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From the Axiomatic Method of Hilbert's Mathematics to Cavaillès's Philosophy of the Concept

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Abstract

We support the thesis of the fruitfulness of the rationalist philosophy of science, as opposed to the naturalist philosophy of science that is recognised as dominant today. Our position, as far as rationalist philosophy is concerned, is that presented by Cavaillès under the title of "philosophy of the concept", inspired by the Hilbertian axiomatic method. Here are a few points of clarification concerning the assertions in our article:

1. Hilbert does not see the axiomatic system as a mere convention. Thus, he thinks that the axiomatic system has a content of its own in itself that is distinct from the particular object. The latter is relative to the axiomatic system (formal system) that formalizes it.

2. The Hilbertian method makes the existence of this formal content possible. Using this method, Hilbert tried to justify infinitist mathematics, such as set theory. For Hilbert, the condition for this justification is to show that it does not entail contradiction (non-contradiction requirement).

3. One of Hilbert's greatest discoveries is that this proper formal, or abstract structure, is relative to the axiomatic system, and we can make this structure analysable through it.

4. The philosophy of the concept comes from the Hilbertian perspective on the axiomatic method. These features are: 1. the abandonment of the absolute foundation (non-foundationalism), 2. the abstract structure relative to the external formal system, 3. the powers of the expansion of the abstract structure, expanding and modifying the external formal system, 4. Relative truth concerning the formal system that specifies it, 5. Progressive revelation of truth with the thematization and expansion of the abstract structure, 6. This thematization, which presupposes the earlier structure, and then contains it in the later structure, is therefore not arbitrary (non-relativism).

Key words: Cavaillès; Hilbert; Axiomatic Method; Structure

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Debates about the foundations of mathematics undermined the possibility of laying any *a priori* foundation for the certainty of mathematical cognition, weakening the persuasiveness of rationalist epistemology. The program for naturalizing epistemology proposed by Quine (1969) in response to this development is becoming a prominent position in contemporary philosophy of science, advancing in step with cognitive science and other fields. This program can be understood as a position close to some form of empiricism. Naturalized epistemology pragmatically assumes that the legitimacy of scientific research is guaranteed. Simultaneously, this epistemology denies its independence as epistemology by reorganizing itself as a part of scientific cognition (especially psychology). As rationalist epistemology is assumed to have become unable to defend its legitimacy sufficiently, this program for naturalizing epistemology has gained a certain degree of persuasiveness.

Incidentally, traditional rationalism, currently thought difficult to maintain assumes that, depending on one's position, it should be possible to guarantee cognitive certainty *a priori*. However, I would argue that it is possible to remove this assumption in practice and that, once removed, the possibility of rationalism remains. The rationalist position I am speaking of here assumes for scientific cognition an "abstract structure of mathematical logic" (i.e., a mode of thinking realized through mathematics and logic) that cannot be reduced to empirical facts. Cavaillès (1938) detailed the facts of Hilbert's program in the foundational debate and proposed a "philosophy of the concept" as a possible rationalist epistemology (Cavaillès [1947], p. 78). This is deserving of further consideration insofar as it builds on the foundational debate's conclusion, which can be interpreted as evidence against maintaining rationalism despite serving as the position from which rationalist epistemology was actively developed. At a point where some opted to give up on rationalism, why was Cavaillès able to go in the opposite direction and actively forge ahead on a rationalist path? We must solve this "riddle."

The first section of this article presents an understanding of rationalism prerequisite for concluding the debate regarding the foundations of mathematics as a reason to abandon rationalism. Next, it presents a way of understanding the abstract structure necessary for this understanding of rationalism. Based on research on the history of mathematics, the second section provides an overview of the historical context of Hilbert's axiomatic method and his understanding of this method. Subsequently, it clarifies Cavaillès' ideas about "abstract axiomatic theory," which he envisioned by rejecting a specific understanding of abstract structure, building on the fruits and failures of Hilbert's axiomatic theory. In this respect, Cavaillès' ideas about the "system of operations" demonstrate the originality of his understanding of abstract structure. The third section discusses the procedures for "thematization," "idealization," and "paradigm" executed for a "system of operation." Further, it explains the mechanisms of developing the "network-type abstract structure" Cavaillès refers to as "mathematics as a whole," exemplifying physical cognition on this basis. By framing the "philosophy of the concept" as rationalist epistemology centering on this idea of the "system of operations," it becomes more intelligible than ever before. This is a novel contribution to Cavaillès's studies.

