}_S(Y)$ be good for χ. Let $\mu\in\mathrm{Ind}_S(T)$ be the Clifford correspondent of χ over α and observe that if δ is any irreducible constituent of $\mu_{T\cap N}$, then $\delta^N\in\mathrm{Irr}(N)$ and $I_G(\delta^N)\cap T=I_T(\delta)$, by Clifford theory.

Suppose first that $N=Y$. By (3.2) and (3.3), in this case we only have to prove that if α and β are two good irreducible constituents of χ_N, then α and β are $C_G(S)$-conjugate. By (3.5), we know that there exists S-invariant nuclei (V,γ) and (W,η) for χ, where V and W are π'-groups, such that α and β are irreducible constituents of γ_N and η_N, respectively. Now the same argument given in (3.6) shows us that $(V, \gamma)^c = (W, \eta)$, for some $c \in C_G(S)$. Therefore, α^c and β are two S-invariant irreducible constituents of η_N. By Glauberman's Lemma (13.9) of [3], in the action in (13.27) of [3], α^c and β are $C_W(S)$-conjugate and hence $C_G(S)$-conjugate.

Now suppose that $Y < N$ and hence $T < G$ (if α is G-invariant, since every irreducible constituent of χ_N has π-defect zero, by (2.3), $Y = N$). Then, by induction, $\mu_{T \cap N}$ has some good irreducible constituent, all of them are $C_T(S)$-conjugate and lie in $\mathrm{Ind}_S(T \cap N)$. If δ is any one of them, notice that $\delta^N \in \mathrm{Ind}_S(N)$. Let $I = I_G(\delta^N)$ and let $\varepsilon \in \mathrm{Irr}(I \cap T \mid \delta)$ be with $\varepsilon^T = \mu$. Since δ is good for μ, it follows that $\varepsilon \in \mathrm{Ind}_S(I \cap T)$. Now, $\varepsilon^G = \chi \in \mathrm{Irr}(G)$, $\varepsilon^I \in \mathrm{Irr}(I \mid \delta^N)$ is the Clifford correspondent of χ over δ^N and also $\varepsilon^I \in \mathrm{Ind}_S(I \cap T)$. Therefore, δ^N is good for χ and lies in $\mathrm{Ind}_S(N)$.

Now suppose that $\tau \in \mathrm{Irr}_S(N)$ is also good for χ and let $\psi \in \mathrm{Irr}(I_G(\tau))$ the Clifford correspondent of χ over τ. Let $\alpha_0 \in \mathrm{Irr}_S(Y)$ be a good consitutent for ψ and observe that α_0 is good for χ and that α_0 lies under τ. By the first part of the proof, α_0 is $C_G(S)$-conjugate to α and hence it is no loss of generality to assume that $\alpha_0 = \alpha$. Since α is good for ψ, let $\xi \in \mathrm{Ind}_S(I_G(\tau) \cap T)$ over α be such that $\xi^{I_G(\tau)} = \psi$. Then $\xi^T = \mu$, by the uniqueness of the Clifford correspondents and, since $(\xi_{T \cap N})^N$ is a multiple of τ, again we have that $\xi_{T \cap N}$ is a multiple of some $\phi \in \mathrm{Irr}(T \cap N)$ with $\phi^N = \tau$. Now, since $I_G(\tau) \cap T = I_T(\phi)$, it follows that ϕ is good for μ. By induction, $\phi = \delta^c$ for some $c \in C_G(S)$. Then $(\delta^N)^c = (\delta^c)^N = \phi^N = \tau$ and the theorem is proved.

Now we want to relate normal subgroups and the correspondence.

(5.2) **Theorem.** *Assume* (3.1). *Let N be a normal S-invariant subgroup of G and let $\theta \in \mathrm{Ind}_S(N)$ be invariant in G. If $\chi \in \mathrm{Ind}_S(G)$, then $[\chi_N, \theta] \neq 0$ if and only if $[\chi^*_{C_N(S)}, \theta^*] \neq 0$.*

Proof. We argue by induction on $|G|$. Suppose first that N is a π'-group. Since $\chi \in \mathrm{Ind}_S(G)$, by the very definition, we may find an S-invariant pair (W, γ) with $N \subseteq W$, with W a π'-group and with $\gamma^G = \chi$. Then $\chi^* = \gamma^{*C_G(S)}$. Notice that $[\chi_N, \theta] \neq 0$ if and only if $[\gamma_N, \theta] \neq 0$. By (5.3) of [9], $[\gamma_N, \theta] \neq 0$ if and only if $[\gamma^*_{C_N(S)}, \theta^*] \neq 0$. Since θ^* is $C_G(S)$-invariant (because $((\theta^*)^x = (\theta^x)^*$ for any automorphism x of NS fixing S), $[\gamma^*_{C_N(S)}, \theta^*] \neq 0$ if and only if $[\chi^*_{C_N(S)}, \theta^*] \neq 0$, as wanted.

