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THE DUALITY BETWEEN SINGULAR POINTS AND INFLECTION POINTS ON WAVE FRONTS

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Abstract

In the previous paper, the authors gave criteria for A_{k+1} -type singularities on wave fronts. Using them, we show in this paper that there is a duality between singular points and inflection points on wave fronts in the projective space. As an application, we show that the algebraic sum of 2-inflection points (i.e. godron points) on an immersed surface in the real projective space is equal to the Euler number of M_- . Here M^2 is a compact orientable 2-manifold, and M_- is the open subset of M^2 where the Hessian of f takes negative values. This is a generalization of Bleecker and Wilson's formula [3] for immersed surfaces in the affine 3-space.

1. Introduction

We denote by \mathbf{K} the real number field \mathbf{R} or the complex number field \mathbf{C} . Let n and m be positive integers. A map $F: \mathbf{K}^n \rightarrow \mathbf{K}^m$ is called \mathbf{K} -differentiable if it is a C^∞ -map when $\mathbf{K} = \mathbf{R}$, and is a holomorphic map when $\mathbf{K} = \mathbf{C}$. Throughout this paper, we denote by $P(V)$ the \mathbf{K} -projective space associated to a vector space V over \mathbf{K} and let $\pi: V \rightarrow P(V)$ be the canonical projection.

Let M^n and N^{n+1} be \mathbf{K} -differentiable manifolds of dimension n and of dimension $n + 1$, respectively. The projectified \mathbf{K} -cotangent bundle

$$P(T^*N^{n+1}) := \bigcup_{p \in N^{n+1}} P(T_p^*N^{n+1})$$

has a canonical \mathbf{K} -contact structure. A \mathbf{K} -differentiable map $f: M^n \rightarrow N^{n+1}$ is called a *frontal* if f lifts to a \mathbf{K} -isotropic map L_f , i.e., a \mathbf{K} -differentiable map $L_f: M^n \rightarrow P(T^*N^{n+1})$ such that the image $dL_f(TM^n)$ of the \mathbf{K} -tangent bundle TM^n lies in the contact hyperplane field on $P(T^*N^{n+1})$. Moreover, f is called a *wave front* or a *front* if it lifts to a \mathbf{K} -isotropic immersion L_f . (In this case, L_f is called a *Legendrian immersion*.) Frontals (and therefore fronts) generalize immersions, as they allow for singular points. A frontal f is said to be *co-orientable* if its \mathbf{K} -isotropic lift L_f can lift up to a \mathbf{K} -differentiable map into the \mathbf{K} -cotangent bundle T^*N^{n+1} , otherwise it is said to be *non-co-orientable*. It should be remarked that, when N^{n+1} is a Riemannian mani-

fold, a front f is co-orientable if and only if there is a globally defined unit normal vector field ν along f .

Now we set $N^{n+1} = \mathbf{K}^{n+1}$. Suppose that a \mathbf{K} -differentiable map $F: M^n \rightarrow \mathbf{K}^{n+1}$ is a frontal. Then, for each $p \in M^n$, there exist a neighborhood U of p and a map

$$\nu: U \rightarrow (\mathbf{K}^{n+1})^* \setminus \{\mathbf{0}\}$$

into the dual vector space $(\mathbf{K}^{n+1})^*$ of \mathbf{K}^{n+1} such that the canonical pairing $\nu \cdot dF(v)$ vanishes for any $v \in TU$. We call ν a *local normal map* of the frontal F . We set $\mathcal{G} := \pi \circ \nu$, which is called a (local) *Gauss map* of F . In this setting, F is a front if and only if

$$L := (F, \mathcal{G}): U \rightarrow \mathbf{K}^{n+1} \times P((\mathbf{K}^{n+1})^*)$$

is an immersion. When F itself is an immersion, it is, of course, a front. If this is the case, for a fixed local \mathbf{K} -differentiable coordinate system (x^1, \dots, x^n) on U , we set

$$(1.1) \quad \nu_p: \mathbf{K}^{n+1} \ni v \mapsto \det(F_{x^1}(p), \dots, F_{x^n}(p), v) \in \mathbf{K} \quad (p \in U),$$

where $F_{x^j} := \partial F / \partial x^j$ ($j = 1, \dots, n$) and ‘det’ is the determinant function on \mathbf{K}^{n+1} . Then we get a \mathbf{K} -differentiable map $\nu: U \ni p \mapsto \nu_p \in (\mathbf{K}^{n+1})^*$, which gives a local normal map of F .

Now, we return to the case that F is a front. Then it is well-known that the local Gauss map \mathcal{G} induces a global map

$$(1.2) \quad \mathcal{G}: M^n \rightarrow P((\mathbf{K}^{n+1})^*)$$

which is called the *affine Gauss map* of F . (In fact, the Gauss map \mathcal{G} depends only on the affine structure of \mathbf{K}^{n+1} .)

We set

$$(1.3) \quad h_{ij} := \nu \cdot F_{x^i x^j} = -\nu_{x^i} \cdot F_{x^j} \quad (i, j = 1, \dots, n),$$

where \cdot is the canonical pairing between \mathbf{K}^{n+1} and $(\mathbf{K}^{n+1})^*$, and

$$F_{x^i x^j} = \frac{\partial^2 F}{\partial x^i \partial x^j}, \quad F_{x^j} = \frac{\partial F}{\partial x^j}, \quad \nu_{x^i} = \frac{\partial \nu}{\partial x^i}.$$

Then

$$(1.4) \quad H := \sum_{i,j=1}^n h_{ij} dx^i dx^j \quad (dx^i dx^j := (1/2)(dx^i \otimes dx^j + dx^j \otimes dx^i))$$

gives a \mathbf{K} -valued symmetric tensor on U , which is called the *Hessian form* of F associated to ν . Here, the \mathbf{K} -differentiable function

$$(1.5) \quad h := \det(h_{ij}): U \rightarrow \mathbf{K}$$

is called the *Hessian* of F . A point $p \in M^n$ is called an *inflection point* of F if it belongs to the zeros of h . An inflection point p is called *nondegenerate* if the derivative dh does not vanish at p . In this case, the set of inflection points $I(F)$ consists of an embedded \mathbf{K} -differentiable hypersurface of U near p and there exists a non-vanishing \mathbf{K} -differentiable vector field ξ along $I(F)$ such that $H(\xi, \nu) = 0$ for all $\nu \in TU$. Such a vector field ξ is called an *asymptotic vector field* along $I(F)$, and $[\xi] = \pi(\xi) \in P(\mathbf{K}^{n+1})$ is called the *asymptotic direction*. It can be easily checked that the definition of inflection points and the nondegeneracy of inflection points are independent of choice of ν and a local coordinate system.

