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ON PRIMITIVE Γ-RINGS WITH MINIMAL ONE-SIDED IDEALS

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Nobusawa developed the notion of a Γ-ring which is more general than a ring. He obtained an analogue of the Wedderburn theorem for simple Γ-ring with minimum condition on one-sided ideals. Recently, Barnes weakened slightly the defining conditions for a Γ-ring, introduced the notions of prime ideals, primary ideals and radical for a Γ-ring, and obtained analogues of the classical Noether-Lasker theorems concerning primary representations of ideals for Γ-rings. In this paper, the notion of primitivity is extended to Γ-ring. The class of primitive Γ-rings having non-zero minimal left ideals is a natural extension of the class of simple Γ-rings satisfying the minimum condition on one-sided ideals. The main theorem here gives a characterization of these Γ-rings as certain Γ-rings of continuous semi-linear transformations. This is a generalization of the well known structure theorem for primitive rings given by Jacobson as well as the result of Nobusawa for simple Γ-rings.

2. Continuous semi-linear transformations. In this section we gather together the basic facts of continuous semi-linear transformations that will be used in the sequel of this paper.

Let (V, W) be a pair of dual spaces over a division ring D. That is, V is a left vector space over D and W is a total vector subspace of the conjugate space of V. If $v \in V$, $w \in W$, we use the notation (v, w) for the image vw in D. There is an associated topology on V, a subbase at zero consisting of the kernels of the functionals in W. The resulting topology is called the W-topology of V.

A mapping T of a left vector space V over a division ring D into a left vector space V' over a division ring D' is called a semi-linear transformation if T is a group homomorphism of $(V, +)$ into $(V', +)$ and if there exists an isomorphism σ of D onto D' such that for all $v \in V$, $d \in D$, we have $(dv)T = (d^{\sigma})(vT)$. When we wish to indicate σ explicitly, we shall speak of the "semi-linear transformation (T, σ)".

Suppose that (V, W) and (V', W') are pairs of dual vector spaces over D and D' respectively and (T, σ) is a semi-linear transformation of V into V'. The mapping T^* of W' into W is called an adjoint of T if $(vT, w')^{\sigma^{-1}} = (v, w'T^*)$

holds for all $v\in V$ and $w'\in W'$. In this case, clearly (T^*, σ^{-1}) is a semi-linear transformation of W' into W. We shall denote by $\mathcal{L}(V, W; V', W')$ or simply by $\mathcal{L}(V, V')$, the additive group of all continuous semilinear transformations of V, topologized by W-topology, into V', topologized by W'-topology. We shall also denote by $\mathcal{F}(V, W; V', W')$ or simply by $\mathcal{F}(V, V')$, the subgroup of $\mathcal{L}(V, V')$ consisting of all continuous semi-linear transformations of V into V' of finite rank.

In the case that $D'=D$, V and V' are of finite dimensions m and n respectively, and σ is the identity mapping on D, then $\mathcal{L}(V,V')=\mathcal{F}(V,V')$ is the additive group of all $m\times n$ matrixes over D. In the case that $D=D'$, $V=V'$, $W=W'$, and σ is the identity mapping on D, then $\mathcal{L}(V, V')$ is the ring of all continuous linear transformations on V topologized by W-topology, and is known to be a primitive ring with minimal one-sided ideals.

The following two theorems are basic throughout our discussion. The proofs can be found in [3] and will be omitted.

Theorem 2.1. *Let (V,W) be a pair of dual vector spaces over a division ring D and f is linear functional on V. Then f is a continuous mapping of V, topologized by W-topology, into D, topologized by discrete topology if and only if there exists $w\in W$ such that $vf=(v, w)$ for all $v\in V$.*

Theorem 2.2. *Let (V,W) and (V',W') be pairs of dual vector spaces over division rings D and D' respectively. A semi-linear transformation (T, σ) of V into V' is continuous if and only if it has an adjoint (T^*, σ^{-1}).*

The next theorem is an analogue of Proposition 1 of [3, p. 74].

