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BORSUK-ULAM TYPE THEOREMS ON STIEFEL MANIFOLDS

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Abstract

In this paper, we study the degree of equivariant maps between Stiefel manifolds by using cohomological index theory. As applications, we have some Borsuk-Ulam type theorems on Stiefel manifolds.

1. Introduction

We are concerned with the following classical version of the Borsuk-Ulam theorem:

(i) If $n > k$ then there is no map $f\colon S^n \to S^k$ such that $f(-x) = -f(x)$ for all x.

This easily follows from the next proposition:

(ii) Let $f\colon S^n \to S^n$ be a map of the sphere such that $f(-x) = -f(x)$ for all x. Then $\deg f \equiv 1 \pmod 2$.

Now let S^n denote the standard n-dimensional sphere with antipodal $\boldsymbol{Z}_2$-action, then the proposition (ii) implies that for any $\boldsymbol{Z}_2$-map $f\colon S^n \to S^n$, the degree of f is odd.

Many authors have been contributing to generalizing and extending the Borsuk-Ulam theorem in various ways. E. Fadell-S. Husseini and J. Jaworowski introduced an ideal-valued cohomological index theory, and generalized the Borsuk-Ulam theorem (see [2], [3] and [5]). Let $V_k(\boldsymbol{R}^m)$ denote the space of orthonormal k-frames in $\boldsymbol{R}^m$ and $O(k)$ the orthogonal group. If we represent an element of $V_k(\boldsymbol{R}^m)$ as a column vector $[v_1 \cdots v_k]^T$, and if $O(k)$ is the orthogonal group of $k \times k$ matrices, then $V_k(\boldsymbol{R}^m)$ is a free $O(k)$-space under the action induced by matrix multiplication $g[v_1 \cdots v_k]^T$, $g \in O(k)$. In [4], Yasuhiro Hara considered the degree of $O(k)$-maps $f\colon V_k(\boldsymbol{R}^m) \to V_k(\boldsymbol{R}^m)$.

In this paper, we will consider the degree of $(\boldsymbol{Z}_2)^k$-maps $f\colon V_k(\boldsymbol{R}^m) \to V_k(\boldsymbol{R}^m)$ where $(\boldsymbol{Z}_2)^k = \boldsymbol{Z}_2 \times \cdots \times \boldsymbol{Z}_2$ (k times) is the subgroup of $O(k)$ which is diagonally imbedded. We will show

Theorem 3.3. *Let $f\colon V_k(\boldsymbol{R}^m) \to V_k(\boldsymbol{R}^m)$ be a $(\boldsymbol{Z}_2)^k$-map. Then the degree of f is odd.*

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By a similar way, $U(k)$ acts freely on the complex Stiefel manifold $V_k(\boldsymbol{C}^m)$. We restrict the $U(k)$-action on $V_k(\boldsymbol{C}^m)$ to the subgroup $(\boldsymbol{Z}_p)^k$ where p is a prime number. Then we will show

Theorem 3.5. *Let $f\colon V_k(\boldsymbol{C}^m) \to V_k(\boldsymbol{C}^m)$ be a $(\boldsymbol{Z}_p)^k$-map. Then the degree of f is not congruent to zero modulo p.*

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2. Index theory

In this section we will recall the definition and basic properties of index theory which was first introduced by Fadell and Husseini and independently by Jaworowski.

Let G be a compact Lie group and X a G-CW complex. We denote the universal principal G-bundle by $EG \to BG$. Then G acts freely on $EG \times X$ by $g(e, x) = (ge, gx)$. We denote the quotient space of this action by $EG \times_G X$. Note that the orbit map $p\colon EG \times X \to EG \times_G X$ is a fiber bundle of the fiber G. The Borel cohomology of X with coefficients in a field $\boldsymbol{K}$ is defined by $H_G^*(X;\boldsymbol{K}) = H^*(EG \times_G X;\boldsymbol{K})$, where $H^*(\ )$ is singular cohomology theory. Let $c_X\colon X \to *$ be a constant map to one-point space. The G*-index* of X, denoted by $\mathrm{Ind}^G(X;\boldsymbol{K})$, is an ideal in $H^*(BG;\boldsymbol{K})$. $\mathrm{Ind}^G(X;\boldsymbol{K})$ is defined to be the kernel of the homomorphism $\overline{c}_X^* = (\mathrm{id} \times_G c_X)^*\colon H^*(BG;\boldsymbol{K}) = H_G^*(*;\boldsymbol{K}) \to H_G^*(X;\boldsymbol{K})$. If X is a free G-space, then $\mathrm{Ind}^G(X)$ coincides with the kernel of the homomorphism $H^*(BG) \to H^*(X/G)$ induced from a classifying map $X/G \to BG$ for the free G-action on X. Furthermore for an integer k we set

$$\mathrm{Ind}_k^G(X;\boldsymbol{K}) = \mathrm{Ind}^G(X;\boldsymbol{K}) \cap H^k(BG;\boldsymbol{K}) = \ker\left(\overline{c}_X^*\colon H^k(BG;\boldsymbol{K}) \to H_G^k(X;\boldsymbol{K})\right).$$

The following proposition is a basic property of the G-index.

