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SEMICOLOCAL PAIRS AND FINITELY COGENERATED
INJECTIVE MODULES

Dedicated to Professor Yukio Tsushima on his 60th birthday

MARI MORIMOTO and TAKESHI SUMIOKA

(Received February 2, 1999)

Let $P$ and $Q$ be rings, and $pM$, $N_Q$ and $pV_Q$ a left $P$-module, a right $Q$-module and a $P$-$Q$-bimodule, respectively. Let $\varphi : M \times N \to V$ be a $P$-$Q$-bilinear map. Then we say that $(pM, N_Q)$ is a pair with respect to $\varphi$ or simply a pair (see [12], [14], [10] or [1, Section 24]). For elements $x \in M$, $y \in N$ and for submodules $pX \leq pM$, $Y_Q \leq N_Q$, by $xy$ we denote the element $\varphi(x, y)$, and by $XY$ we denote the $P$-$Q$-subbimodule of $pV_Q$ generated by $\{xy | x \in X, y \in Y\}$. A pair $(pM, N_Q)$ is said to be colocal if $pMN_Q$ is colocal both as a left $P$-module and as a right $Q$-module. In [10] and [7], we studied colocal pairs related to some results in [5] and [4].

We shall define a semicolocal pair $(pM, N_Q)$ as a generalization of a colocal pair. A $P$-$Q$-bimodule $pU_Q$ is said to be semicolocal if (i) the rings $P$ and $Q$ have complete sets $\{e_1, e_2, \ldots, e_m\}$ and $\{f_1, f_2, \ldots, f_n\}$ of orthogonal idempotents, respectively such that each $e_iU_Q$ and each $pUf_j$ are colocal modules and (ii) the socle of $pU$ coincides with the socle of $U_Q$. Moreover a pair $(pM, N_Q)$ is said to be semicolocal if $pMN_Q$ is semicolocal. Anh and Menini investigated semicolocal modules with some conditions related to duality (see [2]). In this note, we shall give some generalizations of results of [10] and [7] using the term "semicolocal pairs", and in particular give characterizations of finitely cogenerated injective modules (Theorems 2.4 and 2.5).

Throughout this note, $P$, $Q$ and $R$ are rings with identity and all modules are unitary. Let $M$ be a module. Then $L \leq M$ ($L < M$) signifies that $L$ is a (proper) submodule of $M$. By $S(M)$, $T(M)$ and $|M|$, we denote the socle, the top and the composition length of $M$, respectively. Moreover by $\Pi(R)$, we denote the set of primitive idempotents of $R$. Every homomorphism is written on the side opposite to the scalars.

1. Semicolocal pairs

A module $M_R$ is said to be colocal if $M_R$ has an essential simple socle.

Lemma 1.1. Let $f$ be an idempotent of $R$ and $M_R$ a colocal module with $S(M_R) \cong T(hR_f)$ for some $h \in \Pi(Q)$, where $Q = fRf$. Then $Mf_Q$ is a colocal module with $S(Mf_Q) = S(M_R)f = S(M_R)hQ$. 

Proof. Let $0 \neq x = xf \in S(M_R)f$ and $0 \neq y = yf \in M_f$. Then $xR = S(M_R) \leq yR$, so $xQ \subseteq yQ$. This shows that $M_fQ$ is a colocal module and $S(M_fQ) = S(M_R)f$. Moreover $S(M_R)hQ = S(M_R)f$ holds since $0 \neq S(M_R)hQ \leq S(M_R)f$.

A $P$-$Q$-bimodule $pUQ$ is said to be colocal (resp. faithful) if both $pU$ and $UQ$ are colocal (resp. faithful).

**Remark 1.** For a $P$-$Q$-bimodule $pUQ$, the following hold.

1. Both $S(pU)$ and $S(UQ)$ are subbimodules of $pUQ$.
2. If $pUQ$ is a colocal bimodule, then $S(pU) = S(UQ)$.
3. For any idempotents $e \in P$ and $f \in Q$, $S(eUQ) = eS(UQ)$ and $S(pUf) = S(pU)f$.