Theorem 2.3. *Let (V,W) and (V',W') be pairs of dual spaces over division rings D and D' respectively. A semi-linear transformation $(T, \sigma)\in\mathcal{F}(V,V')$ if and only if there exist $w_i\in W$, $v'_i\in V'$, $i=1,2,\dots,n$, such that $vT=\Sigma(v, w_i)^\sigma v_i'$, for all $v\in V$.*

Proof. Assume that $T\in\mathcal{F}(V, V')$. Let $\{v_1', v_2', \dots, v_n'\}$ be a basis of VT over D'. Then for each $v\in V$, $vT=\Sigma\phi_i(v)v_i'$, where $\phi_i(v)\in D'$, $i=1, 2, \dots, n$, is uniquely determined. The mapping $v\rightarrow\phi_i(v)^{\sigma^{-1}}$ is a linear functional on V. Also this mapping is composed of continuous mappings T, a linear functional on VT, and σ^{-1}. Hence it is continuous. By Theorem 2.1 there exists $w_i\in W$, such that $\phi_i(v)^{\sigma^{-1}}=(v, w_i)$, or $\phi_i(v)=(v, w_i)^\sigma$ for each $i=1, 2, \dots, n$. Thus, $vT=\Sigma(v, w_i)^\sigma v_i'$.

Conversely, if T is a mapping of V into V' of this form. It is clear that T is a semi-linear transformation of finite rank. Let T^* be the mapping of W' into W defined by $w'T^*=\Sigma w_i(v_i', w')^{\sigma^{-1}}$. Then we can see easily that $(vT, w')=(v, w'T^*)^\sigma$, for all $v\in V$, $w'\in W'$. Consequently, T^* is an adjoint of T, and hence T is continuous.

3. Γ-rings. Let M and Γ be two abelian additive groups. If, for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, the following conditions are satisfied,

(1) $x\alpha y$ is an element of M,

(2) $(x+y)\alpha z = x\alpha z + y\alpha z$,
$x(\alpha+\beta)z = x\alpha z + x\beta z$,
$x\alpha(y+z) = x\alpha y + x\alpha z$,

(3) $(x\alpha y)\beta z = x\alpha(y\beta z)$,

Then following Barnes [1] M is called a Γ-ring. We may note that $0\alpha y = x0y = x\alpha 0 = 0$ for all $x, y \in M$ and $\alpha \in \Gamma$.

If the defining conditions for a Γ-ring are strengthened to

(1′) $x\alpha y$ is an element of M, $\alpha x\beta$ is an element of Γ,

(2′) same as (2),

(3′) $(x\alpha y)\beta z = x(\alpha y\beta)z = x\alpha(y\beta z)$,

(4′) $x\alpha y = 0$ for all $x, y \in M$ implies $\alpha = 0$,

we then have a Γ-ring in the sense of Nobusawa [4].

A Γ-ring M in the sense of Nobusawa is simple if, for any nonzero elements $x, y \in M$, there exists $\gamma \in \Gamma$, such that $x\gamma y \neq 0$.

An additive subgroup I of M is a right (left) ideal of M if $I\Gamma M \subseteq I (M\Gamma I \subseteq I.)$ Here $I\Gamma M$ denotes the set of all finite sums $\Sigma a_i\gamma_i x_i$, where $a_i \in I$, $\gamma_i \in \Gamma$, $x_i \in M$. If I is both a left and a right ideal of M, then I is a two-sided ideal or simply an ideal of M. A non-zero right (left) ideal I of M is minimal if the only right (left) ideal of M contained in I are 0 and I itself.

Let M be a Γ-ring and let F be the abelian free group generated by the set of all ordered pairs (γ, x), where $\gamma \in \Gamma$, $x \in M$. Let A be the group of all those elements $\Sigma m_i(\gamma_i, x_i)$ in F satisfying $\Sigma m_i x\gamma_i x_i = 0$ for all $x \in M$, where m_i's are integers. Denote by R the factor group F/A, and by $[\gamma, x]$ the coset $A+(\gamma, x)$. Every element in R can then be expressed as a finite sum $\Sigma[\gamma_i, x_i]$. Also it can be verified easily that $[\alpha, x]+[\beta, x] = [\alpha+\beta, x]$ and $[\alpha, x]+[\alpha, y] = [\alpha, x+y]$, for all $\alpha, \beta \in \Gamma$, and $x, y \in M$.
Now, define a multiplication in R by

$$\Sigma_i[\alpha_i, x_i]\Sigma_j[\beta_j, y_j] = \Sigma_{i,j}[\alpha_i, x_i\beta_j y_j]\,.$$