Proposition 2.1 ([2], [5]). *If there exists a G-map $f\colon X \to Y$, then for any $k \in \boldsymbol{Z}$*

$$\mathrm{Ind}_k^G(X) \supset \mathrm{Ind}_k^G(Y).$$

We now consider a basic computation which is important to an application which we give later on.

$V_k(\boldsymbol{R}^m)$ denotes the space of orthonormal k-frames in $\boldsymbol{R}^m$ and $O(k)$ denotes the orthogonal group. Then $O(k)$ acts freely on $V_k(\boldsymbol{R}^m)$ by the usual action gv, $g \in O(k)$ and v is a column vector representing k-frame. We restrict this action to the subgroup $(\boldsymbol{Z}_2)^k$ of diagonal matrices with entries ± 1. Then $V_k(\boldsymbol{R}^m)$ is also a free $(\boldsymbol{Z}_2)^k$-space.

Recall that $B(\boldsymbol{Z}_2)^k = B\boldsymbol{Z}_2 \times \cdots \times B\boldsymbol{Z}_2$ (k times) and

$$H^*\left(B(\boldsymbol{Z}_2)^k; \boldsymbol{Z}_2\right) = H^*(B\boldsymbol{Z}_2) \otimes \cdots \otimes H^*(B\boldsymbol{Z}_2) = \boldsymbol{Z}_2[t_1, \ldots, t_k],$$

where $\dim t_i = 1$. Fadell proved the following in [3].

Proposition 2.2. *The monomial* $t_1^{m-1} t_2^{m-2} \cdots t_k^{m-k}$ *does not belong to* $\mathrm{Ind}^{(\boldsymbol{Z}_2)^k}(V_k(\boldsymbol{R}^m); \boldsymbol{Z}_2)$.

In particular, since $\dim V_k(\boldsymbol{R}^m) = mk - k(k+1)/2$, *we can assert*

$$\mathrm{Ind}^{(\boldsymbol{Z}_2)^k}_{\dim V_k(\boldsymbol{R}^m)}(V_k(\boldsymbol{R}^m); \boldsymbol{Z}_2) \neq H^{\dim V_k(\boldsymbol{R}^m)}\left(B(\boldsymbol{Z}_2)^k; \boldsymbol{Z}_2\right).$$

We have an analogous proposition for complex Stiefel manifolds. $V_k(\boldsymbol{C}^m)$ denotes the space of orthonormal k-frames in $\boldsymbol{C}^m$ and $U(k)$ denotes the unitary group. Then $U(k)$ acts freely on $V_k(\boldsymbol{C}^m)$ by the usual action gv, $g \in U(k)$ and v is a column vector representing k-frame. We restrict this action to the subgroup $(\boldsymbol{Z}_p)^k$ of diagonal matrices with entries p-th root of one and consider $\mathrm{Ind}^{(\boldsymbol{Z}_p)^k}(V_k(\boldsymbol{C}^m); \boldsymbol{Z}_p)$, where p is a prime number.

In case $p = 2$ we show that $t_1^{2(m-1)+1} t_2^{2(m-2)+1} \cdots t_k^{2(m-k)+1}$ is not in $\mathrm{Ind}^{(\boldsymbol{Z}_2)^k}(V_k(\boldsymbol{C}^m); \boldsymbol{Z}_2)$ by induction on k. The computation will be based on the fibration

$$S^{2(m-k)+1} \to V_k(\boldsymbol{C}^m) \xrightarrow{\pi} V_{k-1}(\boldsymbol{C}^m), \tag{1}$$

where π is the projection on the first $k-1$ coordinates. Consider the sequence

$$\boldsymbol{Z}_2 \to (\boldsymbol{Z}_2)^k \to (\boldsymbol{Z}_2)^{k-1}, \tag{2}$$

where $\boldsymbol{Z}_2$ injects on the last coordinate and $(\boldsymbol{Z}_2)^k$ projects on the first $k-1$ coordinates. Dividing out the action of (2) on (1), we obtain

$$\boldsymbol{R}P^{2(m-k)+1} \to V_k(\boldsymbol{C}^m)/(\boldsymbol{Z}_2)^k \to V_{k-1}(\boldsymbol{C}^m)/(\boldsymbol{Z}_2)^{k-1}.$$