In Section 2, we shall define the terminology that

- a \mathbf{K} -differentiable vector field η along a \mathbf{K} -differentiable hypersurface S of M^n is *k-nondegenerate* at $p \in S$, and
- η *meets* S at p with *multiplicity* $k + 1$.

Using this new terminology, $p (\in I(F))$ is called an *A_{k+1} -inflection point* if ξ is *k-nondegenerate* at p but does not meet $I(F)$ with multiplicity $k + 1$. In Section 2, we shall prove the following:

Theorem A. *Let $F: M^n \rightarrow \mathbf{K}^{n+1}$ be an immersed \mathbf{K} -differentiable hypersurface. Then $p \in M^n$ is an A_{k+1} -inflection point ($1 \leq k \leq n$) if and only if the affine Gauss map \mathcal{G} has an A_k -Morin singularity at p . (See the appendix of [10] for the definition of A_k -Morin singularities, which corresponds to A_{k+1} -points under the intrinsic formulation of singularities as in the reference given in Added in Proof.)*

Though our definition of A_{k+1} -inflection points are given in terms of the Hessian, this assertion allows us to define A_{k+1} -inflection points by the singularities of their affine Gauss map, which might be more familiar to readers than our definition. However, the new notion “*k-multiplicity*” introduced in the present paper is very useful for recognizing the duality between singular points and inflection points. Moreover, as mentioned above, our definition of A_k -inflection points works even when F is a front. We have the following dual assertion for the previous theorem. Let $\mathcal{G}: M^n \rightarrow P((\mathbf{K}^{n+1})^*)$ be an immersion. Then $p \in M^n$ is an *A_{k+1} -inflection point* of \mathcal{G} if it is an A_{k+1} -inflection point of $\nu: M^n \rightarrow (\mathbf{K}^{n+1})^*$ such that $\pi \circ \nu = \mathcal{G}$. This property does not depend on a choice of ν .

Proposition A'. *Let $F: M^n \rightarrow \mathbf{K}^{n+1}$ be a front. Suppose that the affine Gauss map $\mathcal{G}: M^n \rightarrow P((\mathbf{K}^{n+1})^*)$ is a \mathbf{K} -immersion. Then $p \in M^n$ is an A_{k+1} -inflection point*

of \mathcal{G} ($1 \leq k \leq n$) if and only if F has an A_{k+1} -singularity at p . (See (1.1) in [10] for the definition of A_{k+1} -singularities.)

In the case that $\mathbf{K} = \mathbf{R}$, $n = 3$ and F is an immersion, an A_3 -inflection point is known as a *cuspidal point of the Gauss map* (cf. [2]).

It can be easily seen that inflection points and the asymptotic directions are invariant under projective transformations. So we can define A_{k+1} -inflection points ($1 \leq k \leq n$) of an immersion $f: M^n \rightarrow P(\mathbf{K}^{n+2})$. For each $p \in M^n$, we take a local \mathbf{K} -differentiable coordinate system $(U; x^1, \dots, x^n) (\subset M^n)$. Then there exists a \mathbf{K} -immersion $F: U \rightarrow \mathbf{K}^{n+2}$ such that $f = [F]$ is the projection of F . We set

$$(1.6) \quad G: U \ni p \mapsto F_{x^1}(p) \wedge F_{x^2}(p) \wedge \cdots \wedge F_{x^n}(p) \wedge F(p) \in (\mathbf{K}^{n+2})^*.$$

Here, we identify $(\mathbf{K}^{n+2})^*$ with $\bigwedge^{n+1} \mathbf{K}^{n+2}$ by

$$\bigwedge^{n+1} \mathbf{K}^{n+2} \ni v_1 \wedge \cdots \wedge v_{n+1} \longleftrightarrow \det(v_1, \dots, v_{n+1}, *) \in (\mathbf{K}^{n+2})^*,$$

where ‘det’ is the determinant function on \mathbf{K}^{n+2} . Then G satisfies

$$(1.7) \quad G \cdot F = 0, \quad G \cdot dF = dG \cdot F = 0,$$

where \cdot is the canonical pairing between \mathbf{K}^{n+2} and $(\mathbf{K}^{n+2})^*$. Since, $g := \pi \circ G$ does not depend on the choice of a local coordinate system, the projection of G induces a globally defined \mathbf{K} -differentiable map

$$g = [G]: M^n \rightarrow P((\mathbf{K}^{n+2})^*),$$

which is called the *dual front* of f . We set

$$h := \det(h_{ij}): U \rightarrow \mathbf{K} \quad (h_{ij} := G \cdot F_{x^i x^j} = -G_{x^i} \cdot F_{x^j}),$$

which is called the *Hessian* of F . The inflection points of f correspond to the zeros of h .

