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SEMICOLOCAL PAIRS AND FINITELY COGENERATED INJECTIVE MODULES

Dedicated to Professor Yukio Tsushima on his 60th birthday

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Let P and Q be rings, and ${}_P M$, N_Q and ${}_P V_Q$ a left P -module, a right Q -module and a P - Q -bimodule, respectively. Let $\varphi : M \times N \rightarrow V$ be a P - Q -bilinear map. Then we say that $({}_P M, N_Q)$ is a pair with respect to φ or simply a pair (see [12], [14], [10] or [1, Section 24]). For elements $x \in M$, $y \in N$ and for submodules ${}_P X \leq {}_P M$, $Y_Q \leq N_Q$, by xy we denote the element $\varphi(x, y)$, and by XY we denote the P - Q -subbimodule of ${}_P V_Q$ generated by $\{xy \mid x \in X, y \in Y\}$. A pair $({}_P M, N_Q)$ is said to be colocal if ${}_P M N_Q$ is colocal both as a left P -module and as a right Q -module. In [10] and [7], we studied colocal pairs related to some results in [5] and [4].

We shall define a semicolocal pair $({}_P M, N_Q)$ as a generalization of a colocal pair. A P - Q -bimodule ${}_P U_Q$ is said to be semicolocal if (i) the rings P and Q have complete sets $\{e_1, e_2, \dots, e_m\}$ and $\{f_1, f_2, \dots, f_n\}$ of orthogonal idempotents, respectively such that each $e_i U_Q$ and each ${}_P U f_j$ are colocal modules and (ii) the socle of ${}_P U$ coincides with the socle of U_Q . Moreover a pair $({}_P M, N_Q)$ is said to be semicolocal if ${}_P M N_Q$ is semicolocal. Anh and Menini investigated semicolocal modules with some conditions related to duality (see [2]). In this note, we shall give some generalizations of results of [10] and [7] using the term “semicolocal pairs”, and in particular give characterizations of finitely cogenerated injective modules (Theorems 2.4 and 2.5).

Throughout this note, P , Q and R are rings with identity and all modules are unitary. Let M be a module. Then $L \leq M$ ($L < M$) signifies that L is a (proper) submodule of M . By $S(M)$, $T(M)$ and $|M|$, we denote the socle, the top and the composition length of M , respectively. Moreover by $\text{Pi}(R)$, we denote the set of primitive idempotents of R . Every homomorphism is written on the side opposite to the scalars.

1. Semicolocal pairs

A module M_R is said to be colocal if M_R has an essential simple socle.

Lemma 1.1. *Let f be an idempotent of R and M_R a colocal module with $S(M_R) \cong T(hR_R)$ for some $h \in \text{Pi}(Q)$, where $Q = fRf$. Then Mf_Q is a colocal module with $S(Mf_Q) = S(M_R)f = S(M_R)hQ$.*

Proof. Let $0 \neq x = xf \in S(M_R)f$ and $0 \neq y = yf \in Mf$. Then $xR = S(M_R) \leq yR$, so $xQ \leq yQ$. This shows that Mf_Q is a colocal module and $S(Mf_Q) = S(M_R)f$. Moreover $S(M_R)hQ = S(M_R)f$ holds since $0 \neq S(M_R)hQ \leq S(M_R)f$. \square

A P - Q -bimodule ${}_P U_Q$ is said to be colocal (resp. faithful) if both ${}_P U$ and U_Q are colocal (resp. faithful).

REMARK 1. For a P - Q -bimodule ${}_P U_Q$, the following hold.

- (1) Both $S({}_P U)$ and $S(U_Q)$ are subbimodules of ${}_P U_Q$.
- (2) If ${}_P U_Q$ is a colocal bimodule, then $S({}_P U) = S(U_Q)$.
- (3) For any idempotents $e \in P$ and $f \in Q$, $S(eU_Q) = eS(U_Q)$ and $S({}_P Uf) = S({}_P U)f$.

A finite set $\{e_1, e_2, \dots, e_n\}$ of orthogonal idempotents of R is said to be complete if $e_1 + e_2 + \dots + e_n = 1 \in R$.