1. Assumptions Inherent in the Conclusion that Rationalist Epistemology is Untenable

Why has it become difficult to believe in the possibility of maintaining rationalist epistemology? Rationalist epistemology holds that "the abstract structure of mathematical logic functions at a foundational level of scientific cognition." Conversely, empirical epistemology recognizes that the source of cognition is individual sensory experience, rejecting any *a priori* conditions for cognition.

Let us assume that rationalism argues that the abstract structure of mathematical logic exists as a certainty. Mathematics and logic are prerequisites in science and are especially certain within cognition. However, what would happen if we discovered an inextricable contradiction within? Moreover, what if there was no guarantee promising *a priori* that such a contradiction would not be found? This is an aspect of the conclusion of the debate about the foundations of mathematics. If this is the case, the abstract structure of mathematical logic supporting rationalist arguments is rendered uncertain and indefinite. In this case, it is no different from empirical science filled with experiments and failures of the trial and error of everyday thinking. This may suggest evidence against rationalism, potentially leading us to believe that the abstract structure of mathematical logic that supports rationalism lacks any epistemological privileges and is merely one aspect of empirical cognition. Consequently, we may lose all faith in rationalism and believe that its arguments can no longer be maintained.

This caricature of a description contains some vague assumptions. Where empiricism does not need to guarantee the certainty of universal cognition, rationalism has the condition of needing to guarantee the certainty of universal cognition because of the existence of particular rationalist positions arguing that such certainty is actually possible. For example, Cavaillès considered Kant's transcendental analytic as such a position (cf. Cavaillès [1947], pp. 8–12) and that it supposes the following conditions: (1) the abstract structure of mathematical logic is inherent in consciousness; (2) the abstract structure of mathematical logic as a whole is determinable. The second condition of "being unambiguously determinable" means that there exists no determination that is neither affirmation nor rejection and that the imperative that is all truth can be constructed by structuring determinations and intuitions within consciousness. It follows that this argument can be interpreted as meaning that *the possibility of solving (solvability of) all mathematical problems is guaranteed beforehand*. The opposite of this—"the solvability of all mathematical problems is not guaranteed"—can be defined as "new problems that cannot be solved with existing conceptual systems constantly appear." Introducing new terminology and conceptual systems as well as proving new theorems to solve those problems can be defined as "developing the abstract structure." If so, hypothesizing that solvability is not guaranteed beforehand (i.e., rejection of (2)) means that those problems can be solved only by going outside of the abstract structure inherent in consciousness, which refutes (1. It follows that (1) implies (2). The major condition that the abstract structure of mathematical logic is unambiguously determinable must be part of the debate on the possibility of guaranteeing cognitive certainty by virtue of it being inherent in consciousness.

Nevertheless, this major condition is not a prerequisite for maintaining rationalism because any rationalist position is based on an understanding of the abstract structure. It follows that it should be possible to explore a tenable rationalist path with an understanding of the abstract structure that excludes this major condition. Nonetheless, if we want to explore a new rationalism, we must reconstruct rationalism based on an understanding of the abstract structure as not unambiguously determinable. Therefore, we can no longer seek a basis in transcendental consciousness like Kant did (cf. Cavaillès [1947], pp. 71–75).

Hilbert acts as a good mediator for establishing this kind of rationalism because his axiomatic method and formal system enable an understanding of the abstract structure of mathematical logic that is not merely conceptual. Furthermore, Hilbert believed that, ideally, the abstract structure of mathematical logic should be unambiguously determinable. Hilbert's program was developed based on such thinking. As such, its failure does not demonstrate the impossibility of rationalism itself but the untenableness of an unambiguously determinable abstract structure and rationalism based on such an understanding (i.e., rationalism as claiming cognitive certainty). This also suggests the possibility of a new rationalism that excludes the major condition of an unambiguously determinable abstract structure. Let us consider the facts of Hilbert's program and Cavaillès' understanding of the abstract structure.