In Section 3, we prove the following

Theorem B. *Let $f: M^n \rightarrow P(\mathbf{K}^{n+2})$ be an immersed \mathbf{K} -differentiable hypersurface. Then $p \in M^n$ is an A_{k+1} -inflection point ($k \leq n$) if and only if the dual front g has an A_k -singularity at p .*

Next, we consider the case of $\mathbf{K} = \mathbf{R}$. In [8], we defined the *tail part* of a swallow-tail, that is, an A_3 -singular point. An A_3 -inflection point p of $f: M^2 \rightarrow P(\mathbf{R}^4)$ is called *positive* (resp. *negative*), if the Hessian takes negative (resp. positive) values on the tail

part of the dual of f at p . Let $p \in M^2$ be an A_3 -inflection point. Then there exists a neighborhood U such that $f(U)$ is contained in an affine space A^3 in $P(\mathbf{R}^4)$. Then the affine Gauss map $\mathcal{G}: U \rightarrow P(A^3)$ has an elliptic cusp (resp. a hyperbolic cusp) if and only if it is positive (resp. negative) (see [2, p. 33]). In [13], Uribe-Vargas introduced a projective invariant ρ and studied the projective geometry of swallowtails. He proved that an A_3 -inflection point is positive (resp. negative) if and only if $\rho > 1$ (resp. $\rho < 1$). The property that h as in (1.5) is negative is also independent of the choice of a local coordinate system. So we can define the set of negative points

$$M_- := \{p \in M^2; h(p) < 0\}.$$

In Section 3, we shall prove the following assertion as an application.

Theorem C. *Let M^2 be a compact orientable C^∞ -manifold without boundary, and $f: M^2 \rightarrow P(\mathbf{R}^4)$ an immersion. We denote by $i_2^+(f)$ (resp. $i_2^-(f)$) the number of positive A_3 -inflection points (resp. negative A_3 -inflection points) on M^2 (see Section 3 for the precise definition of $i_2^+(f)$ and $i_2^-(f)$). Suppose that inflection points of f consist only of A_2 and A_3 -inflection points. Then the following identity holds*

$$(1.8) \quad i_2^+(f) - i_2^-(f) = 2\chi(M_-).$$

The above formula is a generalization of that of Bleecker and Wilson [3] when $f(M^2)$ is contained in an affine 3-space.

Corollary D (Uribe-Vargas [13, Corollary 4]). *Under the assumption of Theorem C, the total number $i_2^+(f) + i_2^-(f)$ of A_3 -inflection points is even.*

In [13], this corollary is proved by counting the parity of a loop consisting of flecnodal curves which bound two A_3 -inflection points.

Corollary E. *The same formula (1.8) holds for an immersed surface in the unit 3-sphere S^3 or in the hyperbolic 3-space H^3 .*

Proof. Let $\pi: S^3 \rightarrow P(\mathbf{R}^4)$ be the canonical projection. If $f: M^2 \rightarrow S^3$ is an immersion, we get the assertion applying Theorem C to $\pi \circ f$. On the other hand, if f is an immersion into H^3 , we consider the canonical projective embedding $\iota: H^3 \rightarrow S_+^3$ where S_+^3 is the open hemisphere of S^3 . Then we get the assertion applying Theorem C to $\pi \circ \iota \circ f$. \square

Finally, in Section 4, we shall introduce a new invariant for 3/2-cusps using the duality, which is a measure for acuteness using the classical cycloid.

This work is inspired by the result of Izumiya, Pei and Sano [4] that characterizes A_2 and A_3 -singular points on surfaces in H^3 via the singularity of certain height functions, and the result on the duality between space-like surfaces in hyperbolic 3-space (resp. in light-cone), and those in de Sitter space (resp. in light-cone) given by Izumiya [5]. The authors would like to thank Shyuichi Izumiya for his impressive informal talk at Karatsu, 2005.

2. Preliminaries and a proof of Theorem A

In this section, we shall introduce a new notion “multiplicity” for a contact of a given vector field along an immersed hypersurface. Then our previous criterion for A_k -singularities (given in [10]) can be generalized to the criteria for k -multiple contactness of a given vector field (see Theorem 2.4).

Let M^n be a \mathbf{K} -differentiable manifold and $S (\subset M^n)$ an embedded \mathbf{K} -differentiable hypersurface in M^n . We fix $p \in S$ and take a \mathbf{K} -differentiable vector field

$$\eta: S \supset V \ni q \mapsto \eta_q \in T_q M^n$$

along S defined on a neighborhood $V \subset S$ of p . Then we can construct a \mathbf{K} -differential vector field $\tilde{\eta}$ defined on a neighborhood $U \subset M^n$ of p such that the restriction $\tilde{\eta}|_S$ coincides with η . Such an $\tilde{\eta}$ is called *an extension of η* . (The local existence of $\tilde{\eta}$ is mentioned in [10, Remark 2.2].)

DEFINITION 2.1. Let p be an arbitrary point on S , and U a neighborhood of p in M^n . A \mathbf{K} -differentiable function $\varphi: U \rightarrow \mathbf{K}$ is called *admissible* near p if it satisfies the following properties

- (1) $O := U \cap S$ is the zero level set of φ , and
- (2) $d\varphi$ never vanishes on O .

One can easily find an admissible function near p . We set $\varphi' := d\varphi(\tilde{\eta}): U \rightarrow \mathbf{K}$ and define a subset $S_2 (\subset O \subset S)$ by

$$S_2 := \{q \in O; \varphi'(q) = 0\} = \{q \in O; \eta_q \in T_q S\}.$$

If $p \in S_2$, then η is said to *meet S with multiplicity 2 at p* or equivalently, η is said to *contact S with multiplicity 2 at p* . Otherwise, η is said to *meet S with multiplicity 1 at p* . Moreover, if $d\varphi'(T_p O) \neq \{0\}$, η is said to *be 2-nondegenerate at p* . The k -th multiple contactness and k -nondegeneracy are defined inductively. In fact, if the j -th multiple contactness and the submanifolds S_j have been already defined for $j = 1, \dots, k$ ($S_1 = S$), we set

$$\varphi^{(k)} := d\varphi^{(k-1)}(\tilde{\eta}): U \rightarrow \mathbf{K} \quad (\varphi^{(1)} := \varphi')$$

and can define a subset of S_k by

$$S_{k+1} := \{q \in S_k; \varphi^{(k)}(q) = 0\} = \{q \in S_k; \eta_q \in T_q S_k\}.$$

We say that η *meets* S with *multiplicity* $k + 1$ at p if η is k -nondegenerate at p and $p \in S_{k+1}$. Moreover, if $d\varphi^{(k)}(T_p S_k) \neq \{0\}$, η is called $(k + 1)$ -nondegenerate at p . If η is $(k + 1)$ -nondegenerate at p , then S_{k+1} is a hypersurface of S_k near p .