Let P and Q be rings. Then a P - Q -bimodule ${}_P U_Q$ is said to be semicolocal if the following conditions (i) and (ii) are satisfied.

- (i) The rings P and Q have complete sets $\{e_1, e_2, \dots, e_m\}$ and $\{f_1, f_2, \dots, f_n\}$ of orthogonal idempotents, respectively such that each $e_i U_Q$ and each ${}_P U f_j$ are colocal modules.
- (ii) $S({}_P U) = S(U_Q)$.

Let ${}_P M$ and N_Q be modules and $({}_P M, N_Q)$ a pair and put $U = {}_P M N_Q$. Then the pair $({}_P M, N_Q)$ is said to be semicolocal if ${}_P U_Q$ is a semicolocal bimodule.

REMARK 2. If ${}_P U_Q$ is a bimodule and e and e' are idempotents of P with $eP \cong e'P$, then $eU_Q \cong e'U_Q$. This is easily seen since there exist elements $a = eae'$ and $b = e'be$ in P such that $ab = e$ and $ba = e'$.

REMARK 3. Let P and Q be semiperfect rings. Then by Remark 2, a bimodule ${}_P U_Q$ is semicolocal if and only if for each $g \in \text{Pi}(P)$ and each $h \in \text{Pi}(Q)$ with $gU \neq 0$ and $Uh \neq 0$, gU_Q and ${}_P U h$ are colocal modules and $S({}_P U) = S(U_Q)$.

Let R be a semiperfect ring and e and f idempotents of R . Then in [16], Xue defined a Nakayama pair (eR, Rf) as a generalization of an i -pair in [4] (also see [5, Theorem 3.1]). We define a Nakayama pair (eU, Uf) for a bimodule ${}_P U_Q$ and idempotents $e \in P$ and $f \in Q$ (see the condition 4 in [2, Theorem 3.3]). An idempotent e of R is said to be local if eRe is a local ring.

Let P and Q be rings and ${}_P U_Q$ a P - Q -bimodule. First, for local idempotents $g \in P$ and $h \in Q$, (gU, Uh) is called a Nakayama pair if gU_Q and ${}_P U h$ are colocal modules and $S(gU_Q) \cong T(hQ_Q)$ and $S({}_P U h) \cong T({}_P P g)$. Generally for idempotents $e \in P$ and $f \in Q$ with semiperfect rings ePe and fQf , (eU, Uf) is called a

Nakayama pair if for each $g \in \text{Pi}(ePe)$ (resp. $h \in \text{Pi}(fQf)$) there exists $h \in \text{Pi}(fQf)$ (resp. $g \in \text{Pi}(ePe)$) such that (gU, Uh) is a Nakayama pair (see Remark 2).

Let ${}_P M_R$ be a P - R -bimodule and f an idempotent of R and put $Q = fRf$. Then we always assume that a pair $({}_P M, Rf_Q)$ signifies the pair with respect to the P - Q -bilinear map $\varphi : M \times Rf \rightarrow Mf$ defined by $\varphi(x, af) = xaf$; $x \in M, af \in Rf$.

Let $({}_P M, N_Q)$ be a pair. Then for any subsets $A \subseteq M$ and $B \subseteq N$, we define submodules $r(A) (= r_N(A)) \leq N_Q$ and $l(B) (= l_M(B)) \leq {}_P M$, as follows: $r(A) = \{y \in N \mid Ay = 0\}$ and $l(B) = \{x \in M \mid xB = 0\}$. We say that the pair $({}_P M, N_Q)$ is left faithful (resp. right faithful) if $l(N) = 0$ (resp. $r(M) = 0$) holds, and $({}_P M, N_Q)$ is faithful if it is left and right faithful.

Let M_R and N_R be semisimple modules. Then by $M_R \sim N_R$, we mean that any simple submodule of M_R is isomorphic to a submodule of N_R and the converse is also satisfied.