2. Hilbert's Axiomatic Mathematics and the Mathematical Theory Cavaillès Derived Therefrom

It would be impossible to cover the entire historical context of Hilbert's axiomatic method here, especially in view of its complexity. Therefore, I provide a simple overview of a few key points needed for subsequent discussion.

Mathematical historians have positioned Hilbert in the middle of two trends (cf. Boniface [2004], Hayashi [2006]). Exemplified by Kronecker, among others, the first is the classical trend of using numeric operations and complex computational algorithms to produce finite arithmetic proofs for algebraic and number theory theorems. The other is the modern trend of Cantor, Dedekind, and Riemann, among others, who sought to develop abstract mathematics by defining free and theoretical concepts such as sets and manifolds. Arguably, his historical position between these two trends determined the essential direction of Hilbert's program (cf. Boniface [2004], p. 51, Hayashi [2006], pp. 155–158).

For Hilbert, the infinite mathematical method[1)] developed by Cantor and Dedekind was extremely important. For example, as Boniface (2004) and Hayashi (2006) have pointed out, he employed the infinite mathematical method in his solution to the Gordan problem, one of his early important accomplishments. However, this infinite mathematical method is extremely powerful as it is. Toward the end of the nineteenth century, when expectations were ballooning about the new world it would inaugurate, the method possessed several paradoxical properties, such as the Burali-Forti paradox and Russell's paradox of the comprehension axiom. Such discoveries came as a big shock to Hilbert. It must have seemed necessary to defend the infinite mathematical method to preserve his methodology.[2)] Such was the historical context of the debate about the foundations of mathematics within which Hilbert's axiomatic method was situated.

Hilbert's intention behind developing the axiomatic method should be understood in the context of defending this conceptual infinite mathematics through a finitist approach, which is incredibly similar to Kant's transcendental conceptualism (cf. Boniface [2001], ch. 6). Dismissing the entire concept of infinite mathematics as groundless and returning from finite mathematics to productive mathematics, as advocated by Kronecker, must have seemed a fatal retrogression in Hilbert's eyes (cf. Hilbert [1918]). Nevertheless, simply viewing the infinite sets dealt with in infinite mathematics as actually existing, as Cantor and Dedekind had, leads to paradoxes. Therefore, Hilbert completely axiomatized its content through axiomatic systems to legitimize infinite mathematics and demonstrate that no contradictions can be derived from those axiomatic systems (i.e., non-contradiction of the axiomatic systems) (cf. Hilbert [1900]). This was the goal of Hilbert's axiomatic method within its historical context. Nonetheless, this pursuit also led to Hilbert's proof theory facing new issues in the form of axiomatic system characteristics, such as inter-axiomatic interdependence, inter-model categoricity, and the syntactic completeness of axiomatic systems (cf. Hilbert [1900]). Incidentally, a real number system cannot be used as a model to prove the ultimate non-contradiction of an axiomatic system, although this is what Hilbert (1899) did. As Poincaré noted, recognizing an infinite object (e.g., a set of real numbers) as a model for proving the non-contradiction of infinite mathematics causes a vicious cycle.

Hilbert made extremely important innovations to avoid this vicious cycle. Notably, he had the idea to use formal systems comprising symbols. A formal system allows for all content-related theorems to be derived from that formal system through mechanical procedures by completely formalizing content-related theories meant to be reproduced in a system of symbols that carry no meaning in themselves. One example is the formal system in Russell and Whitehead's *Principia Mathematica*. Other examples of refined formal systems include the formal system of axiomatic set theory—known as von Neumann-Bernays-Gödel (NBG) set theory—and that of primitive recursive arithmetic (PRA).[3)] With a formal system, there is no direct interaction with the formalized content; therefore, mathematics is only performed via the operating symbols of formalized contents (terms and equations). Such formal systems are syntactically complete (i.e., a contradiction occurs when a proposition neither proven nor disproven is added to the axiomatic system). If all propositions that are true content-wise can be formally derived, the solvability of the problems in that theory is concretely realized beforehand. This suggests that if all mathematical actions are replaced with such collections of formal systems, the entire abstract structure becomes unambiguously determinable.