REMARK 2.2. Here we did not define ‘1-nondegeneracy’ of η . However, from now on, any \mathbf{K} -differentiable vector field η of M^n along S is always 1-nondegenerate by convention. In the previous paper [10], ‘1-nondegeneracy’ (i.e. nondegeneracy) is defined not for a vector field along the singular set but for a given singular point. If a singular point $p \in U$ of a front $f: U \rightarrow \mathbf{K}^{n+1}$ is nondegenerate in the sense of [10], then the function $\lambda: U \rightarrow \mathbf{K}$ defined in [10, (2.1)] is an admissible function, and the null vector field η along $S(f)$ is given. When $k \geq 2$, by definition, k -nondegeneracy of the singular point p is equivalent to the k -nondegeneracy of the null vector field η at p (cf. [10]).

Proposition 2.3. *The k -th multiple contactness and k -nondegeneracy are both independent of the choice of an extension $\tilde{\eta}$ of η and also of the choice of admissible functions as in Definition 2.1.*

Proof. We can take a local coordinate system $(U; x^1, \dots, x^n)$ of M^n such that $x^n = \varphi$. Write

$$\tilde{\eta} := \sum_{j=1}^n c^j \frac{\partial}{\partial x^j},$$

where c^j ($j = 1, \dots, n$) are \mathbf{K} -differentiable functions. Then we have that $\varphi' = \sum_{j=1}^n c^j \varphi_{x^j} = c^n$.

Let ψ be another admissible function defined on U . Then

$$\psi' = \sum_{j=1}^n c^j \frac{\partial \psi}{\partial x^j} = c^n \frac{\partial \psi}{\partial x^n} = \varphi' \frac{\partial \psi}{\partial x^n}.$$

Thus ψ' is proportional to φ' . Then the assertion follows inductively. \square

Corollary 2.5 in [10] is now generalized into the following assertion:

Theorem 2.4. *Let $\tilde{\eta}$ be an extension of the vector field η . Let us assume $1 \leq k \leq n$. Then the vector field η is k -nondegenerate at p , but η does not meet S with*

multiplicity $k + 1$ at p if and only if

$$\varphi(p) = \varphi'(p) = \cdots = \varphi^{(k-1)}(p) = 0, \quad \varphi^{(k)}(p) \neq 0,$$

and the Jacobi matrix of \mathbf{K} -differentiable map

$$\Lambda := (\varphi, \varphi', \dots, \varphi^{(k-1)}): U \rightarrow \mathbf{K}^k$$

is of rank k at p , where φ is an admissible \mathbf{K} -differentiable function and

$$\varphi^{(0)} := \varphi, \varphi^{(1)} (= \varphi') := d\varphi(\tilde{\eta}), \dots, \varphi^{(k)} := d\varphi^{(k-1)}(\tilde{\eta}).$$

The proof of this theorem is completely parallel to that of Corollary 2.5 in [10].

To prove Theorem A by applying Theorem 2.4, we shall review the criterion for A_k -singularities in [10]. Let U^n be a domain in \mathbf{K}^n , and consider a map $\Phi: U^n \rightarrow \mathbf{K}^m$ where $m \geq n$. A point $p \in U^n$ is called a *singular point* if the rank of the differential map $d\Phi$ is less than n . Suppose that the singular set $S(\Phi)$ of Φ consists of a \mathbf{K} -differentiable hypersurface U^n . Then a vector field η along S is called a *null vector field* if $d\Phi(\eta)$ vanishes identically. In this paper, we consider the case $m = n$ or $m = n + 1$. If $m = n$, we define a \mathbf{K} -differentiable function $\lambda: U^n \rightarrow \mathbf{K}$ by

$$(2.1) \quad \lambda := \det(\Phi_{x^1}, \dots, \Phi_{x^n}).$$

On the other hand, if $\Phi: U^n \rightarrow \mathbf{K}^{n+1}$ ($m = n + 1$) and ν is a non-vanishing \mathbf{K} -normal vector field (for a definition, see [10, Section 1]) we set

$$(2.2) \quad \lambda := \det(\Phi_{x^1}, \dots, \Phi_{x^n}, \nu).$$

Then the singular set $S(\Phi)$ of the map Φ coincides with the zeros of λ . Recall that $p \in S(\Phi)$ is called *nondegenerate* if $d\lambda(p) \neq \mathbf{0}$ (see [10] and Remark 2.2). Both of two cases (2.1) and (2.2), the functions λ are admissible near p (cf. Definition 2.1), if p is non-degenerate. When $S(\Phi)$ consists of nondegenerate singular points, then it is a hypersurface and there exists a non-vanishing null vector field η on $S(\Phi)$. Such a vector field η determined up to a multiplication of non-vanishing \mathbf{K} -differentiable functions. The following assertion holds as seen in [10].