Lemma 1.2. *Let M_R be a module and f an idempotent of R such that $({}_P M, Rf_Q)$ is a left faithful pair, where $P = \text{End } M_R, Q = fRf$. If Mf_Q is colocal, then M_R is colocal with $S(M_R) = S(Mf_Q)R$.*

Therefore, if Q is a semiperfect ring and $({}_P M, Rf_Q)$ is a faithful semicolocal pair, then M_R is a direct sum of a finite number of colocal right R -modules and $S(M_R) \sim T(fR_R)$ holds, and in particular M_R is finitely cogenerated.

Proof. Let $0 \neq x = xf \in S(Mf_Q)$ and $0 \neq y \in M_R$. Since $({}_P M, Rf_Q)$ is left faithful, we have $ya \neq 0$ for some $a = af \in Rf$. Hence $xQ \leq yaQ$, so $xR \leq yaR \leq yR$. This shows that M_R is a colocal module with $S(M_R) = S(Mf_Q)R$. Assume that Q is a semiperfect ring and $({}_P M, Rf_Q)$ is a faithful semicolocal pair. Since ${}_P Mf_Q$ is a faithful bimodule, P and Q have complete sets $G = \{g_1, g_2, \dots, g_m\}$ and $H = \{h_1, h_2, \dots, h_n\}$ of orthogonal primitive idempotents, respectively such that each $g_i Mf_Q$ and each ${}_P Mf h_j$ are colocal modules and $S({}_P Mf) = S(Mf_Q)$. Hence for any $g \in G$ (resp. $h \in H$) there exists $h \in H$ (resp. $g \in G$) such that $S(gMf_Q)h = g S(Mf_Q)h = g S({}_P Mf)h \neq 0$, so $S(gM_R) \cong T(hR_R)$ by using the first assertion. Thus $S(M_R) \sim T(fR_R)$ holds. □

For local idempotents g and h of R , (gR, Rh) is a Nakayama pair if and only if $({}_g R_g R, Rh_h Rh)$ is a faithful colocal pair (e.g. see [7, Lemma 3.2]). In the following proposition, the equivalence (1) \iff (3) is a generalization of this fact.

Proposition 1.3. *Let ${}_P U_Q$ be a bimodule and g and h local idempotents of P and Q , respectively. Then the following are equivalent.*

- (1) (gU, Uh) is a Nakayama pair.
- (2) Both gU_Q and ${}_P Uh$ are colocal and $g S(U_Q)h = g S({}_P U)h \neq 0$ holds.
- (3) $({}_g P_g gU, Qh_h Qh)$ is a left faithful pair and $({}_g P_g gP, Uh_h Qh)$ is a right faithful

pair and ${}_g P g U h {}_Q h$ is a colocal bimodule.

Proof. (1) \implies (2). By assumption, $g S(U_Q)h = S(gU_Q)h \neq 0$. Since $S(U_Q)h$ is a non-zero submodule of a colocal module ${}_P U h$, $S({}_P U)h = S({}_P U h) \leq S(U_Q)h$, so $g S({}_P U)h \leq g S(U_Q)h$. Hence $g S({}_P U)h = g S(U_Q)h \neq 0$ by symmetry.

(2) \implies (3). By Lemma 1.1.

(3) \implies (1). By Lemma 1.2. \square

In the following proposition we give characterizations of a semicolocal bimodule ${}_P U_Q$ for semiperfect rings P and Q . The proposition is essentially due to [2, Theorems 3.3 and 3.4] (also see [16, Theorem 3.4], [8, Proposition 1.11] and [9, Theorem 2.2]).

Proposition 1.4. *Let P and Q be semiperfect rings and ${}_P U_Q$ a bimodule such that $gU \neq 0$ and $Uh \neq 0$ for any $g \in \text{Pi}(P)$ and any $h \in \text{Pi}(Q)$. Then the following are equivalent.*

- (1) ${}_P U_Q$ is semicolocal.
- (2) $(U, U) (= ({}_P U, U {}_Q))$ is a Nakayama pair.
- (3) Both ${}_P U$ and U_Q have essential socles, and ${}_P U_Q$ -duals of simple modules are simple.
- (4) For each $g \in \text{Pi}(P)$ and each $h \in \text{Pi}(Q)$, gU_Q and ${}_P U h$ are colocal, and ${}_P S({}_P U) \sim {}_P T({}_P P)$ and $S(U_Q)_Q \sim T(Q)_Q$.