Suppose now that $Y = O_{\pi'}(N) < N$ and let $\alpha \in \mathrm{Irr}_S(Y)$ be good for χ. Let $T = I_G(\alpha)$ and, by (3.3), let $\mu \in \mathrm{Ind}_S(T)$ the Clifford correspondent of χ over α. Observe, again, that if δ is any irreducible constituent of $\mu_{T \cap N}$, then $\delta^N \in \mathrm{Irr}(N)$ and $I_G(\delta^N) \cap T = I_T(\delta)$, by Clifford theory. By the definition of the map we have that $\chi^* = (\mu^*)^{C_G(S)}$ and $(\delta^N)^* = (\delta^*)^{C_N(S)}$. Also $T < G$.

Suppose first that θ lies under χ. Since $(\mu^{TN})_N$ is a multiple of θ, $\mu_{T\cap N}$ is a multiple of some $\delta\in\mathrm{Irr}(T\cap N)$, where δ is the Clifford correspondent of θ over α. By (5.1), observe that $\delta\in\mathrm{Ind}_S(T\cap N)$. By induction, we have that μ^* lies over δ^*. Since $\mu^{*C_G(S)}=\chi^*$ and $\delta^{*C_G(S)}=\theta^*$, χ^* lies over θ^*, as wanted.

Suppose now that χ^* lies over θ^*. We know that θ^* is $C_G(S)$-invariant, and thus $\chi^*_{C_N(S)}$ is a multiple of θ^*. By (5.1), let $\eta\in\mathrm{Ind}_S(N)$ be under χ. By the first part of the proof, η^* lies under χ^*. Therefore, $\eta^*=\theta^*$ and hence $\eta=\theta$, as wanted.

With the help of Theorem (5.2), we can now show that if the Hall π-subgroups of Γ are nilpotent, then $\mathrm{Ind}_S(G)^*$ is exactly the set of π-defect zero characters in $\mathrm{Irr}(C_G(S))$.

Firxt, we need an easy fact about B_π-characters.

(5.3) **Lemma.** *Let G be a π-separable group and let $\chi\in B_\pi(G)$. Suppose that $1=G_0\lhd G_1\lhd\cdots\lhd G_s=G$ is a normal series of G where every G_i/G_{i+1} is a π-group or a π'-group. If χ_{G_i} is homogeneous for every i, then χ has π-degree.*

Proof. We argue by induction on $|G|$. Write $\chi_{G_1}=e\theta$, where $\theta\in B_\pi(G_1)$ (by (7.5) of [4]) and θ has π-degree by induction. If G/G_1 is a π-group, then e is a π-number and so is $\chi(1)$. If G/G_1 is a π'-group, by (6.5) of [4], $e=1$ and the result follows.

(5.4) **Theorem.** *Assume* (3.1). *Let $\alpha\in\mathrm{Irr}(C_G(S))$ be a π-defect zero character. If the Hall π-subgroups of Γ are nilpotent, there exists $\chi\in\mathrm{Ind}_S(G)$ with $\chi^*=\alpha$.*

Proof. Let N be a normal S-invariant subgroup of G and suppose that $\alpha_{C_N(S)}$ is not homogeneous. Let $\nu\in\mathrm{Irr}(C_N(S))$ be a constituent of $\alpha_{C_N(S)}$ and let $\tau\in\mathrm{Irr}(I\mid\nu)$ be such that $\tau^{C_G(S)}=\alpha$, where $I=I_{C_G(S)}(\nu)$. By (2.2), ν and τ have π-defect zero. By induction, let $\theta\in\mathrm{Ind}_S(N)$ be such that $\theta^*=\nu$ and write $T=I_G(\theta)$. Since $T\cap C_G(S)=I<C_G(S)$, it follows that $T<G$. By induction, let $\psi\in\mathrm{Ind}_S(T)$ be such that $\psi^*=\tau$. By (5.2), ψ lies over θ and hence $\psi^G\in\mathrm{Irr}(G)$. By the definition of $\mathrm{Ind}_S(G)$ and the map, $\psi^G\in\mathrm{Ind}_S(G)$ and $(\psi^G)^*=(\psi^*)^{C_G(S)}=\tau^{C_G(S)}=\alpha$. So we may assume that for any normal S-invariant subgroup N of G, $\alpha_{C_N(S)}$ is homogeneous.

Since Γ is π-separable, we may produce a normal series in $C_G(S)$ with π or π'-factors by intersecting with $C_G(S)$ a chief series of Γ. Thus, by (2.4) and (5.3), α has π'-degree. Since α has π-defect zero, it follows that $C_G(S)$ is a π'-group. If G itself is a π'-group the Theorem is true by the Glauberman-Isaacs Correspondence. Otherwise, if $H>1$ is an S-invariant Hall π-subgroup of G, since HS is nilpotent, we have $C_H(S)>1$, which is a contradiction.