We then have an induced diagram of fibrations

$$\begin{array}{ccc}
\boldsymbol{R}P^{2(m-k)+1} & \xrightarrow{\alpha_{m-k+1,1}} & B\boldsymbol{Z}_2 \\
{\scriptstyle i_m}\downarrow & & {\scriptstyle i_\infty}\downarrow \\
V_k(\boldsymbol{C}^m)/(\boldsymbol{Z}_2)^k & \xrightarrow{\alpha_{m,k}} & B(\boldsymbol{Z}_2)^k \\
{\scriptstyle p_m}\downarrow & & {\scriptstyle p_\infty}\downarrow \\
V_{k-1}(\boldsymbol{C}^m)/(\boldsymbol{Z}_2)^{k-1} & \xrightarrow{\alpha_{m,k-1}} & B(\boldsymbol{Z}_2)^{k-1}
\end{array}$$

where the $\alpha_{i,j}$ are classifying maps. Recall that our coefficients are $\boldsymbol{Z}_2$, and since i_∞^* and $\alpha_{m-k+1,1}^*$ are surjective, $i_m^* \colon H^*(V_k(\boldsymbol{C}^m)/(\boldsymbol{Z}_2)^k) \to H^*(\boldsymbol{R}P^{2(m-k)+1})$ is surjective.

Thus, the Leray-Hirsch theorem applies and we have a diagram

$$\begin{array}{ccc} H^*\left(V_{k-1}(\boldsymbol{C}^m)/(\boldsymbol{Z}_2)^{k-1}\right)\otimes H^*(\boldsymbol{R}P^{2(m-k)+1}) & \xrightarrow{\varphi_m} & H^*\left(V_k(\boldsymbol{C}^m)/(\boldsymbol{Z}_2)^k\right) \\ {\scriptstyle \alpha^*_{m,k-1}\otimes\alpha^*_{m-k+1,1}}\uparrow & & {\scriptstyle \alpha^*_{m,k}}\uparrow \\ H^*\left(B(\boldsymbol{Z}_2)^{k-1}\right)\otimes H^*(\boldsymbol{R}P^\infty) & \xrightarrow{\varphi_\infty} & H^*\left(B(\boldsymbol{Z}_2)^k\right) \end{array}$$

with φ_m and φ_∞ isomorphisms. Then

$$\begin{aligned} &\alpha^*_{m,k}\left[t_1^{2(m-1)+1}t_2^{2(m-2)+1}\cdots t_k^{2(m-k)+1}\right] \\ &= \alpha^*_{m,k}\circ\varphi_\infty\left[t_1^{2(m-1)+1}t_2^{2(m-2)+1}\cdots t_{k-1}^{2(m-k+1)+1}\otimes t_k^{2(m-k)+1}\right] \\ &= \varphi_m\left[\alpha^*_{m,k-1}\left(t_1^{2(m-1)+1}t_2^{2(m-2)+1}\cdots t_{k-1}^{2(m-k+1)+1}\right)\otimes\alpha^*_{m-k+1,1}\left(t_k^{2(m-k)+1}\right)\right]. \end{aligned}$$

But $\alpha^*_{m-k+1,1}\left(t_k^{2(m-k)+1}\right)\neq 0$ and assuming by induction that

$$\alpha^*_{m,k-1}\left(t_1^{2(m-1)+1}t_2^{2(m-2)+1}\cdots t_{k-1}^{2(m-k+1)+1}\right)\neq 0,$$

we have

$$\alpha^*_{m,k}\left[t_1^{2(m-1)+1}t_2^{2(m-2)+1}\cdots t_k^{2(m-k)+1}\right]\neq 0.$$

Thus $t_1^{2(m-1)+1}t_2^{2(m-2)+1}\cdots t_k^{2(m-k)+1}$ is not in $\ker\alpha^*_{m,k}$.

When p is an odd prime, $H^*(B(\boldsymbol{Z}_p)^k;\boldsymbol{Z}_p)=\boldsymbol{Z}_p[x_1,x_2,\ldots,x_k]\otimes E(y_1,y_2,\ldots,y_k)$, where $\boldsymbol{Z}_p[x_1,x_2,\ldots,x_k]$ denotes the $\boldsymbol{Z}_p$-polynomial algebra on 2-dimensional generators x_i and $E(y_1,y_2,\ldots,y_k)$ denotes the $\boldsymbol{Z}_p$-exterior algebra on 1-dimensional generators y_i. The ring is graded-commutative, i.e. $xy=(-1)^{\deg(x)\deg(y)}yx$. We next show that $x_1^{m-1}y_1x_2^{m-2}y_2\cdots x_k^{m-k}y_k$ is not in $\mathrm{Ind}^{(\boldsymbol{Z}_p)^k}(V_k(\boldsymbol{C}^m);\boldsymbol{Z}_p)$ by induction on k. Consider the sequence