A finite set $\{e_1, e_2, \ldots, e_n\}$ of orthogonal idempotents of $R$ is said to be complete if $e_1 + e_2 + \cdots + e_n = 1 \in R$.

Let $P$ and $Q$ be rings. Then a $P$-$Q$-bimodule $pUQ$ is said to be semicolocal if the following conditions (i) and (ii) are satisfied.

(i) The rings $P$ and $Q$ have complete sets $\{e_1, e_2, \ldots, e_m\}$ and $\{f_1, f_2, \ldots, f_n\}$ of orthogonal idempotents, respectively such that each $e_iUQ$ and each $pUf_j$ are colocal modules.

(ii) $S(pU) = S(UQ)$.

Let $pM$ and $N_Q$ be modules and $(pM, N_Q)$ a pair and put $U = pMN_Q$. Then the pair $(pM, N_Q)$ is said to be semicolocal if $pUQ$ is a semicolocal bimodule.

**Remark 2.** If $pUQ$ is a bimodule and $e$ and $e'$ are idempotents of $P$ with $eP \cong e'P$, then $eUQ \cong e'UQ$. This is easily seen since there exist elements $a = eae'$ and $b = e'be$ in $P$ such that $ab = e$ and $ba = e'$.

**Remark 3.** Let $P$ and $Q$ be semiperfect rings. Then by Remark 2, a bimodule $pUQ$ is semicolocal if and only if for each $g \in \Pi(P)$ and each $h \in \Pi(Q)$ with $gU \neq 0$ and $Uh \neq 0$, $gUQ$ and $pUh$ are colocal modules and $S(pU) = S(UQ)$.

Let $R$ be a semiperfect ring and $e$ and $f$ idempotents of $R$. Then in [16], Xue defined a Nakayama pair $(eR, Rf)$ as a generalization of an $i$-pair in [4] (also see [5, Theorem 3.1]). We define a Nakayama pair $(eU, Uf)$ for a bimodule $pUQ$ and idempotents $e \in P$ and $f \in Q$ (see the condition 4 in [2, Theorem 3.3]). An idempotent $e$ of $R$ is said to be local if $eRe$ is a local ring.

Let $P$ and $Q$ be rings and $pUQ$ a $P$-$Q$-bimodule. First, for local idempotents $g \in P$ and $h \in Q$, $(gU, Uh)$ is called a Nakayama pair if $gUQ$ and $pUh$ are colocal modules and $S(gUQ) \cong T(hQ)Q$ and $S(pUh) \cong T(pPg)$. Generally for idempotents $e \in P$ and $f \in Q$ with semiperfect rings $ePe$ and $fQf$, $(eU, Uf)$ is called a
Nakayama pair if for each $g \in \text{Pi}(ePe)$ (resp. $h \in \text{Pi}(fQf)$) there exists $h \in \text{Pi}(fQf)$ (resp. $g \in \text{Pi}(ePe)$) such that $(gU, Uh)$ is a Nakayama pair (see Remark 2).

Let $\rho M_R$ be a $P$-$R$-bimodule and $f$ an idempotent of $R$ and put $Q = fRf$. Then we always assume that a pair $(\rho M, RfQ)$ signifies the pair with respect to the $P$-$Q$-bilinear map $\varphi : M \times Rf \rightarrow Mf$ defined by $\varphi(x, af) = xaf$; $x \in M$, $af \in Rf$.

Let $(\rho M, N_Q)$ be a pair. Then for any subsets $A \subseteq M$ and $B \subseteq N$, we define submodules $r(A) = r(A) = \{y \in N | Ay = 0\}$ and $l(B) = l(B) = \{x \in M | xB = 0\}$. We say that the pair $(\rho M, N_Q)$ is left faithful (resp. right faithful) if $l(N) = 0$ (resp. $r(M) = 0$) holds, and $(\rho M, N_Q)$ is faithful if it is left and right faithful.