Clearly, the multiplication is well defined and R forms a ring. Furthermore, if we define a composition on $M \times R$ into M by $x\,\Sigma[\alpha_i, x_i] = \Sigma x\alpha_i x_i$, for $x \in M$, and $\Sigma[\alpha_i, x_i] \in R$, then M forms a right R-module. We shall call R the right operator ring of the Γ-ring M. Using similar pattern, we may construct a left operator ring L of M so that M is a left L-module. $(L, +)$ is the factor group G/B, where G is the abelian free group generated by the set of all ordered pairs (x, γ) with $x \in M$ and $\gamma \in \Gamma$, and B is the subgroup of G consisting of all elements $\Sigma m_i(x_i, \gamma_i)$ with the property that $\Sigma m_i x_i\gamma_i x = 0$ for all $x \in M$. Without causing any ambiguity we shall denote by $[x, \gamma]$ the coset $B+(x, \gamma)$. The multiplication in L is defined by $\Sigma_i[x_i, \alpha_i]\Sigma_j[y_j, \beta_j]=\Sigma_{i,j}[x_i\alpha_i y_j, \beta_j]$. It might be worth to note that I is a right (left) ideal of M if and only if I is a R-submodule (L-submodule) of M.

Let M be a Γ-ring and $S\subseteq M$. For any positive integer n, we shall denote by S^n the set $S\Gamma S\Gamma\cdots\Gamma S$ (all finite sums $\Sigma x_1\gamma_1 x_2\gamma_2\cdots\gamma_{n-1}x_n$ with $x_i\in S$, $\gamma_i\in\Gamma$). If $\Delta\subseteq\Gamma$, we shall denote by $[S, \Delta]$ the set of all finite sums $\Sigma[x_i, \gamma_i]$, where $x_i\in S$, $\gamma_i\in\Delta$. The notation $[\Delta, S]$ will be defined analogously.

An one-sided ideal I of M is strongly nilpotent if $I^n=0$ for some positive integer n.

Theorem 3.1. *Let M be a Γ-ring. If M has no non-zero strongly nilpotent left ideals then M has no non-zero strongly niplotent right ideals.*

Proof. Let I be a non-zero strongly nilpotent right ideal of M and $I^n=0$. Then $K=I+M\Gamma I$ is a left ideal of M. By induction on k, it can be shown that $K^k\subseteq I^k+M\Gamma I^k$, and hence $K^n\subseteq I^n+M\Gamma I^n=0$, so K is a non-zero strongly nilpotent left ideal of M.

Theorem 3.2. *Let M be a Γ-ring, and I be a minimal left ideal of M. Then either $I^2=0$, or $I=M\gamma e$ for some $\gamma\in\Gamma$, $e\in I$, where $e\gamma e=e$.*

Proof. If $I^2\neq 0$, then there exist $\gamma\in\Gamma$, $a\in I$, such that $I\gamma a=I$, so e exists in I such that $e\gamma a=a$. Thus we have $e\gamma e\gamma a=e\gamma a$, or $(e\gamma e-e)\gamma a=0$. Let $K=\{x\in I\colon x\gamma a=0\}$. Clearly K is a left ideal of M properly contained in I. Hence, $K=0$, so $e\gamma e-e=0$, or $e=e\gamma e$.

Theorem 3.3. *Let M be a Γ-ring, and $e\gamma e=e$, where $e\in M$, $\gamma\in\Gamma$. Then $M\gamma e$ is a minimal left ideal of M if and only if $L[e, \gamma]$ is a minimal left ideal of L, where L is the left operator ring of M.*

Proof. Assume that $M\gamma e$ is minimal. If I is a non-zero left ideal of L contained in $L[e, \gamma]$, then $0\neq Ie\subseteq M\Gamma e\gamma e=M\Gamma e$. By the minimality of $M\Gamma e$, we obtain $Ie=M\Gamma e$. Hence, $I=I[e, \gamma]=L[e, \gamma]$, and $L[e, \gamma]$ is minimal.

Conversely, assume that $L[e, \gamma]$ is a minimal left ideal of L. If I is a non-zero left ideal of M contained in $M\gamma e$, then $0\neq[I, \gamma]\subseteq[M, \gamma][e, \gamma]\subseteq L[e, \gamma]$. By the minimality of $L[e, \gamma]$, we obtain $[I, \gamma]=L[e, \gamma]$. Consequently, $I=I\gamma e=M\Gamma e\gamma e=M\Gamma e$, and $M\Gamma e$ is minimal.