Proof. (1) \implies (2) \implies (4). These are clear (see Remark 3).

(4) \implies (1). By assumption, for any $h \in \text{Pi}(Q)$ we have $S({}_P U)h = S({}_P U h) \leq {}_P S(U_Q)h$ since ${}_P S(U_Q)h$ is a non-zero submodule of a colocal module ${}_P U h$. This shows $S({}_P U) \leq S(U_Q)$ and by symmetry $S({}_P U) = S(U_Q)$.

(1) \implies (3). Let $g \in \text{Pi}(P)$. Then we have $\text{Hom}_P(T({}_P P g), U)_Q \cong \text{gr}_U(\text{rad}(P))_Q = g S({}_P U)_Q$. Hence $\text{Hom}_P(T({}_P P g), U)_Q \cong S(gU_Q)_Q$ is simple, and by symmetry ${}_P \text{Hom}_Q(T(hQ_Q), U)$ is simple for any $h \in \text{Pi}(Q)$.

(3) \implies (1). Let $g \in \text{Pi}(P)$. Since $\text{Hom}_P(T({}_P P g), U)_Q \cong g S({}_P U)_Q$, $g S({}_P U)_Q$ is a simple submodule of gU_Q . Hence we have $g S({}_P U)_Q \leq S(gU_Q)_Q = g S(U_Q)_Q$. This shows $S({}_P U)_Q \leq S(U_Q)_Q$ and by symmetry $S({}_P U) = S(U_Q)$. Therefore $S(gU_Q)_Q = g S({}_P U)_Q$ is simple and similarly ${}_P S({}_P U h)$ is simple for any $h \in \text{Pi}(Q)$. Thus gU_Q and ${}_P U h$ are colocal. \square

In Proposition 1.4, the condition (3) is equivalent to the following condition (3)' since in the proof of (3) \implies (1), for any $g \in \text{Pi}(P)$, $g S({}_P U)_Q$ is a simple submodule of a colocal module gU_Q and $g S({}_P U)_Q = g S(U_Q)_Q$ holds.

(3)' For each $g \in \text{Pi}(P)$ and each $h \in \text{Pi}(Q)$, gU_Q and ${}_P U h$ are colocal, and ${}_P U_Q$ -duals of simple left P -modules are simple.

Lemma 1.5. *Let $({}_P M, N_Q)$ be a semicolocal pair and $Y' < Y \leq N_Q$ with $Y' = rl(Y')$. If Y/Y'_Q is simple, then ${}_P l(Y')/l(Y)$ is also simple and $Y = rl(Y)$.*

Proof. Put $U = {}_P M N_Q$, $X = l(Y)$ and $X' = l(Y')$. Since ${}_P U_Q$ is semicolocal and Y/Y' is simple, there exist an idempotent $f \in Q$ and an element $y = yf \in Y$ such that ${}_P U f$ is colocal and $Y = yQ + Y' \leq N_Q$. From $rl(Y') = Y' < Y \leq rl(Y)$, we obtain $X = l(Y) < l(Y') = X'$. For any $x \in X'$, the left multiplication map $\hat{x} : Y/Y'_Q \rightarrow xY_Q$ by x is an epimorphism. This shows that $xY_Q \leq S(U_Q)$, so $X'Y_Q \leq S(U_Q)$. Therefore we have $0 \neq X'y \leq S(U_Q)f = S({}_P U)f = S({}_P U f)$. Thus ${}_P X'y = S({}_P U f)$ is a simple left P -module. On the other hand, the map $\eta : {}_P X'/X \rightarrow {}_P X'y$ defined by $(x + X)\eta = xy$ is a monomorphism. Thus ${}_P l(Y')/l(Y) (= {}_P X'/X)$ is simple. By the same argument, it follows that $rl(Y)/rl(Y')_Q$ is simple. Hence we have $Y = rl(Y)$ from $rl(Y') = Y' < Y \leq rl(Y)$. □

We say that a pair $({}_P M, N_Q)$ satisfies l -ann (resp. r -ann) if $lr(X) = X$ (resp. $rl(Y) = Y$) hold for any $X \leq {}_P M$ (resp. $Y \leq N_Q$), and $({}_P M, N_Q)$ is dual if $({}_P M, N_Q)$ satisfies l -ann and r -ann.