Let $M_R$ and $N_R$ be semisimple modules. Then by $M_R \sim N_R$, we mean that any simple submodule of $M_R$ is isomorphic to a submodule of $N_R$ and the converse is also satisfied.

Lemma 1.2. Let $M_R$ be a module and $f$ an idempotent of $R$ such that $(\rho M, RfQ)$ is a left faithful pair, where $P = \text{End}_M R$, $Q = fRf$. If $MfQ$ is colocal, then $M_R$ is colocal with $S(M_R) = S(MfQ)R$.

Therefore, if $Q$ is a semiperfect ring and $(\rho M, RfQ)$ is a faithful semicolocal pair, then $M_R$ is a direct sum of a finite number of colocal right $R$-modules and $S(M_R) \sim T(fR_R)$ holds, and in particular $M_R$ is finitely cogenerated.

Proof. Let $0 \neq x = xf \in S(MfQ)$ and $0 \neq y \in M_R$. Since $(\rho M, RfQ)$ is left faithful, we have $ya \neq 0$ for some $a = af \in Rf$. Hence $xQ \leq yaQ$, so $xR \leq yaR \leq yR$. This shows that $M_R$ is a colocal module with $S(M_R) = S(MfQ)R$. Assume that $Q$ is a semiperfect ring and $(\rho M, RfQ)$ is a faithful semicolocal pair. Since $\rho MfQ$ is a faithful bimodule, $P$ and $Q$ have complete sets $G = \{g_1, g_2, \ldots, g_m\}$ and $H = \{h_1, h_2, \ldots, h_n\}$ of orthogonal primitive idempotents, respectively such that each $g_iMf$ and each $\rho Mf h_j$ are colocal modules and $S(\rho Mf) = S(MfQ)$. Hence for any $g \in G$ (resp. $h \in H$) there exists $h \in H$ (resp. $g \in G$) such that $S(\rho Mf)h = g S(MfQ)h = g S(fMf)h \neq 0$, so $S(gM_R) \cong T(hR_R)$ by using the first assertion. Thus $S(M_R) \sim T(fR_R)$ holds.

For local idempotents $g$ and $h$ of $R$, $(gR, Rh)$ is a Nakayama pair if and only if $(gRgR, RhR_h)$ is a faithful colocal pair (e.g. see [7, Lemma 3.2]). In the following proposition, the equivalence $(1) \iff (3)$ is a generalization of this fact.

Proposition 1.3. Let $\rho U_Q$ be a bimodule and $g$ and $h$ local idempotents of $P$ and $Q$, respectively. Then the following are equivalent.

1. $(gU, Uh)$ is a Nakayama pair.
2. Both $gU_Q$ and $\rho Uh$ are colocal and $g S(U_Q)h = g S(\rho U)h \neq 0$ holds.
3. $(g(pgU, QhQ)h)$ is a left faithful pair and $(g(pgU, Uh)hQh)$ is a right faithful
pair and \( g_{P} g_{U} h_{Q} \) is a colocal bimodule.

Proof. (1) \( \implies \) (2). By assumption, \( g S(U) h = S(g U) h \neq 0 \). Since \( S(U) h \) is a non-zero submodule of a colocal module \( P U h \), \( S(P U) h = S(P U h) \leq S(U) h \), so \( g S(P U) h \leq g S(U) h \). Hence \( g S(P U) h = g S(U) h \neq 0 \) by symmetry.

(2) \( \implies \) (3). By Lemma 1.1.

(3) \( \implies \) (1). By Lemma 1.2.

In the following proposition we give characterizations of a semicolocal bimodule \( P U Q \) for semiperfect rings \( P \) and \( Q \). The proposition is essentially due to [2, Theorems 3.3 and 3.4] (also see [16, Theorem 3.4], [8, Proposition 1.11] and [9, Theorem 2.2]).