Fact 2.5. *Suppose $m = n$ and Φ is a C^∞ -map (resp. $m = n + 1$ and Φ is a front). Then Φ has an A_k -Morin singularity (resp. A_{k+1} -singularity) at $p \in M^n$ if and only if η is k -nondegenerate at p but does not meet $S(\Phi)$ with multiplicity $k + 1$ at p . (Here multiplicity 1 means that η meets $S(\Phi)$ at p transversally, and 1-nondegeneracy is an empty condition.)*

As an application of the fact for $m = n$, we now give a proof of Theorem A: Let $F: M^n \rightarrow \mathbf{K}^{n+1}$ be an immersed \mathbf{K} -differentiable hypersurface. Recall that a point $p \in$

M^n is called a *nondegenerate inflection point* if the derivative dh of the local Hessian function h (cf. (1.5)) with respect to F does not vanish at p . Then the set $I(F)$ of inflection points consists of a hypersurface, called the *inflectional hypersurface*, and the function h is an admissible function on a neighborhood of p in M^n . A nondegenerate inflection point p is called an A_{k+1} -*inflection point* of F if the asymptotic vector field ξ is k -nondegenerate at p but does not meet $I(F)$ with multiplicity $k+1$ at p .

Proof of Theorem A. Let ν be a map given by (1.1), and $\mathcal{G}: M^n \rightarrow P((\mathbf{K}^{n+1})^*)$ the affine Gauss map induced from ν by (1.2). We set

$$\mu := \det(\nu_{x^1}, \nu_{x^2}, \dots, \nu_{x^n}, \nu),$$

where ‘det’ is the determinant function of $(\mathbf{K}^{n+1})^*$ under the canonical identification $(\mathbf{K}^{n+1})^* \cong \mathbf{K}^{n+1}$, and (x^1, \dots, x^n) is a local coordinate system of M^n . Then the singular set $S(\mathcal{G})$ of \mathcal{G} is just the zeros of μ . By Theorem 2.4 and Fact 2.5, our criteria for A_{k+1} -inflection points (resp. A_{k+1} -singular points) are completely determined by the pair $(\xi, I(F))$ (resp. the pair $(\eta, S(\mathcal{G}))$). Hence it is sufficient to show the following three assertions (1)–(3).

- (1) $I(F) = S(\mathcal{G})$.
- (2) For each $p \in I(F)$, p is a nondegenerate inflection point of F if and only if it is a nondegenerate singular point of \mathcal{G} .
- (3) The asymptotic direction of each nondegenerate inflection point p of F is equal to the null direction of p as a singular point of \mathcal{G} .

Let $H = \sum_{i,j=1}^n h_{ij} dx^i dx^j$ be the Hessian form of F . Then we have that

$$(2.3) \quad \begin{pmatrix} h_{11} & \dots & h_{1n} & * \\ \vdots & \ddots & \vdots & \vdots \\ h_{n1} & \dots & h_{nn} & * \\ 0 & \dots & 0 & \nu \cdot {}^t\nu \end{pmatrix} = \begin{pmatrix} \nu_{x^1} \\ \vdots \\ \nu_{x^n} \\ \nu \end{pmatrix} (F_{x^1}, \dots, F_{x^n}, {}^t\nu),$$

where $\nu \cdot {}^t\nu = \sum_{j=1}^{n+1} (\nu^j)^2$ and $\nu = (\nu^1, \dots, \nu^n)$ as a row vector. Here, we consider a vector in \mathbf{K}^n (resp. in $(\mathbf{K}^n)^*$) as a column vector (resp. a row vector), and ${}^t(\cdot)$ denotes the transposition. We may assume that $\nu(p) \cdot {}^t\nu(p) \neq 0$ by a suitable affine transformation of \mathbf{K}^{n+1} , even when $\mathbf{K} = \mathbf{C}$. Since the matrix $(F_{x^1}, \dots, F_{x^n}, {}^t\nu)$ is regular, (1) and (2) follow by taking the determinant of (2.3). Also by (2.3), $\sum_{i=1}^n a_i h_{ij} = 0$ for all $j = 1, \dots, n$ holds if and only if $\sum_{i=1}^n a_i \nu_{x^i} = \mathbf{0}$, which proves (3). \square

Proof of Proposition A’. Similar to the proof of Theorem A, it is sufficient to show the following properties, by virtue of Theorem 2.4.

- (1’) $S(F) = I(\mathcal{G})$, that is, the set of singular points of F coincides with the set of inflection points of the affine Gauss map.

(2') For each $p \in I(\mathcal{G})$, p is a nondegenerate inflection point if and only if it is a nondegenerate singular point of F .

(3') The asymptotic direction of each nondegenerate inflection point coincides with the null direction of p as a singular point of F .

Since \mathcal{G} is an immersion, (2.3) implies that

$$\begin{aligned} I(\mathcal{G}) &= \{p; (F_{x^1}, \dots, F_{x^n}, {}^t v) \text{ are linearly dependent at } p\} \\ &= \{p; \lambda(p) = 0\} \quad (\lambda := \det(F_{x^1}, \dots, F_{x^n}, {}^t v)). \end{aligned}$$

Hence we have (1'). Moreover, $h = \det(h_{ij}) = \delta\lambda$ holds, where δ is a function on U which never vanishes on a neighborhood of p . Thus (2') holds. Finally, by (2.3), $\sum_{j=1}^n b_j h_{ij} = 0$ for $i = 1, \dots, n$ if and only if $\sum_{j=1}^n b_j F_{x^j} = \mathbf{0}$, which proves (3'). \square

EXAMPLE 2.6 (A_2 -inflection points on cubic curves). Let $\gamma(t) := {}^t(x(t), y(t))$ be a \mathbf{K} -differentiable curve in \mathbf{K}^2 . Then $v(t) := (-\dot{y}(t), \dot{x}(t)) \in (\mathbf{K}^2)^*$ gives a normal vector, and

$$h(t) = v(t) \cdot \ddot{\gamma}(t) = \det(\dot{\gamma}(t), \ddot{\gamma}(t))$$

is the Hessian function. Thus $t = t_0$ is an A_2 -inflection point if and only if

$$\det(\dot{\gamma}(t_0), \ddot{\gamma}(t_0)) = 0, \quad \det(\dot{\gamma}(t_0), \ddot{\gamma}(t_0)) \neq 0.$$