Theorem 3.4. *Let M be a Γ-ring. If $M\gamma e$ is a minimal left ideal of M, where $e\in M$, $\gamma\in\Gamma$, and $e\gamma e=e$, then*

(i) *$R[\gamma, e]$ is a minimal left ideal of R,*
(ii) *$[e, \gamma]L[e, \gamma]$ is a division ring.*
(iii) *$[\gamma, e]R[\gamma, e]$ is a division ring,*

where R and L are respectively the right and left operator rings of M.

Proof. To prove (i), let I be a non-zero left ideal of R contained in $R[\gamma, e]$. From $0 \neq MI = MI[\gamma, e] \subseteq M\gamma e$, we have $MI = M\gamma e$, and $R[\gamma, e] = RI \subseteq I$. Thus $R[\gamma, e] = I$, and $R[\gamma, e]$ is minimal.

(ii) and (iii) are immediate consequences of Theorem 3.3 and (i) of this theorem (see [3, p. 65]).

Theorem 3.5. *Let M be a Γ-ring having no non-zero strongly nilpotent one-sided ideal. If $D = [e, \gamma]L[e, \gamma]$ is a division ring, where $e\gamma e = e$, then $M\gamma e$ is a minimal left ideal of M, where L is the left operator ring of M.*

Proof. It is easy to see that if M has no non-zero strongly nilpotent one-sided ideals, then L has no non-zero nilpotent one-sided ideals. Hence if D is a division ring then $L[e, \gamma]$ is a minimal left ideal of L. By Theorem 3.3, $M\gamma e$ is therefore a minimal left ideal of M.

We should note that Theorems 3.1, 3.2, 3.3, and 3.5 all remain true if we replace L by R, R by L, left by right, right by left, $M\gamma e$ by $e\gamma M$, $L[e, \gamma]$ by $[\gamma, e]R$, and $R[\gamma, e]$ by $[e, \gamma]L$ at the same time.

We conclude this section by the following

Theorem 3.6. *Let M be a Γ-ring having no non-zero strongly nilpotent one-sided ideals. Let L and R be respectively the left and right operator rings of M. If $e\gamma e = e$, where $e \in M$, $\gamma \in \Gamma$, then the following statements are equivalent:*

(i) *$M\gamma e$ is a minimal left ideal of M,*
(ii) *$e\gamma M$ is a minimal right ideal of M,*
(iii) *$L[e, \gamma]$ is a minimal left ideal of L,*
(iv) *$[\gamma, e]R$ is a minimal right ideal of R,*
(v) *$[e, \gamma]\ L$ is a minimal right ideal of L,*
(vi) *$R[\gamma, e]$ is a minimal left ideal of R,*
(vii) *$[e.\ \gamma]L[e, \gamma]$ is a division ring,*
(viii) *$[\gamma, e]R[\gamma, e]$ is a division ring.*

Moreover, the division rings $[e, \gamma]L[e, \gamma]$ and $[\gamma\ ,e]R[\gamma, e]$ are isomorphic if any of the above statements occurs.

Proof. The equivalence of the above eight statements is an immediate consequence of Theorems 3.1 through 3.5. We shall show now that $D = [e, \gamma]$ $L[e, \gamma]$ is isomorphic to $D' = [\gamma, e]R[\gamma, e]$. Note that every element of D can be expressed as $[e, \gamma][x, \gamma][e, \gamma]$ for some $x \in M$. Consider the mapping σ of D onto D' defined by $([e, \gamma][x, \gamma][e, \gamma])\sigma = [\gamma, e][\gamma, x][\gamma, e]$. The mapping σ is well defined, for if $[e, \gamma][x, \gamma][e, \gamma] = [e, \gamma][y, \gamma][e, \gamma]$ then $[e, \gamma][x-y, \gamma][e, \gamma] = 0$. This implies $M\gamma e\gamma(x-y)\gamma e = M\gamma e\gamma(x-y)\gamma e\gamma e = 0$, or $[\gamma, e][\gamma, x-y][\gamma, e] = 0$. Hence $[\gamma, e][\gamma, x][\gamma, e] = [\gamma, e][\gamma, y][\gamma, e]$. The reader can easily verify that σ is an isomorphism of D onto D'.