This idea is rooted in algebra and is extremely close to Kronecker's ideas (cf. Boniface [2003], pp. 232–235, Hayashi [2006], p. 134). I use a similar algebraic example to make it easier to visualize a formal system. Cantor and Dedekind defined irrational numbers as infinite sequences of rational numbers (e.g., Dedekind's fixed chain definition and Cantor's basic sequence definition). Conversely, Kronecker added the equality $x^2 = 5$ to the entire algebraic expression of rational coefficients as an axiom. The whole algebraic system with "indeterminate x" defined by this axiom is considered a new number system with √5 added to the rational numbers as a whole (cf. Hayashi [2004], p. 153). Expanding this number system allows for the solution to be found *only through finite transformations between equations*. According to Boniface (2004), many differences exist between Hilbert's ideas and those of Kronecker. Nonetheless, a key difference is that where Hilbert used symbols that carry no meaning themselves but are morphologically distinguishable, Kronecker used natural numbers as givens. For Hilbert, there was a significant difference between using natural numbers and symbols as givens. After all, if he did not recognize symbols as givens, he would have been unable to propose a proof theory whose givens were the terms and equations of the signified formal systems or a proof diagram comprising the finite sequences of an inference diagram. Natural numbers are not sufficient to formalize a proof theory or set theory. If this is not possible, there is a risk of not being able to achieve the initial goal of defending infinite mathematics.

To achieve his goal, Hilbert could not presuppose infinite operations of the formal systems themselves because tolerating infinite operations of formal systems was assumed to trigger new vicious cycles. Therefore, he formalized the operations needed in the formal systems with intuitively distinguishable symbols in the form of logical expressions and proof diagrams and thought of their finite combinations as new "content-related" (*inhaltlich*) (Hilbert [1926]) mathematical operations. Thus, he could attempt to prove the non-contradiction of those formal systems through finite inference despite the formal systems of first-order arithmetic, the set theory, and the mathematical analysis formally containing contents beyond the finite. This was Hilbert's second innovation and the cornerstone of his proof theory. If a formal system's non-contradiction can be proven through a content-related (finite) procedure, it guarantees the existence of an abstract structure that contains "what is ideal" (*idealen*) beyond the content-related operations (cf. Hilbert [1926], Giaquinto [2002], pp. 145–157).

The objective of Hilbert's program may be interpreted as evidencing how "not being able to derive actual propositions considered falls from a finite position from the arithmetically suitable formal system T" as the "real-soundness" of a formal system, demonstrating this through a finite method and substituting this T with a formal system of mathematical analysis or set theory to establish the same for each of them (cf. Giaquinto [2002], pp. 179–180). If T were subsequently found to be contradictory, then $\neg (0 = 0)$ or any other proposition considered false from a finite position can be derived with T; conversely, if real propositions considered false from a finite position cannot be derived with T, then T is non-contradictory. The stipulation of "real-soundness" means that a system is sound if no real propositions considered falls from a finite position can be derived from a given system. As such, the non-contradiction of T can be derived from its real-soundness through finite inference such that finite proof of real-soundness can be expanded to finite proof of non-contradiction (cf. Giaquinto [2002], pp. 179–180).

However, Gödel's second incompleteness theorem showed that there exists no proof of T's finite non-contradiction. As such, no finite proof of T's real-soundness exists either. Furthermore, Hilbert's idea of syntactic completeness of first-order arithmetic—that all propositions or their rejections can be proven—is also undermined by the first incompleteness theorem. This clarified that completely describing contents that you seek to formalize as a formal system (syntactic completeness) is impossible with mathematics that include first-order arithmetic. It follows that as long as first-order arithmetic is included in mathematics as a whole, mathematical actions as a whole cannot be unambiguously determined ("I believe that I ought to discard the notion of defining mathematics" [Cavaillès (1939), p. 599]).

The central notion of this paper is not the mathematical conclusion of Hilbert's program, but the fact that the following understanding of the abstract structure became untenable because of the failure of the program. First, regarding the abstract structure as an abstract notion is no longer a sufficiently trustworthy proposition, a point further demonstrated when considering the paradox of Cantor's simple set theory. Second, if we consider how the axioms of infinity and reducibility in Russel and Whitehead's formal systems cannot be reduced to logical tautologies, it is impossible to reduce the abstract structure to a first-order predicate logic tautology. As such, it is also impossible to define all abstract structures that include arithmetic from a finite position—PRA—unambiguously and completely through formal systems. Reducing the contents of mathematics to procedures for deriving finite theorems is not possible.