Finally, we point out that when S is a p-group (and G is p-solvable) our map coincides with Nagao's. If we assume (3.1) and C is a complete set of representatives of G-conjugacy classes of complements of G in Γ, first we show that the set of Γ-invariant π-defect zero characters of G is exactly the disjoint union $\bigcup_{Q\in C}\mathrm{Ind}_Q(G)$. Secondly, we will show that if $P, Q\in C$, and $C_G(P)=C_G(Q)$ has a p-defect zero character then $P=Q$. Nagao's map will be the "disjoint union" of our maps.

If χ is a Γ-invariant π-defect zero character of G, we know that $\chi\in B_{\pi'}(G)$ and that there is a unique $\hat{\chi}\in B_{\pi'}(\Gamma)$ extending χ. If (V, γ) is a nucleus for, $\hat{\chi}$ then by (6.2) of [4], $(V\cap G, \gamma_{V\cap G})$ is a nucleus for χ, where $V\cap G$ is a π'-group (because χ has π-defect zero). Now, if Q is a Hall π-subgroup of V, then $V=(V\cap G)Q$ with $Q\cap G=1$ and hence Q is a complement of G in Γ (because $(\gamma^{\Gamma})_G$ is irreducible). By conjugating by an appropriate element we may assume that $Q\in C$ and therefore, that $\chi\in\mathrm{Ind}_Q(G)$. Also, if $\chi\in\mathrm{Ind}_P(G)\cap\mathrm{Ind}_Q(G)$, where $P, Q\in C$, by (3.5), we know that P and Q are Hall π-subgroups of two nucleus of $\hat{\chi}$. By (3.5), the nuclei of $\hat{\chi}$ are Γ-conjugate. Since $GP=GQ=\Gamma$, it follows that Q and P are G-conjugate, as wanted.

For the second part, since groups with a p-defect zero character have no nontrivial normal p-subgroups, it suffices to show the following.

(5.5) **Lemma.** *Suppose that G is a normal complemented subgroup of Γ, where Γ/G is a p-group. Let P and Q be complements of G in Γ and assume that $C_G(P)=C_G(Q)=D$. If $O_p(D)=1$, then P and Q are G-conjugate.*

Proof. Let $M=C_\Gamma(D)$. Since M contains both P and Q, $M=P(M\cap G)=Q(M\cap G)$. Now, $C_{M\cap G}(Q)=D\cap M\cap G=C_D(D)=Z(D)$ is a p'-group. Now we claim that $|M\cap G|$ is not divisible by p, and observe that if the claim is proved, by the Schur-Zassenhaus Theorem, the lemma follows. Let T be a Sylow p-subgroup of M containing Q. Then $T\cap M\cap G$ is a Q-invariant Sylow p-subgroup of $M\cap G$. If $M\cap G$ is divisible by p, then $C_{T\cap M\cap G}(Q)$ is nontrivial and this is a contradiction with the fact that $C_{M\cap G}(Q)$ is a p'-group.

To end, by (12.1) of [7], it suffices to porve the following. (Recall that in the Glauberman correspondence, when the group acting is a p-group, the correspondent of χ is the unique irreducible constituent χ^* of $\chi_{C_G(S)}$ with multiplicity not divisible by p. Also $[\chi_{C_G(S)}, \chi^*]\equiv 1 \bmod p$ ((13.14) and (13.21) of [3])).

(5.6) **Theorem.** *Assume* (3.1) *with S a p-group. Let $\chi\in\mathrm{Irr}_S(G)$ and let $\eta\in\mathrm{Ind}_S(G)$. Then $[\chi_{C_G(S)}, \eta^*]\equiv[\chi, \eta] \bmod p$.*

Proof. Write $\eta=\delta^G$, where $\delta\in\mathrm{Irr}_S(J)$ and J is an S-invariant π'-subgroup of G. Since χ_J is S-invariant, we certainly may write

$$\chi_J = \Delta+\textstyle\sum_{\Lambda\in\Lambda} a_\Lambda(\sum_{\mu\in\Lambda}\mu)\,,$$

where every irreducible constituent of Δ is S-invariant, and $\mathbf{\Lambda}$ is the set of the nontrivial S-orbits of irreducible constituents of χ_J. If we pick $\mu_\Lambda \in \Lambda$ for every $\Lambda \in \mathbf{\Lambda}$, we may write

$$\chi_{C_J(S)} = \Delta_{C_J(S)} + \sum_{\Lambda \in \mathbf{\Lambda}} a_\Lambda |\Lambda| (\mu_\Lambda)_{C_J(S)} .$$

Now $[\chi_{C_G(S)}, \eta^*] = [\chi_{C_G(S)}, (\delta^*)^{C_G(S)}] = [\chi_{C_J(S)}, \delta^*] \equiv [\Delta_{C_J(S)}, \delta^*] \equiv [\Delta, \delta] \bmod p$, where the last congruence follows by (13.14) and (13.21) of [3].

Since δ is S-invariant, $[\Delta, \delta] \equiv [\chi_J, \delta] = [\chi, \eta] \bmod p$, as wanted.

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