4. Primitive Γ-rings. Let M be a Γ-ring and L and R be respectively the left and right operator rings of M. M is said to be left (right) primitive if (i) $L(R)$ is a left (right) primitive ring, and (ii) $x\Gamma M=0$ ($M\Gamma x=0$) implies $x=0$.

Theorem 4.1. *Let M be a left primitive Γ-ring. If I_1 and I_2 are two non-zero left ideals of M, then $I_1\Gamma I_2\neq 0$.*

Proof. Let N be a faithful irreducible left L-module. Then there exists $\gamma\in\Gamma$, such that $[I_2, \gamma]N\neq 0$, for otherwise, $[I_2, \Gamma]=0$ would imply $I_2\Gamma M=0$ and $I_2=0$. Since $[I_2, \gamma]N$ is an L-submodule of N and N is irreducible, we have $[I_2, \gamma]N=N$. Now suppose contrarily that $I_1\Gamma I_2=0$. Then $[I_1, \Gamma]N=[I_1, \Gamma][I_2, \gamma]N=[I_1\Gamma I_2, \gamma]N=0$. It would follow that $[I_1, \Gamma]=0$, or $I_1\Gamma M=0$, so $I_1=0$.

From Theorems 3.1 and 4.1, we immediately have

Corollary. *A left primitive Γ-ring has no non-zero strongly nilpotent one-sided ideals.*

Theorem 4.2. *Let M be a Γ-ring having minimal one-sided ideals. Then M is left primitive if and only if M is right primitive.*

Proof. By the left-right symmetricity, it will suffice to show that left primitivity implies right primitivity. Let us assume that M is a left primitive Γ-ring, and that $M\gamma e$ is a minimal left ideal of M, $e\gamma e=e$. Then by Theorem 3.6, $[\gamma, e]R$ is a minimal right ideal of R and hence is an irreducible right R-module, where R is the right operator ring of M. We assert further that $[\gamma, e]\,R$ is faithful as a R-module. For, if $[\gamma, e]R\,\Sigma[\gamma_i, x_i]=0$, where $\Sigma[\gamma_i, x_i]\in R$, then $(M[\gamma, e])\Gamma(M\Sigma[\gamma_i, x_i])=0$, while $M[\gamma, e]\neq 0$ and $M\Sigma[\gamma_i, x_i]$ are left ideals of M. Hence by Theorem 4.1, $M\Sigma[\gamma_i, x_i]=0$ or $\Sigma[\gamma_i, x_i]=0$. Thus R is a right primitive ring. Moreover, if $M\Gamma x=0$, $x\in M$, then $(x\Gamma M)^2=0$. Since M has no non-zero strongly nilpotent right ideals, $x\Gamma M=0$. By the left primitivity of M, $x=0$. Therefore, M is a right primitive Γ-ring. This completes the proof.

Now, let M be a Γ-ring and M' be a Γ'-ring. If θ is a group isomorphism of M onto M' and ϕ is a group isomorphism of Γ onto Γ' then the pair (θ, ϕ) is called an isomorphism of Γ-ring M onto Γ'-ring M' if $(x\alpha y)\theta=(x\theta)(\alpha\phi)(y\theta)$ for all $x, y\in M$, $\alpha\in\Gamma$.

The definition here for isomorphism is slightly general than that given by Barnes [1] in which he assumed $\Gamma=\Gamma'$, and ϕ is the identity mapping on Γ.

A Γ-ring M and a Γ'-ring M' are said to be isomorphic if there exists an isomoprhism (θ, ϕ) of M onto M'. In this case if M is a Γ-ring in the sense of Nobusawa, then, for $\alpha', \beta'\in\Gamma'$, $x'\in M'$, the composition $\alpha'x'\beta'$ can be introduced so that M' forms a Γ'-ring in the sense of Nobusawa, and $(\alpha x\beta)\phi=(\alpha\phi)(x\theta)(\beta\phi)$ holds for all $\alpha, \beta\in\Gamma$, $x\in M$.

Now we are ready to prove the main theorem.