However, Cavaillès translated these outcomes into interpretations that affirm the abstract structure. As Cavaillès (1939) wrote, "The necessarily incomplete property of all mathematical logic demonstrates that each development of that theory requires the intervention of new rules of inference" (p. 600). He likely could argue as such because he interpreted the impossibility of unambiguously determining mathematical actions as a whole as the *potential for the appearance of new problems* by formalizing the properties and operations of existing theory through concepts. According to Cavaillès (1939), "Imported terms are demanded by the solving of problems, and the appearance of new terms from already existing terms in turn leads those new terms to establish new problems" (p. 594). The positive substance of mathematical contents not being reducible to form should be that formalization through new conceptual systems generates unsolvable new problems. It follows that contents not reducible to form (i.e., problems) demand further formalization, advancing the abstract structure. For example, formalization by manifolds created the problem of inter-coordinate bijection, which required set theory. This set theory created the problem of having to prove the non-contradiction of an axiomatic system, in response to which formal systems and proof theory were formulated. The formulation of formal systems and proof theory *made it possible to propose* new problems related to formal systems, such as the problems of computability in the theory of computation and computational complexity. *If the possibility of all problems is established*, all mathematical problems, regardless of whether they are singular or multiple, can be reduced to procedures for deriving theorems in formal systems. Nonetheless, *if this is not the case*, particular problems in mathematics can be solved through formalization; simultaneously, *new problems can be proposed through that formalization*. This, in turn, requires additional formalization, prompting the further development of the abstract structure.

Hilbert appears to have been insistent on the syntactically complete axiomatic system (e.g., this appears to have been suggested by the addition of the "axiom of completeness" in the fourth edition of *The Foundations of Geometry*). Cavaillès referred to this as "concrete axiomatic theory," distinguishing it from his “abstract axiomatic theory.” According to Cavaillès (1947), “Concrete axiomatic theory like Hilbert’s axiomatic theory with regard to geometry […] still retains a dimension of categorical argument that suggests the notion of comprehensive definition. Together with abstract axiomatic theory, matters appropriate to this and that are appearing in sufficient light. […] However, the definition is no longer comprehensive at this point. The definition presumes the constructive operations of already existing systems—together with operational characteristics and outcomes” (p. 73).

The position of viewing syntactic completeness as a prerequisite for the axiomatic method—based on the premise that establishing syntactic completeness is not a problem in itself, as in the case of Boolean algebra or propositional logic—is similar to the ideas of the Bourbaki group, a contemporary with whom Cavaillès was likely personally acquainted, and Emmy Noether, with whom he published *Exchanges Cantor-Dedekind* (cf. Sinaceur [1987], Sinaceur [1994], pp. 19–20). They had the following two points in common (cf. Bourbaki [1948]). First, by not viewing syntactic completeness as a prerequisite for the axiomatic method, axiomatic systems of groups, topologies, sequential structures, and such also become part of the axiomatic method. Second, by connecting axiomatic systems, combining described theories, and such *to solve problems*, connections among theories are actively added. The fact that axiomatic systems become associated with other theories when they have models also holds for syntactically complete theories. Importantly, axiomatic systems do not become complete in themselves; rather, associations with other theories are actively imagined to solve problems. Such mutual communication among theories through formalization is assumed to grant the abstract structure as a whole real existence. For example, topological algebra (i.e., continuity defined between algebraic calculations) and algebraic topology (i.e., topological relations subject to algebraic calculations) may be thought of as complex structures. Additionally, there can be, for example, repetitions of similar structures like group structures within multiple theories. Thus, there is mutual communication among theories, and these lines of communication are reorganized so that previously unsolvable problems frequently become solvable. For Cavaillès, the abstract structure “makes up a sufficiently developed organic system” (Cavaillès [1939], p. 600). I refer to this abstract structure comprising multiple complexly intertwined theories, where mutual connections are reorganized either overall or in part every time new formalizations take place, as a “network-type abstract structure.”