In the following theorem, the implications (1) \iff (2) \implies (3) are essentially due to [12, Theorem 1.1] (and [14, Theorem 1.1]).

Theorem 1.6. *Let P and Q be rings and $({}_P M, N_Q)$ a faithful semicolocal pair, and consider the following conditions.*

- (1) $|N_Q| < \infty$.
- (2) $|{}_P M| < \infty$.
- (3) $({}_P M, N_Q)$ is a dual pair.

Then the implications (1) \iff (2) \implies (3) hold, and in case either P or Q is a perfect ring, the conditions are equivalent.

Proof. The implications (1) \iff (2) \implies (3) are easily seen from Lemma 1.5 (see the proof of [10, Theorem 1.4]).

Assume that $({}_P M, N_Q)$ is a dual pair and P is a perfect ring. Then any factor module of ${}_P M$ has finite Goldie dimension (see [3, Corollary 1.6] or [11, Theorem 1.7]). Hence by the proof of [13, Propositions 2.9 and 2.12] (or [11, Lemma 1.9]) ${}_P M$ has finite length. □

2. Finitely cogenerated injective modules

Throughout this section, we always assume that R is a semiperfect ring.

Let M_R and L_R be right R -module modules. Following Harada [6], M is said to be L -simple-injective if for any submodule K of L_R , any homomorphism $\theta : K_R \rightarrow M_R$ can be extended to a homomorphism $\eta : L_R \rightarrow M_R$. Moreover M is said to be simple-injective if M is N -simple-injective for any right R -module N .

Lemma 2.1 (see [7, Lemma 4.1]). *Let M_R be a finitely cogenerated module with $S(M_R) \sim T(fR_R)$ and assume that Mf_Q has finite Loewy length, where f is an idempotent of R and $Q = fRf$. If M_R is R -simple-injective, then M_R is injective.*

Proof. Since $S(M_R)$ is essential in M_R and $S(M_R) \sim T(M_R)$, $l_M(Rf) = 0$ holds. Hence $Lf \neq 0$ for any non-zero submodule $L \leq M_R$ because $Lf = LRf$. Let I be a non-zero right ideal of R and $\theta : I \rightarrow M$ a non-zero homomorphism and put $J = \text{rad}(R)$. Then $0 \neq \theta(I(fJf)^k) = \theta(I)(fJf)^k \leq S(Mf_Q)$ for some integer $k \geq 0$. Put $K = I(fJf)^k R$. Since $S(Mf_Q)R = l_{Mf}(fJf)R \leq l_M(J) = S(M_R)$, $\theta(K) \leq S(M_R)$ holds. By assumption we have $S(M_R) = S_1 \oplus \cdots \oplus S_n$ for a finite number of simple modules S_i ($1 \leq i \leq n$). Hence the restriction map $\theta|_K : K \rightarrow M$ of θ can be represented as $\theta|_K = \theta_1 + \cdots + \theta_n$ for some homomorphisms $\theta_i : K \rightarrow M$ with $\text{Im } \theta_i \leq S_i$ ($1 \leq i \leq n$). Therefore we have $(\theta - \hat{x})(K) = 0$ with left multiplication $\hat{x} : R \rightarrow M$ by some element $x \in M$. If $\theta - \hat{x} : I \rightarrow M$ is a non-zero homomorphism, then $(\theta - \hat{x})(I)f \neq 0$ (i.e. $k \geq 1$) and $0 \neq (\theta - \hat{x})(I(fJf)^m) \leq S(Mf_Q)$ for some integer m with $k > m \geq 0$. Iterating the above argument, we have $(\theta - \hat{y})(I) = 0$ for some element $y \in M$. Thus M_R is injective. □

The following lemma is related to [9, Theorem 1.6].