**Proposition 1.4.** Let \( P \) and \( Q \) be semiperfect rings and \( P U Q \) a bimodule such that \( g U \neq 0 \) and \( U h \neq 0 \) for any \( g \in \text{Pi}(P) \) and any \( h \in \text{Pi}(Q) \). Then the following are equivalent.

1. \( P U Q \) is semicolocal.
2. \((U, U) = (1_P U, U 1_Q))\) is a Nakayama pair.
3. Both \( P U \) and \( U Q \) have essential socles, and \( P U Q \)-duals of simple modules are simple.
4. For each \( g \in \text{Pi}(P) \) and each \( h \in \text{Pi}(Q) \), \( g U Q \) and \( P U h \) are colocal, and \( S(P U) \leq S(U Q) \) and \( S(U Q) \leq S(P U) \).

Proof. (1) \( \implies \) (2). By assumption, for any \( h \in \text{Pi}(Q) \) we have \( S(P U) h = S(P U h) \leq P S(U Q) h \) since \( P S(U Q) h \) is a non-zero submodule of a colocal module \( P U h \). This shows \( S(P U) \leq S(U Q) \) and by symmetry \( S(P U) = S(U Q) \).

(1) \( \implies \) (3). Let \( g \in \text{Pi}(P) \). Then we have \( \text{Hom}_P(T(P g), U)_Q \cong g S(P U)_Q \). Hence \( \text{Hom}_P(T(P g), U)_Q \cong g S(U Q)_Q \) is simple, and by symmetry \( P \text{Hom}_Q(T(h Q), U) \) is simple for any \( h \in \text{Pi}(Q) \).

(3) \( \implies \) (1). Let \( g \in \text{Pi}(P) \). Since \( \text{Hom}_P(T(P g), U)_Q \cong g S(P U)_Q \), \( g S(P U)_Q \) is a simple submodule of \( g U Q \). Hence we have \( g S(P U)_Q \leq S(g U Q)_Q = g S(U Q)_Q \). This shows \( S(P U)_Q \leq S(U Q)_Q \) and by symmetry \( S(P U) = S(U Q) \). Therefore \( S(g U Q)_Q = g S(P U)_Q \) is simple and similarly \( P S(P U h) \) is simple for any \( h \in \text{Pi}(Q) \). Thus \( g U Q \) and \( P U h \) are colocal.

In Proposition 1.4, the condition (3) is equivalent to the following condition (3)’ since in the proof of (3) \( \implies \) (1), for any \( g \in \text{Pi}(P) \), \( g S(P U)_Q \) is a simple submodule of a colocal module \( g U Q \) and \( g S(P U)_Q = g S(U Q)_Q \) holds.

(3)’ For each \( g \in \text{Pi}(P) \) and each \( h \in \text{Pi}(Q) \), \( g U Q \) and \( P U h \) are colocal, and \( P U Q \)-duals of simple left \( P \)-modules are simple.
Lemma 1.5. Let \((pM, NQ)\) be a semicolocal pair and \(Y' < Y \leq N_Q\) with \(Y' = rl(Y')\). If \(Y/Y'_Q\) is simple, then \(pI(Y')/I(Y)\) is also simple and \(Y = rl(Y)\).