Considering $\mathbf{K}^2 \subset P(\mathbf{K}^3)$ as an affine subspace, this criterion is available for curves in $P(\mathbf{K}^3)$. When $\mathbf{K} = \mathbf{C}$, it is well-known that non-singular cubic curves in $P(\mathbf{C}^3)$ have exactly nine inflection points which are all of A_2 -type. One special singular cubic curve is $2y^2 - 3x^3 = 0$ in $P(\mathbf{C}^3)$ with homogeneous coordinates $[x, y, z]$, which can be parameterized as $\gamma(t) = [\sqrt[3]{2}t^2, \sqrt{3}t^3, 1]$. The image of the dual curve of γ in $P(\mathbf{C}^3)$ is the image of γ itself, and γ has an A_2 -type singular point $[0, 0, 1]$ and an A_2 -inflection point $[0, 1, 0]$.

These two points are interchanged by the duality. (The duality of fronts is explained in Section 3.)

EXAMPLE 2.7 (The affine Gauss map of an A_4 -inflection point). Let $F: \mathbf{K}^3 \rightarrow \mathbf{K}^4$ be a map defined by

$$F(u, v, w) = {}^t \left(w, u, v, -u^2 - \frac{3v^2}{2} + uw^2 + vw^3 - \frac{w^4}{4} + \frac{w^5}{5} - \frac{w^6}{6} \right) \quad (u, v, w \in \mathbf{K}).$$

If we define $\mathcal{G}: \mathbf{K}^3 \rightarrow P(\mathbf{K}^4) \cong P((\mathbf{K}^4)^*)$ by

$$\mathcal{G}(u, v, w) = [-2uw - 3vw^2 + w^3 - w^4 + w^5, 2u - w^2, 3v - w^3, 1]$$

using the homogeneous coordinate system, \mathcal{G} gives the affine Gauss map of F . Then the Hessian h of F is

$$\det \begin{pmatrix} -2 & 0 & 2w \\ 0 & -3 & 3w^2 \\ 2w & 3w^2 & 2u + 6vw - 3w^2 + 4w^3 - 5w^4 \end{pmatrix} = 6(2u + 6vw - w^2 + 4w^3 - 2w^4).$$

The asymptotic vector field is $\xi = (w, w^2, 1)$. Hence we have

$$\begin{aligned} h &= 6(2u + 6vw - w^2 + 4w^3 - 2w^4), \\ h' &= 12(3v + 6w^2 - w^3), \quad h'' = 144w, \quad h''' = 144, \end{aligned}$$

where $h' = dh(\xi)$, $h'' = dh'(\xi)$ and $h''' = dh''(\xi)$. The Jacobi matrix of (h, h', h'') at $\mathbf{0}$ is

$$\begin{pmatrix} 2 & * & * \\ 0 & 36 & * \\ 0 & 0 & 144 \end{pmatrix}.$$

This implies that ξ is 3-nondegenerate at $\mathbf{0}$ but does not meet $I(F) = h^{-1}(0)$ at p with multiplicity 4, that is, F has an A_4 -inflection point at $\mathbf{0}$. On the other hand, \mathcal{G} has the A_3 -Morin singularity at $\mathbf{0}$. In fact, by the coordinate change

$$U = 2u - w^2, \quad V = 3v - w^3, \quad W = w,$$

it follows that \mathcal{G} is represented by a map germ

$$(U, V, W) \mapsto -(UW + VW^2 + W^4, U, V).$$

This coincides with the typical A_3 -Morin singularity given in (A.3) in [10].

3. Duality of wave fronts

Let $P(\mathbf{K}^{n+2})$ be the $(n+1)$ -projective space over \mathbf{K} . We denote by $[x] \in P(\mathbf{K}^{n+2})$ the projection of a vector $x = {}^t(x^0, \dots, x^{n+1}) \in \mathbf{K}^{n+2} \setminus \{\mathbf{0}\}$. Consider a $(2n+3)$ -submanifold of $\mathbf{K}^{n+2} \times (\mathbf{K}^{n+2})^*$ defined by

$$\tilde{C} := \{(x, y) \in \mathbf{K}^{n+2} \times (\mathbf{K}^{n+2})^*; x \cdot y = 0\},$$

and also a $(2n+1)$ -submanifold of $P(\mathbf{K}^{n+2}) \times P((\mathbf{K}^{n+2})^*)$

$$C := \{([x], [y]) \in P(\mathbf{K}^{n+2}) \times P((\mathbf{K}^{n+2})^*); x \cdot y = 0\}.$$

As C can be canonically identified with the projective tangent bundle $PTP(\mathbf{K}^{n+2})$, it has a canonical contact structure: Let $\pi: \tilde{C} \rightarrow C$ be the canonical projection, and define a 1-form

$$\omega := \sum_{j=0}^{n+1} (x^j dy^j - y^j dx^j),$$

which is considered as a 1-form of \tilde{C} . The tangent vectors of the curves $t \mapsto (tx, y)$ and $t \mapsto (x, ty)$ at $(x, y) \in \tilde{C}$ generate the kernel of $d\pi$. Since these two vectors also belong to the kernel of ω and $\dim(\ker \omega) = 2n + 2$,