Lemma 2.2 (see [10, Corollary 2.6]). *Let U_Q be a module with $P = \text{End } U_Q$ and g and h local idempotents of P and Q , respectively. If gU_Q is a U -simple-injective module and $0 \neq x = gxh \in S(gU_Q)$, then gU_Q and ${}_P U h$ are colocal modules with $S(gU_Q) = xQ \cong T(hQ_Q)$ and $S({}_P U h) = Px \cong T({}_P P g)$. Therefore, for any idempotents $e \in P$ and $f \in Q$ with semiperfect rings ePe and fQf , if eU_Q is a U -simple-injective module and $S(eU_Q)$ is essential in eU_Q with $S(eU_Q) \sim T(fQ_Q)$, then (eU, Uf) is a Nakayama pair.*

Proof. By [10, Lemma 2.2] (or [7, Lemma 3.6]), gU_Q is a colocal module with $S(gU_Q) = xQ$. Let $0 \neq y \in U h$. Then we have $r_{hQ}(y) \leq hJ = r_{hQ}(x)$, where $J = \text{rad}(Q)$. Hence the map $\theta : yQ \rightarrow gU$ via $\theta(y c) = x c$ ($c \in Q$) is well-defined. Therefore by U -simple-injectivity of gU_Q we have $x = ay$ for some $a \in \text{Hom}_Q(U, gU) = gP$. Thus $x \in Py$, which implies that ${}_P U h$ is a colocal module with $S({}_P U h) = Px$. □

Lemma 2.3. *Let M be a finitely cogenerated simple-injective right R -module with $S(M_R) \sim T(fR_R)$, where f is an idempotent of R , and assume that $\text{End } M$ is a semiperfect ring. Then $({}_P M, Rf_Q)$ is a faithful semicolocal pair, where $P = \text{End } M$ and $Q = fRf$.*

Proof. By Lemma 2.2 for each $g \in \text{Pi}(P)$ (resp. $h \in \text{Pi}(Q)$), there exists $h \in \text{Pi}(Q)$ (resp. $g \in \text{Pi}(P)$) such that (gM, Mh) is a Nakayama pair. Therefore by Lemma 1.1 for a bimodule ${}_P M f_Q$, (gMf, Mfh) is a Nakayama pair. On the other hand by the proof of [10, Lemma 2.1], $({}_g P_g gM, Rf h_h Q_h)$ is faithful. This shows that $({}_P M, Rf_Q)$ is a faithful semicolocal pair by Proposition 1.4. \square

Generalizing [10, Theorem 2.7] and [7, Theorem 4.2], we have the following theorem.

Theorem 2.4. *Let M be a finitely cogenerated right R -module with $S(M_R) \sim T(fR_R)$, where f is an idempotent of R , and put $P = \text{End } M_R$ and $Q = fRf$. Consider the following conditions.*

- (1) M_R is injective.
- (2) M_R is simple-injective and P is a semiperfect ring.
- (3) $({}_P M, Rf_Q)$ is a faithful semicolocal pair satisfying r -ann.
- (4) M_R is R -simple-injective.

Then the implications (1) \implies (2) \implies (3) \implies (4) hold. Moreover, in case Mf_Q has finite Loewy length, these conditions are equivalent.

Proof. Note that in case $({}_P M, Rf_Q)$ is left faithful, $l_M(I) = l_M(I f)$ holds for any right ideal of I of R .

(1) \implies (2). This is clear.

(2) \implies (3). By Lemma 2.3 $({}_P M, Rf_Q)$ is a faithful semicolocal pair. Let L_Q be a submodule of Rf_Q . Assume that $L < rl(L)$. Then $(rl(L)R/LR)f \neq 0$, so $(rl(L)R/LR)h \neq 0$ for some $h \in \text{Pi}(Q)$. Hence there exist right ideals I and K of R such that $LR \leq K < I \leq rl(L)R_R$ and $I/K_R \cong T(hR)$. Therefore $l(L) \geq l(Kf) \geq l(I f) \geq lrl(L) = l(L)$. Thus $l_M(K) = l_M(Kf) = l_M(I f) = l_M(I)$. On the other hand $I/K_R (\cong T(hR))$ is isomorphic to a direct summand of $S(M)$. Hence we have a map $\theta : I \rightarrow M$ such that $\text{Im } \theta$ simple and $\text{Ker } \theta = K$. Then by simple-injectivity of M , there exists an element x of M such that $xc = \theta(c)$ for each $c \in I$. This implies that $x \in l_M(K) - l_M(I)$, a contradiction. Thus $L = rl(L)$ and $({}_P M, Rf_Q)$ satisfies r -ann.