Simultaneously, there are fundamental differences between Bourbaki, Noether, and Cavaillès. The first difference is that Cavaillès thought of this “network-type abstract structure” as historical (cf. Cavaillès [1939], p. 600, Sinaceur [1994], pp. 22–25).

The second difference, which I consider more important, is that Cavaillès seemingly assumed set theory as an inadequate basis for this abstract structure. Conversely, Bourbaki is recognized as a classic example of postwar mathematics, which regarded set theory as the ontological basis for abstract structure. As such, this is a key difference. Cavaillès was interested in understanding what mathematicians do when they formalize concepts in existing theories to engage with new problems. While set theory is a convenient tool for presenting solution outcomes, it is ill-suited for understanding the procedures conducted when connecting theories and discovering new distillations to reach solutions. This is because set theory must fundamentally begin with existing elemental sets, suggesting that the solution is already given.

Cavaillès thought that the unification of mathematical objects can only happen relative to operations. He referred to what produces the unification of objects through the performance of operations as the "system of operations" (*système d' opération*) (Cavaillès [1947], p. 74). In short, he sought the unification of mathematical theory not in the objects but in the amalgamation of the "operations" that facilitate the production of objects. This "operational homogeneity" (Cavaillès [1947], p. 47) was described in an axiomatic format. One example is the system of operations of directed graphs in category theory. Another is the iterative operation of cumulative hierarchies of sets. The advantages of an "operation system," as shown above, are that it can potentially 1) avoid the ontological issues of the foundations of mathematics and 2) describe the mechanisms for forming a network-type abstract structure. The next section explains the second advantage in greater detail.

3. Cavaillès' "Philosophy of the Concept"

As discussed above, Cavaillès expanded the insights he gleaned from analyzing Hilbert's axiomatic method and formal systems into scientific cognition in general in his "philosophy of the concept." While these ideas were initially little more than a research program, their legitimacy was confirmed in numerous studies of the history and philosophy of science after his death.[4)]

Cavaillès referred to the aforementioned network-type abstract structure as "the entirety of mathematics." According to Cavaillès (1947), "The entirety of mathematics (*le corps entier de mathématique*) develops itself through a single motion based on multiple forms, passing through multiple stages. This entirety of mathematics can also involve technological artefacts so that it may or may not execute the same functions as cognition. If cognition is granted by mathematics, it cannot be anything other than deductive reasoning" (p. 74).

To understand this cognition based on abstract structure, shifting the focus of rationalist epistemology from the psychological processes that gain cognition to the objective network-type abstract structure is necessary because it is impossible to determine the entirety of a network-type abstract structure unambiguously so that developments and alterations by formalization essentially become triggers for the abstract structure. Moreover, the "philosophy of the concept" does not analyze the processes of human consciousness that conduct cognition but the mechanisms for the development and alteration of the abstract structure and the gradual expansion of deductive cognition they enable.

The existence of problems that cannot be solved by existing formalization prompted the development of the abstract structure into a network. Unsolved problems encourage formalization by creating new concepts and introducing new terms needed for the solutions. Such formalization is performed through existing mathematical operations. Described in an axiomatic format, the "system of operations" is executed to produce unification with the theoretical objects. The formalization of the system of operations allows the various parts of the abstract structure to enter into mutual communication and reorganizes those lines of communication. Cavaillès referred to the procedures of such formalization as "thematization," "idealization," and "paradigm." While discussing the details of these procedures here is not possible, Cavaillès' explanation of their definitions is noteworthy. "Thematization means to transform a certain operation into the element of a higher-order operational domain" (Cavaillès [1938], p. 177). "Idealization means demanding that an operation executed while randomly restricted by some external circumstance is made free of that external restriction, with the external restriction being one stemming from the creation of an object system no longer subject to intuition" (Cavaillès [1939], p. 602). "Paradigm" means a movement that "manifests the internal principles of a transmutation" (Cavaillès [1947], p. 27) and recovers its form by replacing what is given with a "variable" (Cavaillès [1947], p. 29). Therefore, Cavaillès argued that "the entirety of mathematics (*le corps entier de mathématique*) develops itself through a single motion based on multiple forms, passing through multiple stages" (Cavaillès [1947], p. 74).