Theorem 4.3. *Let M be a Γ-ring. Then M is a left primitive Γ-ring (in the sense of Nobusawa) having minimal one-sided ideals if and only if there exist two pairs of dual spaces (V, W) and (V', W') over isomorphic division rings D and D' respectively, such that M is isomorphic to a Γ'-ring M', where $\mathcal{F}(V, V')\subseteq M'\subseteq \mathcal{L}(V, V')$ and $\mathcal{F}(V', V,)\subseteq\Gamma'\subseteq\mathcal{L}(V', V)$, and the composition $x\alpha y$ for $x, y\in M'$, $\alpha\in\Gamma'$ is the composition of mappings. Moreover, $\mathcal{F}(V, V')$ is the unique minimal two-sided ideal of M'.*

Proof. Sufficiency. Assume that (V, W) and (V', W') are pairs of dual spaces over D and D' respectively, and M is a Γ-ring with $\mathcal{F}(V, V')\subseteq M\subseteq \mathcal{L}(V, V')$, $\mathcal{F}(V', V)\subseteq\Gamma\subseteq\mathcal{L}(V', V)$. Let σ be the isomorphism of D onto D'. Let $0\neq v_0'\in V'$ and $I=\{x\in\mathcal{L}(V, V')\colon Vx\subseteq\langle v_0'\rangle\}$, where $\langle v_0'\rangle$ denotes the subspace of V' generated by v_0' over D'. Then clearly I is a non-zero left ideal of the Γ-ring M. Now, we claim that I is minimal. Suppose $K\neq 0$ is a left ideal of M contained in I. Let $y\in K$ and $y\neq 0$. By Theorem 2.3, there exists $w_1\in W$, $w_1\neq 0$, such that $vy=(v, w_1)^\sigma v_0'$. Let x be an arbitrary element in I and $vx=(v, w_0)^\sigma v_0'$, where $w_0\in W$. By the non-degeneracy of bilinear forms on dual spaces, there exist $v_1\in V$, $w_0'\in W'$, such that $(v_1, w_1)=1$ and $(v_0', w_0')=1$. Define $\gamma_1\in\mathcal{F}(V', V)$ by $v'\gamma_1=(v', w_0')^{\sigma^{-1}}v_1$, for all $v\in V$, $v'\in V'$. It is easy to see that $vx\gamma_1 y=vx$ for all $v\in V$. Hence $x=x\gamma_1 y\in K$. Thus $K=I$, and I is minimal. To show that I being an irreducible L-module is faithful, we assume that $0\neq\Sigma[x_i, \gamma_i]\in L$ and $\Sigma[x_i, \gamma_i]I=0$ (where L is the left operator ring of M). Then there exists $z\in M$, $v_1\in V$, such that $v_1\Sigma x_i\gamma_i z\neq 0$, and hence there exists $w_i'\in W'$ such that $(v_1\Sigma x_i\gamma_i z, w_i')=1$. Let $v_0\in V$, $w_1\in W$, so that $(v_0, w_1)=1$, and let $\gamma\in\Gamma$, $x\in I$ be defined by $v'\gamma=(v', w_i')^{\sigma^{-1}}v_0$, and $vx=(v, w_1)^\sigma v_0'$, for all $v'\in V'$ and $v\in V$. By noting that $z\gamma x\in I$, we obtain $0=v_1(\Sigma[x_i, \gamma_i]z\gamma x)=v_0x=v_0'$, a contradiction. Thus I is a faithful left L-module. Also it is clear that $x\Gamma M=0$ implies $x=0$. Therefore, M is a left primitive Γ-ring.

Now we shall show that $\mathcal{F}(V, V')$ is the unique minimal two-sided ideal of M. It will be sufficient to show that every non-zero ideal of M contains every continuous semi-linear transformation of rank one, since, according to Theorem 2.3, each $x\in\mathcal{F}(V, V')$ can be expressed as a sum of finitely many x_i's in $\mathcal{F}(V, V')$ of rank one.