(3) \implies (4). Let I be a right ideal of R and $\theta : I \rightarrow M$ a homomorphism with $\text{Im } \theta$ simple, and put $K = \text{Ker } \theta$. Then $I/K \cong T(hR)$ for some $h \in \text{Pi}(Q)$. Hence we have $Kf < I f$ because of $Kh < Ih$. Since $({}_P M, Rf_Q)$ satisfies r -ann, $l_M(K) = l_M(Kf) > l_M(I f) = l_M(I)$. Thus we have an element $x \in l_M(K) - l_M(I)$. Since $I/K \cong T(hR)$, $I = aR + K$ for some $a = ah \in I$. Put $y = \theta(a)$ and $z = xa$. Then y and z are non-zero elements of ${}_P S(M_R)h$. By assumption ${}_P S(M_R)f = l_M(J)f \leq l_{Mf}(fJf) = {}_P S(Mf_Q) = {}_P S({}_P Mf)$ holds; where $J = \text{rad}(R)$, and ${}_P S({}_P Mf)h = {}_P S({}_P Mh)$ is simple. Hence ${}_P S(M_R)h = {}_P S(M_R)f h = {}_P S({}_P Mf)h$ is simple, so we have $Py = Pz (= {}_P S(M_R)h)$ and in particular $y = \varphi(z)$ for some $\varphi \in P$. Therefore we have $\theta(a) = \varphi(z) = \varphi(x)a$. Thus M is R -simple-injective.

(4) \implies (1). By Lemma 2.1. □

REMARK 4. In Theorem 2.4, the condition “ $P = \text{End } M_R$ ” can be replaced by the condition “ ${}_P M_R$ is a P - R -bimodule” except for the implications (1) \implies (2) \implies (3).

REMARK 5. For the implication (2) \implies (3) in Theorem 2.4, we can give another proof by using Propositions 1.4 and 1.3, [11, Lemma 2.4] and [7, Lemma 3.4].

The following theorem is related to [16, Theorem 3.4].

Theorem 2.5. *Let M be a right R -module and f an idempotent of R and put $P = \text{End } M_R$, $Q = fRf$. If $|Rf_Q| < \infty$ is satisfied, then the following are equivalent.*

- (1) M_R is finitely cogenerated injective with $S(M_R) \sim T(fR_R)$.
- (2) $({}_P M, Rf_Q)$ is a faithful semicolocal pair.

Proof. In case (2) is satisfied, by Theorem 1.6 and Lemma 1.2, $({}_P M, Rf_Q)$ satisfies r -ann and M_R is a finitely cogenerated module with $S(M_R) \sim T(fR_R)$. Thus the assertion follows from Theorem 2.4 since $|Q_Q| \leq |Rf_Q| < \infty$. □

The following proposition is related to [5, Theorem 3.1], [4, Theorem 3], [16, Theorem 3.4] and [2, Theorem 3.4]. The “only if” part of this proposition is well-known (see e.g. [1, Theorem 30.4 or Exercise 24.8]). However, for the benefit of the reader we provide a direct proof.

Proposition 2.6. *Let M_R be a finitely generated right R -module and f an idempotent of R and assume that $({}_P M, Rf_Q)$ is a faithful pair with $|Rf_Q| < \infty$, where $P = \text{End } M_R$ and $Q = fRf$. Then the bimodule ${}_P Mf_Q$ defines a Morita duality if and only if $({}_P M, Rf_Q)$ is semicolocal.*

Proof. “If” part. By Theorem 2.5, M_R is injective. Hence $P \cong \text{End } Mf_Q$ by [5, Lemma 2.1] (this lemma is valid for a semiperfect ring R). By assumption, Q is a right artinian ring and Mf_Q is finitely generated. Since $({}_P Mf, Q_Q)$ is a faithful semicolocal pair with $|Q_Q| < \infty$, Mf_Q is injective by Theorem 2.5. Thus the bimodule ${}_P Mf_Q$ defines a Morita duality.