Let us consider an example of physical cognitive development through such an abstract structure. Cavaillès wrote, "The processes of an experiment—since [mathematical chains] belongs to another essence—develop so as to produce new processes through *unique* chains independent of mathematical chains, at least as long as they remain as such" (Cavaillès [1947], p. 40). For example, while an experiment can show that electric power and magnetism are mutually related, it cannot explain the mechanisms of their relationship. To identify the mechanisms of the mutuality of electric power and magnetism, you need Maxwell's equations. As Cavaillès (1947) noted, "Physical relationships appear where these two processes (mathematical and physical chains) intersect" (p. 40). These equations describe the experimental behavior of electric power and magnetism as well as the rules governing that behavior. To solve algebraic equations, it is necessary to imagine the site as a vector space, for example, and employ vector analysis (it does not necessarily have to be vector analysis, providing the operations are executable). This abstract structure is usable because of the possibility of "coordination between the two"—between physical process chains and abstract structure process chains (Cavaillès [1947], p. 40). The "coordination" of the abstract structure and physical chains allows for the electromagnetic field, enabling us to discern the mutuality and unification of electric power and magnetism as something beyond a mere experimental fact. Furthermore, the recognition of "exist" and "not exist" as scientific objects depends on the network-type abstract structure; hence, epistemology and ontology are developed in parallel regarding such scientific objects.

The "philosophy of the concept" is characterized by the idea that scientific cognition is considered correlatively to systems of operation. Scientific cognition is executed with some kind of operation system as its medium. Although dependent on the abstract structure as the entirety of the actually useable systems of operation, this abstract structure develops while being reorganized by the "thematization," "idealization," and "paradigm" procedures executed to solve problems. As such, the network-type abstract structure cannot be contained within human consciousness *as long as the problems posed cannot be solved outside existing concepts* (Cavaillès [1947], pp. 71–75). Advancing our scientific understanding means deriving new cognition by yielding to the reorganization and development of the abstract structure. For this kind of rationalist epistemology, "One essential issue is that development […] is permanent alterations of contents through deepening and erasure" (Cavaillès [1947], p. 78).

Considering this position of the "philosophy of the concept," it is easy to understand natural science requires the mediation of artificial observational instruments, symbolically expressing and analyzing the quantities and relationships gained from the observational instruments are essential, and science is not an individual act but collective and transmissive. After all, scientific objects are not recognized through individual sensory experience but in the system of operations describing them with symbols and the network-type abstract structure that binds such systems together. For instance, length, time, weight, and other physical objects are all scientific objects recognized through an abstract structure consisting of such systems of operation. Further, scientific thought is made possible not by individual inner consciousness but by objective and effective network-type abstract structure; thus, shared understanding becomes possible among different individuals, allowing for the pursuit of common research interests across time and space.

To defend the final reasonable conclusion, employing a philosophical view not particularly commonsensical is necessary. What Cavaillès called the "philosophy of the concept" and promoted as a new rationalist epistemology is such a view. In lieu of a conclusion, the following list clarifies the characteristics of this view:

1. It is necessary to abandon some kind of absolute basis or reduction for certainty. There exists no absolute formal system to which the absolute structure can be completely reduced. The certainty of cognition only gradually grows in step with the abstract structure's development; hence, no final basis can be expected.

2. Where scientific cognition cannot be reduced to an individual's inner consciousness or mind, this is possible within the entirety of the network of the abstract structure comprising axiomatically described systems of operation.

3. As new problems are posed through the formalization of systems of operation, new formalization is demanded. This enables and demands the further development and reorganization of the abstract structure.

4. The formalization of a system of operation is executed via "thematization," idealization," and "paradigm," through which multiple theories enter into mutual communication, and the lines of communication are reorganized.

5. Regardless of how basic they are, scientific objects are correlative to the system of operation, with their recognition taking place within the network-type abstract structure.

6. The actual existence of scientific objects is only recognized correlatively to operations within the system of operation, which effectively makes them presentable. It follows that this position is incompatible with one that demands the actual existence of the scientific objects themselves (Platonism).