$$\Pi := d\pi(\ker \omega)$$

is a $2n$ -dimensional vector subspace of $T_{\pi(x,y)}C$. We shall see that Π is the contact structure on C . One can check that it coincides with the canonical contact structure of $PTP(\mathbf{K}^{n+2})$ ($\cong C$). Let U be an open subset of C and $s: U \rightarrow \mathbf{K}^{n+2} \times (\mathbf{K}^{n+2})^*$ a section of the fibration π . Since $d\pi \circ ds$ is the identity map, it can be easily checked that Π is contained in the kernel of the 1-form $s^*\omega$. Since Π and the kernel of the 1-form $s^*\omega$ are the same dimension, they coincide. Moreover, suppose that $p = \pi(x, y) \in C$ satisfies $x^i \neq 0$ and $y^j \neq 0$. We then consider a map of $\mathbf{K}^{n+1} \times (\mathbf{K}^{n+1})^* \cong \mathbf{K}^{n+1} \times \mathbf{K}^{n+1}$ into $\mathbf{K}^{n+2} \times (\mathbf{K}^{n+2})^* \cong \mathbf{K}^{n+2} \times \mathbf{K}^{n+2}$ defined by

$$(a^0, \dots, a^n, b^0, \dots, b^n) \mapsto (a^0, \dots, a^{i-1}, 1, a^{i+1}, \dots, a^n, b^0, \dots, b^{j-1}, 1, b^{j+1}, \dots, b^n),$$

and denote by $s_{i,j}$ the restriction of the map to the neighborhood of p in C . Then one can easily check that

$$s_{i,j}^* \left[\omega \wedge \left(\bigwedge^n d\omega \right) \right]$$

does not vanish at p . Thus $s_{i,j}^*\omega$ is a contact form, and the hyperplane field Π defines a canonical contact structure on C . Moreover, the two projections from C into $P(\mathbf{K}^{n+2})$ are both Legendrian fibrations, namely we get a double Legendrian fibration. Let $f = [F]: M^n \rightarrow P(\mathbf{K}^{n+2})$ be a front. Then there is a Legendrian immersion of the form $L = ([F], [G]): M^n \rightarrow C$. Then $g = [G]: M^n \rightarrow P((\mathbf{K}^{n+2})^*)$ satisfies (1.6) and (1.7). In particular, $L := \pi(F, G): M^n \rightarrow C$ gives a Legendrian immersion, and f and g can be regarded as mutually dual wave fronts as projections of L .

Proof of Theorem B. Since our contact structure on C can be identified with the contact structure on the projective tangent bundle on $P(\mathbf{K}^{n+2})$, we can apply the criteria of A_k -singularities as in Fact 2.5. Thus a nondegenerate singular point p is an A_k -singular point of f if and only if the null vector field η of f (as a wave front) is

$(k-1)$ -nondegenerate at p , but does not meet the hypersurface $S(f)$ with multiplicity k at p . Like as in the proof of Theorem A, we may assume that ${}^tF(p) \cdot F(p) \neq 0$ and $G(p) \cdot {}^tG(p) \neq 0$ simultaneously by a suitable affine transformation of \mathbf{K}^{n+2} , even when $\mathbf{K} = \mathbf{C}$. Since $(F_{x^1}, \dots, F_{x^n}, F, {}^tG)$ is a regular $(n+2) \times (n+2)$ -matrix if and only if $f = [F]$ is an immersion, the assertion immediately follows from the identity

$$(3.1) \quad \begin{pmatrix} h_{11} & \dots & h_{1n} & 0 & * \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ h_{n1} & \dots & h_{nn} & 0 & * \\ 0 & \dots & 0 & 0 & G \cdot {}^tG \\ * & \dots & * & {}^tF \cdot F & 0 \end{pmatrix} = \begin{pmatrix} G_{x^1} \\ \vdots \\ G_{x^n} \\ G \\ {}^tF \end{pmatrix} (F_{x^1}, \dots, F_{x^n}, F, {}^tG). \quad \square$$

Proof of Theorem C. Let $g: M^2 \rightarrow P((\mathbf{R}^4)^*)$ be the dual of f . We fix $p \in M^2$ and take a simply connected and connected neighborhood U of p .

Then there are lifts $\hat{f}, \hat{g}: U \rightarrow S^3$ into the unit sphere S^3 such that

$$\hat{f} \cdot \hat{g} = 0, \quad d\hat{f}(v) \cdot \hat{g} = d\hat{g}(v) \cdot \hat{f} = 0 \quad (v \in TU),$$

where \cdot is the canonical inner product on $\mathbf{R}^4 \supset S^3$. Since $\hat{f} \cdot \hat{f} = 1$, we have

$$d\hat{f}(v) \cdot \hat{f}(p) = 0 \quad (v \in T_p M^2).$$

Thus

$$d\hat{f}(T_p M^2) = \{\zeta \in S^3; \zeta \cdot \hat{f}(p) = \zeta \cdot \hat{g}(p) = 0\},$$

which implies that $df(TM^2)$ is equal to the limiting tangent bundle of the front g . So we apply (2.5) in [9] for g : Since the singular set $S(g)$ of g consists only of cuspidal edges and swallowtails, the Euler number of $S(g)$ vanishes. Then it holds that

$$\chi(M_+) + \chi(M_-) = \chi(M^2) = \chi(M_+) - \chi(M_-) + i_2^+(f) - i_2^-(f),$$

which proves the formula. \square

When $n = 2$, the duality of fronts in the unit 2-sphere S^2 (as the double cover of $P(\mathbf{R}^3)$) plays a crucial role for obtaining the classification theorem in [6] for complete flat fronts with embedded ends in \mathbf{R}^3 . Also, a relationship between the number of inflection points and the number of double tangents on certain class of simple closed regular curves in $P(\mathbf{R}^3)$ is given in [11]. (For the geometry and a duality of fronts in S^2 , see [1].) In [7], Porteous investigated the duality between A_k -singular points and A_k -inflection points when $k = 2, 3$ on a surface in S^3 .