Proof. Put \(U = pMN_Q, X = I(Y)\) and \(X' = I(Y')\). Since \(pUQ\) is semicolocal and \(Y/Y'\) is simple, there exist an idempotent \(f \in Q\) and an element \(y = yf \in Y\) such that \(pUf\) is colocal and \(Y = yQ + Y' \leq N_Q\). From \(rl(Y') = Y' < Y \leq rl(Y)\), we obtain \(X = I(Y) < I(Y') = X'\). For any \(x \in X'\), the left multiplication map \(\hat{x} : Y/Y'_Q \to xy_Q\) by \(x\) is an epimorphism. This shows that \(xy_Q \leq S(U_Q)\), so \(X'Y_Q \leq S(U_Q)\). Therefore we have \(0 \neq X'Y \leq S(U_Q)f = S(pUf) = S(pUf)\). Thus \(pX'y = S(pUf)\) is a simple left \(P\)-module. On the other hand, the map \(\eta : pX'/X \to pX'y \text{ defined by } (x + X)\eta = xy\) is a monomorphism. Thus \(pI(Y')/I(Y) = pX'/X\) is simple. By the same argument, it follows that \(rl(Y)/rl(Y'_Q)\) is simple. Hence we have \(Y = rl(Y)\) from \(rl(Y') = Y' \leq Y \leq rl(Y)\).

We say that a pair \((pM, N_Q)\) satisfies \(l\text{-ann} \text{ (resp. } r\text{-ann} \text{) if } lr(X) = X \text{ (resp. } rl(Y) = Y\) hold for any \(X \leq pM\) (resp. \(Y \leq N_Q\)), and \((pM, N_Q)\) is dual if \((pM, N_Q)\) satisfies \(l\text{-ann} \text{ and } r\text{-ann} \).

In the following theorem, the implications \((1) \iff (2) \implies (3)\) are essentially due to [12, Theorem 1.1] (and [14, Theorem 1.1]).

Theorem 1.6. Let \(P\) and \(Q\) be rings and \((pM, N_Q)\) a faithful semicolocal pair, and consider the following conditions.

1. \(|N_Q| < \infty\).
2. \(|pM| < \infty\).
3. \((pM, N_Q)\) is a dual pair.

Then the implications \((1) \iff (2) \implies (3)\) hold, and in case either \(P\) or \(Q\) is a perfect ring, the conditions are equivalent.

Proof. The implications \((1) \iff (2) \implies (3)\) are easily seen from Lemma 1.5 (see the proof of [10, Theorem 1.4]).

Assume that \((pM, N_Q)\) is a dual pair and \(P\) is a perfect ring. Then any factor module of \(pM\) has finite Goldie dimension (see [3, Corollary 1.6] or [11, Theorem 1.7]). Hence by the proof of [13, Propositions 2.9 and 2.12] (or [11, Lemma 1.9]) \(pM\) has finite length.

2. Finitely cogenerated injective modules

Throughout this section, we always assume that \(R\) is a semiperfect ring.

Let \(M_R\) and \(L_R\) be right \(R\)-module modules. Following Harada [6], \(M\) is said to be \(L\)-simple-injective if for any submodule \(K\) of \(L_R\) any homomorphism \(\theta : K_R \to M_R\) can be extended to a homomorphism \(\eta : L_R \to M_R\). Moreover \(M\) is said to be simple-injective if \(M\) is \(N\)-simple-injective for any right \(R\)-module \(N\).
Lemma 2.1 (see [7, Lemma 4.1]). Let $M_R$ be a finitely cogenerated module with $S(M_R) \sim T(fR_R)$ and assume that $M f_Q$ has finite Loewy length, where $f$ is an idempotent of $R$ and $Q = fRf$. If $M_R$ is $R$-simple-injective, then $M_R$ is injective.