Cavaillès argued that an essential problem of the philosophy of science is the gradual development of the abstract structure that accompanies formalization. The goal of his "philosophy of the concept" as philosophy of science is to clarify this problem of formalization (i.e., the source, limitations, and so on of its essential characteristics and productivity) by analyzing diverse academic domains. The foregoing is the path of Cavaillès' "philosophy of the concept," a new rationalist epistemology, that only came into view after the foundational debate.

Notes

1) The term "infinite mathematical method" is a rephrasing of "infinite arithmetization" used by Hayashi (2006). It can be understood as solutions that use set theory or transfinite arguments.
2) As pointed out by Hayashi (2006) and Boniface (2004), Hilbert was aware of the importance of the finite arithmetic method. This can be seen, for example, in how Hilbert himself proposed a finite solution to the Gordan problem. This likely influenced Hilbert's finitist position on proof theory, as explained later, and the issue of first-order arithmetic completeness (cf. Hayashi [2006], pp. 238–239, Boniface [2004], pp. 128–129).
3) The length of this article does not allow for a concrete examination of these formal systems. For more about the ZF system of axioms in the NBG and Zermelo-Fraenkel set theories, see Kurata (1996); for PRA, see Hayashi (2006); for LK and NK, which are formal systems of classical theory, see Ono (1994).
4) For mathematics, Granger (1994) is important. For physics, recent major outputs include Panza (2005) and Harada (2006).

References

Boniface J. (2003), De l'indéterminée à la variable, (coordoné par J. Boniface), *Calculus & Formes de l'Actovoté Mathématique*, Ellipes Édition Marketing S. A.

Boniface J. (2004), *Hilbert et la notion d'existence en mathématiques*, J. Vrin.

Bourbaki (1948), Architecture de mathématique, présentés par Le Lionnais F., *Les grand courants de la pensée mathématique*, Cahiers du Sud.

Cavaillés J. (1938), *Méthode axiomatique et formalism*, Hermann.

Cavaillés J. (1939), "La pensée mathématique", rééd. dans Cavaillès (1994), pp. 593–630.

Cavaillés J. (1947), *Sur la logique et la théorie de la science*, PUF.

Cavaillés J. (1994), *Œuvres completes de philosophie des sciences*, Hermann.

Granger G.-G. (1994), *Formes operations objets*, J. Vrin.

Giaquinto M. (2002), *The Search for Certinity*, Oxford University Press.

Harada M. (2006), *La physique au carrefour de l'intuitif et du symbolique*, J. Vrin.

Hayashi S. (2006), Dai-2-bu kaisetsu (Part 2 Commentary), Gödel (Hayashi/Yasugi trans.), *Fukanzensei teiri (Axiom of Incompleteness)*, Iwanami Shoten, pp. 73–305.

Hilbert D. (1899), *Grundlagen der Geometrie*, B. G. Teubner, 1956.

Hilbert D. (1900), Über den Zahlbegriff, *Jahresbericht der Deutschen Mathematiker Vereinigung*, t. 8.

Hilbert D. (1918), Axiomatisches Densken, *Mathematische Annalen*, t. 78, pp. 405–415.

Hilbert D. (1926), Über das Unendliche, *Mathematische Annalen*, t. 95, pp. 161–190.

Kurata R., Shinoda J. (1996), *Koriteki shugoron (Axiomatic Set Theory)*, Kawai Institute for Culture and Education.

Kobayashi M. (1996), *Kagaku tetsugaku (Philosophy of Science)*, Sangyo Tosho.

Ono H. (1994), *Joho kagaku ni okeru ronri (Logic in Information Science)*, Nippon Hyoron Sha.

Panza M. (2005), *Newton et les origins de l'analyse: 1664–1666*, Éditions Albert Blanchard.

Quine W. V. (1969), Epistemology naturalized, in *Ontological Relativity and Other Essays*, Columbia University Press.

Sinaceur H. (1987), Structure et concept dans l'épistémologie mathématique de Jean Cavaillès, *Revue d'histoire des sciences*, 1987, t. 40, n. 1, pp. 5–30.

Sinaceur H. (1994), *Jean Cavaillès Philosophie mathématique*, PUF.