Let K be an arbitrary non-zero ideal of M and $0\neq x\in K$. Let $y\in\mathcal{F}(V, V')$ be of rank one, say $vy=(v, w_0)^\sigma v_i'$, for all $v\in V$. Let $v_1\in V$ such that $v_1x\neq 0$. Then there exist $w_1'\in W'$, $w_1\in W$, such that $(v_1x, w_1')=1$, and $(v_1, w_1)=1$. Let $x_1, x_2\in M$, $\gamma\in\Gamma$ be defined by $vx_1=(v, w_0)^\sigma(v_1x)$, $vx_2=(v, w_1)^\sigma v_1'$, $v'\gamma=(v', w_1')^{\sigma^{-1}}v_1$, for all $v\in V$, $v'\in V'$. Then it is easy to see that $x_1\gamma x\gamma x_2=y$. Hence $y\in K$, and $\mathcal{F}(V, V')\subseteq K$. Therefore, $\mathcal{F}(V, V')$ is the unique minimal ideal of M.

It remains to show that M is a Γ-ring in the sense of Nobusawa. Let $0\neq$

$\gamma\in\Gamma$, and $v_0'\in V'$ with $v_0'\gamma\neq 0$. Then there exists $w\in W$ such that $(v_0'\gamma, w)=1$. Define $x\in M$ by $vx=(v, w)^\sigma v_0'$ for all $v\in V$. We obtain $(v_0'\gamma)x\gamma x=v_0'\neq 0$. Hence, $x\gamma x\neq 0$, and M is a Γ-ring in the sense of Nobusawa.

Necessity. The proof of this part is similar to that of Jacobson structure theorem for primitive ring given by Kaplansky, but is slightly complicated. We assume that M is a left primitive Γ-ring having minimal left ideals in the sense of Nobusawa. Let $I\neq 0$ be a minimal left ideal of M. By the Corollary to Theorem 4.1, $I^2\neq 0$ and hence by Theorem 3.2, $I=M\gamma e$, $e\gamma e=e$ for some $e\in M$. $\gamma\in\Gamma$. From Theorem 3.6, $D=[e, \gamma]L[e, \gamma]$, $D'=[\gamma, e]R[\gamma, e]$ are isomorphic division rings, where L and R are respectively the left and right operator rings of M. Consider that $V=[e, \gamma]L$ and $W=L[e, \gamma]$ are respectively left and right vector spaces over D and that $V'=[\gamma, e]R$ and $W'=R[\gamma, e]$ are respectively left and right vector spaces over D'. Clearly $V=[e, \Gamma]$ and $W'=[\Gamma, e]$. Now we define non-degenerate bilinear mappings of $V\times W$ into D and of $V'\times W'$ into D' by

$([e, \alpha], \Sigma[y_i, \beta_i][e, \gamma])=[e, \alpha]\Sigma[y_i, \beta_i][e, \gamma]$, and

$([\gamma, e]\Sigma[\alpha_i, x_i], [\alpha, e])=[\gamma, e]\Sigma[\alpha_i, x_i][\alpha, e]$, for all

$\alpha\in\Gamma$, $\Sigma[y_i, \beta_i]\in L$ and $\Sigma[\alpha_i, x_i]\in R$. Consequently, (V, W) and (V', W') are pairs of dual vector spaces over D and D' respectively. Let

σ: $[e, \gamma][x, \gamma][e, \gamma]\to[\gamma, e][\gamma, x][\gamma, e]$ be the isomorphism of D onto D'. For each $x\in M$, we define the mapping T_x: $V\to V'$ by $[e, \alpha]T_x=[\gamma, e][\alpha, x]$ for all $[e, \alpha]\in V$. We can see easily that T_x is a semi-linear transformation. Moreover, T_x is a continuous mapping of V, topologized by W-topology, into V', topologized by W'-topology. In fact, for $\alpha, \beta\in\Gamma$,

$([e, \alpha]T_x, [\beta, e])=[\gamma, e][\alpha, x][\beta, e]=[\gamma, e][\gamma, e\alpha x\beta e][\gamma, e]$

$=([e, \gamma][e\alpha x\beta e, \gamma][e, \gamma])^\sigma=([e, \alpha], [x, \beta][e, \gamma])^\sigma=([e, \alpha], [\beta, e]T_x{}^*)^\sigma$,

where $T_x{}^*$: $W'\to W$ defined by $[\beta, e]T_x{}^*=[x, \beta][e, \gamma]$

is a semi-linear transformation.

Hence, by Theorem 2.2, $T_x\in\mathcal{L}(V, V')$.