Proof. Since $S(M_R)$ is essential in $M_R$ and $S(M_R) \sim \Omega(fR_R)$ holds. Hence $L f \neq 0$ for any non-zero submodule $L \leq M_R$ because $L f = L R f$. Let $I$ be a non-zero right ideal of $R$ and $\theta : I \rightarrow M$ a non-zero homomorphism and put $J = \text{rad}(R)$. Then $0 \neq \theta(I(fJf)^k) = \theta(I)(fJf)^k \leq S(M f_Q)$ for some integer $k \geq 0$. Put $K = I(fJf)^k R$. Since $S(M f_Q) R = l_{M_R}(fJf)^k R \leq l_{M_R}(J) = S(M_R)$, $\theta(K) \leq S(M_R)$ holds. By assumption we have $S(M_R) = S_1 \oplus \cdots \oplus S_n$ for a finite number of simple modules $S_i$ ($1 \leq i \leq n$). Hence the restriction map $\theta|_K : K \rightarrow M$ of $\theta$ can be represented as $\theta|_K = \theta_1 + \cdots + \theta_n$ for some homomorphisms $\theta_i : K \rightarrow M$ with $\text{Im} \theta_i \leq S_i$ ($1 \leq i \leq n$). Therefore we have $(\theta - \hat{x})(K) = 0$ with left multiplication $\hat{x} : R \rightarrow M$ by some element $x \in M$. If $\theta - \hat{x} : I \rightarrow M$ is a non-zero homomorphism, then $(\theta - \hat{x})(I)f \neq 0$ (i.e. $k \geq 1$) and $0 \neq (\theta - \hat{x})(I(fJf)^m) \leq S(M f_Q)$ for some integer $m$ with $k > m \geq 0$. Iterating the above argument, we have $(\theta - \hat{x})(I) = 0$ for some element $y \in M$. Thus $M_R$ is injective. \hfill $\Box$

The following lemma is related to [9, Theorem 1.6].

Lemma 2.2 (see [10, Corollary 2.6]). Let $U_Q$ be a module with $P = \text{End} U_Q$ and $g$ and $h$ local idempotents of $P$ and $Q$, respectively. If $gU_Q$ is a $U$-simple-injective module and $0 \neq x = gxh \in S(gU_Q)$, then $gU_Q$ and $P U h$ are colocal modules with $S(gU_Q) = xQ \cong T(hQ_Q)$ and $S(P U h) = Px \cong T(Pg)$. Therefore, for any idempotents $e \in P$ and $f \in Q$ with semiperfect rings $eP e$ and $fQ f$, if $eU_Q$ is a $U$-simple-injective module and $S(eU_Q)$ is essential in $eU_Q$ with $S(eU_Q) \sim T(fQ_Q)$, then $(eU, U f)$ is a Nakayama pair.

Proof. By [10, Lemma 2.2] (or [7, Lemma 3.6]), $gU_Q$ is a colocal module with $S(gU_Q) = xQ$. Let $0 \neq y \in U h$. Then we have $r_{hQ}(y) \leq hJ = r_{hQ}(x)$, where $J = \text{rad}(Q)$. Hence the map $\theta : yQ \rightarrow gU$ via $\theta(y c) = xc$ ($c \in Q$) is well-defined. Therefore by $U$-simple-injectivity of $gU_Q$ we have $x = ay$ for some $a \in \text{Hom}_{Q}(U, gU) = gP$. Thus $x \in P y$, which implies that $P U h$ is a colocal module with $S(P U h) = Px$. \hfill $\Box$

Lemma 2.3. Let $M$ be a finitely cogenerated simple-injective right $R$-module with $S(M_R) \sim T(fR_R)$, where $f$ is an idempotent of $R$, and assume that $\text{End} M$ is a semiperfect ring. Then $(P M, R f Q)$ is a faithful semicolocal pair, where $P = \text{End} M$ and $Q = fRf$. \hfill $\Box$
Proof. By Lemma 2.2 for each \( g \in \Pi(P) \) (resp. \( h \in \Pi(Q) \)) such that \((gM, Rh)\) is a Nakayama pair. Therefore by Lemma 1.1 for a bimodule \( _PM_fQ \), \((gM, Mf)\) is a Nakayama pair. On the other hand by the proof of [10, Lemma 2.1], \((s_\Phi s_M, RfhQ_\Phi)\) is faithful. This shows that \((PM, Rf_\Phi)\) is a faithful semicolocal pair by Proposition 1.4. \( \square \)

Generalizing [10, Theorem 2.7] and [7, Theorem 4.2], we have the following theorem.