4. Cuspidal curvature on 3/2-cusps

Relating to the duality between singular points and inflection points, we introduce a curvature on 3/2-cusps of planar curves:

Suppose that (M^2, g) is an oriented Riemannian manifold, $\gamma: I \rightarrow M^2$ is a front, $\nu(t)$ is a unit normal vector field, and I an open interval. Then $t = t_0 \in I$ is a 3/2-cusp if and only if $\dot{\gamma}(t_0) = \mathbf{0}$ and $\Omega(\ddot{\gamma}(t_0), \ddot{\gamma}(t_0)) \neq 0$, where Ω is the unit 2-form on M^2 , that is, the Riemannian area element, and the dot means the covariant derivative. When $t = t_0$ is a 3/2-cusp, $\dot{\nu}(t)$ does not vanish (if $M^2 = \mathbf{R}^2$, it follows from Proposition A'). Then we take the (arclength) parameter s near $\gamma(t_0)$ so that $|\nu'(s)| = \sqrt{g(\nu'(s), \nu'(s))} = 1$ ($s \in I$), where $\nu' = d\nu/ds$. Now we define the *cuspidal curvature* μ by

$$\mu := 2 \operatorname{sgn}(\rho) \sqrt{\left| \frac{ds}{d\rho} \right|} \Big|_{s=s_0} \quad (\rho = 1/\kappa_g),$$

where we choose the unit normal $\nu(s)$ so that it is smooth around $s = s_0$ ($s_0 = s(t_0)$). If $\mu > 0$ (resp. $\mu < 0$), the cusp is called *positive* (resp. *negative*). It is an interesting phenomenon that the left-turning cusps have negative cuspidal curvature, although the left-turning regular curves have positive geodesic curvature (see Fig. 4.1). Then it holds that

$$(4.1) \quad \mu = \frac{\Omega(\ddot{\gamma}(t), \ddot{\gamma}(t))}{|\ddot{\gamma}(t)|^{5/2}} \Big|_{t=t_0} = 2 \frac{\Omega(\nu(t), \dot{\nu}(t))}{\sqrt{|\Omega(\ddot{\gamma}(t), \nu(t))|}} \Big|_{t=t_0}.$$

We now examine the case that (M^2, g) is the Euclidean plane \mathbf{R}^2 , where $\Omega(v, w)$ ($v, w \in \mathbf{R}^2$) coincides with the determinant $\det(v, w)$ of the 2×2 -matrix (v, w) . A *cycloid* is a rigid motion of the curve given by $c(t) := a(t - \sin t, 1 - \cos t)$ ($a > 0$), and here a is called the *radius* of the cycloid. The cuspidal curvature of $c(t)$ at $t \in 2\pi\mathbf{Z}$ is equal to $-1/\sqrt{a}$. In [12], the second author proposed to consider the curvature as the inverse of radius of the cycloid which gives the best approximation of the given 3/2-cusp. As shown in the next proposition, μ^2 attains this property:

Proposition 4.1. *Suppose that $\gamma(t)$ has a 3/2-cusp at $t = t_0$. Then by a suitable choice of the parameter t , there exists a unique cycloid $c(t)$ such that*

$$\gamma(t) - c(t) = o((t - t_0)^3),$$

where $o((t - t_0)^3)$ denotes a higher order term than $(t - t_0)^3$. Moreover, the square of the absolute value of cuspidal curvature of $\gamma(t)$ at $t = t_0$ is equal to the inverse of the radius of the cycloid c .

Proof. Without loss of generality, we may set $t_0 = 0$ and $\gamma(0) = \mathbf{0}$. Since $t = 0$ is a singular point, there exist smooth functions $a(t)$ and $b(t)$ such that $\gamma(t) = t^2(a(t), b(t))$.

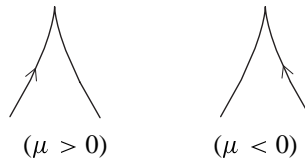


Fig. 4.1. A positive cusp and a negative cusp.

Since $t = 0$ is a $3/2$ -cusp, $(a(0), b(0)) \neq \mathbf{0}$. By a suitable rotation of γ , we may assume that $b(0) \neq 0$ and $a(0) = 0$. Without loss of generality, we may assume that $b(0) > 0$. By setting $s = t\sqrt{b(t)}$, $\gamma(s) = \gamma(t(s))$ has the expansion

$$\gamma(s) = (\alpha s^3, s^2) + o(s^3) \quad (\alpha \neq 0).$$

Since the cuspidal curvature changes sign by reflections on \mathbf{R}^2 , it is sufficient to consider the case $\alpha > 0$. Then, the cycloid

$$c(t) := \frac{2}{9\alpha^2}(t - \sin t, 1 - \cos t)$$

is the desired one by setting $s = t/(3\alpha)$. □

It is well-known that the cycloids are the solutions of the brachistochrone problem. We shall propose to call the number $1/|\mu|^2$ the *cuspidal curvature radius* which corresponds the radius of the best approximating cycloid c .

REMARK 4.2. During the second author's stay at Saitama University, Toshizumi Fukui pointed out the followings: Let $\gamma(t)$ be a regular curve in \mathbf{R}^2 with non-vanishing curvature function $\kappa(t)$. Suppose that t is the arclength parameter of γ . For each $t = t_0$, there exists a unique cycloid c such that a point on c gives the best approximation of $\gamma(t)$ at $t = t_0$ (namely c approximates γ up to the third jet at t_0). The angle $\theta(t_0)$ between the axis (i.e. the normal line of c at the singular points) of the cycloid and the normal line of γ at t_0 is given by

$$(4.2) \quad \sin \theta = \frac{\kappa^2}{\sqrt{\kappa^4 + \dot{\kappa}^2}},$$

and the radius a of the cycloid is given by

$$(4.3) \quad a := \frac{\sqrt{\kappa^4 + \dot{\kappa}^2}}{|\kappa|^3}.$$

One can prove (4.2) and (4.3) by straightforward calculations. The cuspidal curvature radius can be considered as the limit.

ADDED IN PROOF. In a recent authors' preprint, "The intrinsic duality of wave fronts (arXiv:0910.3456)", A_{k+1} -singularities are defined intrinsically. Moreover, the duality between fronts and their Gauss maps is also explained intrinsically.

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