“Only if” part. Since Q is a right artinian ring, Mf_Q is a finitely generated injective cogenerator. Hence by Lemma 2.3, $({}_P Mf, Q_Q)$ or equivalently $({}_P M, Rf_Q)$ is a semicolocal pair. □

REMARK 6. Let $({}_P M, N_Q)$ be a pair which satisfies (i) $({}_P M, N_Q)$ is a semicolocal dual pair with a faithful bimodule ${}_P U_Q$, where ${}_P U_Q = {}_P M N_Q$, (ii) Q is a right artinian ring and (iii) N_Q has finite length. However, this situation does not necessar-

ily imply that ${}_P U_Q$ is a dual bimodule or equivalently $(P, {}_P U_Q, Q)$ is a Baer duality (see [8] and [2], respectively for the definitions a dual bimodule and a Baer duality). Let R be a right artinian ring such that an injective hull E_R of $T(R_R)$ is not finitely generated, (see e.g. [15, Remark 2.9] for such a ring R). Then by Lemma 2.3 and Theorem 1.6, $({}_P E, R_R)$ is a semicolocal dual pair, where $P = \text{End } E_R$. But by Theorem 1.6, $({}_P P, E_R)$ is not a dual pair, so ${}_P E_R$ is not a dual bimodule. Moreover, this example shows that in Proposition 2.6, the assumption “ M_R is finitely generated” can not be removed.

References

- [1] F.W. Anderson and K. R. Fuller: *Rings and Categories of Modules*, Second Edition, Springer-Verlag, Berlin-New York, 1992.
- [2] P.N. Anh: *Morita duality, linear compactness and AB5**: A survey, Proc. Conf. Abelian Groups and Modules, Padova (1994), 17–28.
- [3] P.N. Anh, D. Herbera and C. Menini: *AB-5* and linear compactness*, J. Algebra, **200** (1998), 99–117.
- [4] Y. Baba and K. Oshiro: *On a theorem of Fuller*, J. Algebra, **154** (1993), 86–94.
- [5] K.R. Fuller: *On indecomposable injectives over artinian rings*, Pacific J. Math. **29** (1969), 115–135.
- [6] M. Harada: *Note on almost relative projectives and almost relative injectives*, Osaka J. Math. **29** (1992), 435–446.
- [7] M. Hoshino and T. Sumioka: *Colocal pairs in perfect rings*, to appear. Osaka J. Math.
- [8] Y. Kurata and K. Hashimoto: *On dual-bimodules*, Tsukuba. J. Math. **16** (1992), 85–105.
- [9] Y. Kurata and K. Hashimoto: *Dual-Bimodules and Nakayama Permutations*, preprint.
- [10] M. Morimoto and T. Sumioka: *Generalizations of theorems of Fuller*, Osaka J. Math. **34** (1997), 689–701.
- [11] M. Morimoto and T. Sumioka: *On dual pairs and simple-injective modules*, preprint.
- [12] K. Morita and H. Tachikawa: *Character modules, submodules of a free module, and quasi-Frobenius rings*, Math. Z. **65** (1956), 414–428.
- [13] F.L. Sandomierski: *Linearly compact modules and local Morita duality*, Proc. Conf. Ring Theory, Park City 1971, Academic Press, New York, 1972, 333–346.
- [14] H. Tachikawa: *Duality theorem of character modules for rings with minimum condition*, Math. Z. **68** (1958), 479–487.
- [15] W. Xue: *Rings with Morita duality*, Springer-Verlag Lecture Note. Math., Vol. 1523, 1992.
- [16] W. Xue: *Characterizations of Morita duality via idempotents for semiperfect rings*, Algebra Colloq. **5** (1998), 99–110.

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