**Theorem 2.4.** Let \( M \) be a finitely cogenerated right \( R \)-module with \( S(MR) \sim T(fR_R) \), where \( f \) is an idempotent of \( R \), and put \( P = \text{End} _MR \) and \( Q = fRf \). Consider the following conditions.

(1) \( MR \) is injective.

(2) \( MR \) is simple-injective and \( P \) is a semiperfect ring.

(3) \((PM, Rf_\Phi)\) is a faithful semicolocal pair satisfying r-ann.

(4) \( MR \) is \( R \)-simple-injective.

Then the implications \((1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)\) hold. Moreover, in case \( MF_\Phi \) has finite Loewy length, these conditions are equivalent.

Proof. Note that in case \((PM, Rf_\Phi)\) is left faithful, \( l_M(I) = l_M(If) \) holds for any right ideal of \( I \) of \( R \).

(1) \( \Rightarrow \) (2). This is clear.

(2) \( \Rightarrow \) (3). By Lemma 2.3 \((PM, Rf_\Phi)\) is a faithful semicolocal pair. Let \( L_\Phi \) be a submodule of \( Rf_\Phi \). Assume that \( L < rl(L) \). Then \((rl(L)R/LR)f \neq 0\), so \((rl(L)R/LR)h \neq 0\) for some \( h \in \Pi(Q) \). Hence there exist right ideals \( I \) and \( K \) of \( R \) such that \( LR < K < I \leq rl(L)R_R \) and \( 1/K_R \cong T(hR) \). Therefore \( l(L) \geq l(Kf) \geq l(If) \geq lrl(L) = l(L) \). Thus \( l_M(K) = l_M(Kf) = l_M(If) = l_M(I) \). On the other hand \( 1/K_R \cong T(hR) \) is isomorphic to a direct summand of \( S(M) \). Hence we have a map \( \theta : I \rightarrow M \) such that \( \text{Im} \theta \) simple and \( \text{Ker} \theta = K \). Then by simple-injectivity of \( M \), there exists an element \( x \) of \( M \) such that \( xc = \theta(c) \) for each \( c \in I \). This implies that \( x \in l_M(K) - l_M(I) \), a contradiction. Thus \( L = rl(L) \) and \((PM, Rf_\Phi)\) satisfies r-ann.

(3) \( \Rightarrow \) (4). Let \( I \) be a right ideal of \( R \) and \( \theta : I \rightarrow M \) a homomorphism with \( \text{Im} \theta \) simple, and put \( K = \text{Ker} \theta \). Then \( 1/K \cong T(hR) \) for some \( h \in \Pi(Q) \). Hence we have \( Kf < If \) because of \( Kh < Ih \). Since \((PM, Rf_\Phi)\) satisfies r-ann, \( l_M(K) = l_M(Kf) > l_M(If) = l_M(I) \). Thus we have an element \( x \in l_M(K) - l_M(I) \). Since \( 1/K \cong T(hR) \), \( I = aR + K \) for some \( a = fh \in I \). Put \( y = \theta(a) \) and \( z = xa \). Then \( y \) and \( z \) are non-zero elements of \( _P S(MR)h \). By assumption \( _P S(MR)f = l_M(J)f \leq l_M(fJf) = _P S(Mf_\Phi) = _P S(p_Mf) \) holds; where \( J = \text{rad}(R) \), and \( _P S(Mf_\Phi)h = _P S(p_Mh) \) is simple. Hence \( _P S(Mf_\Phi)h = _P S(Mf_\Phi)h = _P S(Mf_\Phi)h \) is simple, so we have \( Py = Pz = _P S(MR)h \) and in particular \( y = \varphi(z) \) for some \( \varphi \in P \). Therefore we have \( \theta(a) = \varphi(z) = \varphi(x)a \). Thus \( M \) is \( R \)-simple-injective.
Remark 4. In Theorem 2.4, the condition \( P = \text{End} M_R \) can be replaced by the condition \( \rho M_R \) is a \( P\text{-}\text{R-bimodule} \) except for the implications \((1) \implies (2) \implies (3)\).