Let θ: $M\to\mathcal{L}(V, V')$ be defined by $x\theta=T_x$. Clearly, θ is a group homomorphism. Moreover, θ is one-to-one. For, if $T_x=T_0$, where 0 is the zero element in M, then $[\gamma, e][\alpha, x]=0$ for all $\alpha\in\Gamma$, so we have $M\gamma e\Gamma x=0$, and hence $(x\Gamma M\gamma e\gamma M)^2=0$. Since M has no non-zero strongly nilpotent right ideals, $x\Gamma M\gamma e\gamma M=0$, or $[x, \Gamma][M, \gamma][e, \gamma]=0$ in L. Let N be a faithful irreducible left L-module. Then $[M, \gamma][e, \gamma]LN$ is a non-zero L-submodule of N and hence $[M, \gamma][e, \gamma]LN=N$. Thus, $0=[x, \Gamma][M, \gamma][e, \gamma]LN=[x, \Gamma]N$, so $[x, \Gamma]=0$, or $x\Gamma M=0$. Hence by the primitivity of M, $x=0$. Thus, θ is a group monomorphism of M into $\mathcal{L}(V, V')$.

Similarly, for each $\alpha\in\Gamma$, we may define $T_\alpha\in\mathcal{L}(V', V)$ by $([\gamma, e]\Sigma[\alpha_i, x_i])T_\alpha=[e, \Sigma\alpha_i x_i\alpha]$. Also the mapping ϕ: $\alpha\to T_\alpha$ is a group monomorphism of Γ into $\mathcal{L}(V', V)$. That ϕ preserves addition is obvious. We shall show that ϕ

is one-to-one. To this end, we assume that $T_\alpha = T_0$, where 0 is the zero element in Γ. Then $[e, \Gamma][M, \alpha]=0$, so $e\Gamma M\alpha M=0$. It follows that $(M\alpha M\Gamma e)^2=0$, and hence $M\alpha M\Gamma e=0$. By the irreducibility of N again, we have $L[e, \gamma]N=N$, so $0=[M, \alpha]L[e, \gamma]N=[M, \alpha]N$. Consequently, $[M, \alpha]=0$, or $M\alpha M=0$. Using the Nobusawa's condition (4′), we get $\alpha=0$.

It is also easy to see that $T_{x\alpha y}=T_x T_\alpha T_y$, or $(x\alpha x)\theta=(x\theta)(\alpha\phi)(y\theta)$ for all $\alpha\in\Gamma$, and $x, y\in M$.

It remains to show that $M\theta\supseteq\mathcal{F}(V', V)$ and $\Gamma\phi\supseteq\mathcal{F}(V', V)$. We shall show $M\theta\supseteq\mathcal{F}(V, V')$. That $\Gamma\phi\supseteq\mathcal{F}(V', V)$ can be verified similarly. Let $T\in\mathcal{F}(V, V')$ be of rank one, say $VT=\langle[\gamma, e]\Sigma[\beta_j, y_j]\rangle$, the subspace of V' generated by $[\gamma, e]\Sigma[\beta_j, y_j]$ over D'. By Theorem 2.3, there exists $\Sigma[x_i, \alpha_i][e, \gamma]\in W$, such that, for all $\alpha\in\Gamma$,

$$\begin{aligned}
[e, \alpha]T &= ([e, \alpha], \Sigma[x_i, \alpha_i][e, \gamma])^\sigma[\gamma, e]\Sigma[\beta_j, y_j] \\
&= ([e, \alpha]\Sigma[x_i, \alpha_i][e, \gamma])^\sigma[\gamma, e]\Sigma[\beta_j, y_j] \\
&= ([e, \gamma][\Sigma e\alpha x_i\alpha_i e, \gamma][e, \gamma])^\sigma[\gamma, e]\Sigma[\beta_j, y_j] \\
&= [\gamma, e][\gamma, \Sigma e\alpha x_i\alpha_i e][\gamma, e][\gamma, e]\Sigma[\beta_j, y_j] \\
&= [\gamma, e][\alpha, \Sigma_{i,j}x_i\alpha_i e\beta_j y_j] = [e, \alpha]T_x\,,
\end{aligned}$$

where $x=\Sigma_{i,j}x_i\alpha_i e\beta_j y_j$. Thus $T=T_x$, $M\theta$ contains all continuous semilinear transformations of rank one, and hence $M\theta\supseteq\mathcal{F}(V, V')$. This completes the proof.

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