Remark 5. For the implication \((2) \implies (3)\) in Theorem 2.4, we can give another proof by using Propositions 1.4 and 1.3, [11, Lemma 2.4] and [7, Lemma 3.4].

The following theorem is related to [16, Theorem 3.4].

**Theorem 2.5.** Let \( M \) be a right \( R \)-module and \( f \) an idempotent of \( R \) and put \( P = \text{End} M_R, \ Q = f R f \). If \( |Rf_Q| < \infty \) is satisfied, then the following are equivalent.

1. \( M_R \) is finitely cogenerated injective with \( S(M_R) \sim T(f R_R) \).
2. \( (p M, Rf_Q) \) is a faithful semicolocal pair.

Proof. In case (2) is satisfied, by Theorem 1.6 and Lemma 1.2, \( (p M, Rf_Q) \) satisfies \( r\text{-ann} \) and \( M_R \) is a finitely generated module with \( S(M_R) \sim T(f R_R) \). Thus the assertion follows from Theorem 2.4 since \( |Q_Q| \leq |Rf_Q| < \infty \).

The following proposition is related to [5, Theorem 3.1], [4, Theorem 3], [16, Theorem 3.4] and [2, Theorem 3.4]. The \( \text{"only if"} \) part of this proposition is well-known (see e.g. [1, Theorem 30.4 or Exercise 24.8]). However, for the benefit of the reader we provide a direct proof.

**Proposition 2.6.** Let \( M_R \) be a finitely generated right \( R \)-module and \( f \) an idempotent of \( R \) and assume that \( (p M, Rf_Q) \) is a faithful pair with \( |Rf_Q| < \infty \), where \( P = \text{End} M_R \) and \( Q = f R f \). Then the bimodule \( p Mf_Q \) defines a Morita duality if and only if \( (p M, Rf_Q) \) is semicolocal.

Proof. \( \text{"If"} \) part. By Theorem 2.5, \( M_R \) is injective. Hence \( P \cong \text{End} Mf_Q \) by [5, Lemma 2.1] (this lemma is valid for a semiperfect ring \( R \)). By assumption, \( Q \) is a right artinian ring and \( Mf_Q \) is finitely generated. Since \( (p Mf, Q_Q) \) is a faithful semicolocal pair with \( |Q_Q| < \infty \), \( Mf_Q \) is injective by Theorem 2.5. Thus the bimodule \( p Mf_Q \) defines a Morita duality.

\( \text{"Only if"} \) part. Since \( Q \) is a right artinian ring, \( Mf_Q \) is a finitely generated injective cogenerator. Hence by Lemma 2.3, \( (p Mf, Q_Q) \) or equivalently \( (p M, Rf_Q) \) is a semicolocal pair.

Remark 6. Let \( (p M, N_Q) \) be a pair which satisfies (i) \( (p M, N_Q) \) is a semicolocal dual pair with a faithful bimodule \( p U_Q \), where \( p U_Q = p MN_Q \), (ii) \( Q \) is a right artinian ring and (iii) \( N_Q \) has finite length. However, this situation does not necessar-
ily imply that $pU_Q$ is a dual bimodule or equivalently $(P, pU_Q, Q)$ is a Baer duality (see [8] and [2], respectively for the definitions a dual bimodule and a Baer duality). Let $R$ be a right artinian ring such that an injective hull $E_R$ of $T(R_R)$ is not finitely generated, (see e.g. [15, Remark 2.9] for such a ring $R$). Then by Lemma 2.3 and Theorem 1.6, $(pE_R, R_R)$ is a semicolocal dual pair, where $P = \text{End} E_R$. But by Theorem 1.6, $(pP, E_R)$ is not a dual pair, so $pE_R$ is not a dual bimodule. Moreover, this example shows that in Proposition 2.6, the assumption “$M_R$ is finitely generated” can not be removed.

References