$$\boldsymbol{Z}_p\to\left(\boldsymbol{Z}_p\right)^k\to\left(\boldsymbol{Z}_p\right)^{k-1}, \tag{3}$$

where $\boldsymbol{Z}_p$ injects on the last coordinate and $\left(\boldsymbol{Z}_p\right)^k$ projects on the first $k-1$ coordinates. Dividing out the action of (3) on (1), we obtain

$$S^{2(m-k)+1}\big/\boldsymbol{Z}_p\to V_k(\boldsymbol{C}^m)\big/\left(\boldsymbol{Z}_p\right)^k\to V_{k-1}(\boldsymbol{C}^m)\big/\left(\boldsymbol{Z}_p\right)^{k-1}.$$

We then have an induced diagram of fibrations

$$\begin{array}{ccc} L_p^{2(m-k)+1} & \xrightarrow{\alpha_{m-k+1,1}} & B\boldsymbol{Z}_p \\ {\scriptstyle i_m}\big\downarrow & & {\scriptstyle i_\infty}\big\downarrow \\ V_k(\boldsymbol{C}^m)\big/\left(\boldsymbol{Z}_p\right)^k & \xrightarrow{\alpha_{m,k}} & B\left(\boldsymbol{Z}_p\right)^k \\ {\scriptstyle p_m}\big\downarrow & & {\scriptstyle p_\infty}\big\downarrow \\ V_{k-1}(\boldsymbol{C}^m)\big/\left(\boldsymbol{Z}_p\right)^{k-1} & \xrightarrow{\alpha_{m,k-1}} & B\left(\boldsymbol{Z}_p\right)^{k-1} \end{array}$$

where the orbit space $L_p^{2(m-k)+1} = S^{2(m-k)+1}/\boldsymbol{Z}_p$ is the lens space and the $\alpha_{i,j}$ are classifying maps. Recall that our coefficients are $\boldsymbol{Z}_p$, and since i_∞^* and $\alpha_{m-k+1,1}^*$ are surjective, $i_m^*\colon H^*\big(V_k(\boldsymbol{C}^m)\big/\left(\boldsymbol{Z}_p\right)^k\big) \to H^*\big(L_p^{2(m-k)+1}\big)$ is surjective. Thus, the Leray-Hirsch theorem applies and we have a diagram

$$\begin{array}{ccc} H^*\left(V_{k-1}(\boldsymbol{C}^m)\big/\left(\boldsymbol{Z}_p\right)^{k-1}\right) \otimes H^*\left(L_p^{2(m-k)+1}\right) & \xrightarrow{\varphi_m} & H^*\left(V_k(\boldsymbol{C}^m)\big/\left(\boldsymbol{Z}_p\right)^k\right) \\ {\scriptstyle \alpha_{m,k-1}^*\otimes\alpha_{m-k+1,1}^*}\big\uparrow & & {\scriptstyle \alpha_{m,k}^*}\big\uparrow \\ H^*\left(B\left(\boldsymbol{Z}_p\right)^{k-1}\right) \otimes H^*(B\boldsymbol{Z}_p) & \xrightarrow{\varphi_\infty} & H^*\left(B\left(\boldsymbol{Z}_p\right)^k\right) \end{array}$$

with φ_k and φ_∞ isomorphisms. Then

$$\begin{aligned} &\alpha_{m,k}^*\left[x_1^{m-1}y_1x_2^{m-2}y_2\cdots x_k^{m-k}y_k\right] \\ &= \alpha_{m,k}^*\circ\varphi_\infty\left[x_1^{m-1}y_1x_2^{m-2}y_2\cdots x_{k-1}^{m-k+1}y_{k-1}\otimes x_k^{m-k}y_k\right] \\ &= \varphi_m\left[\alpha_{m,k-1}^*\left(x_1^{m-1}y_1x_2^{m-2}y_2\cdots x_{k-1}^{m-k+1}y_{k-1}\right)\otimes\alpha_{m-k+1,1}^*\left(x_k^{m-k}y_k\right)\right]. \end{aligned}$$

But $\alpha_{m-k+1,1}^*\big(x_k^{m-k}y_k\big) \neq 0$ and assuming by induction that

$$\alpha_{m,k-1}^*\left(x_1^{m-1}y_1x_2^{m-2}y_2\cdots x_{k-1}^{m-k+1}y_{k-1}\right) \neq 0,$$

we have

$$\alpha_{m,k}^*\left[x_1^{m-1}y_1x_2^{m-2}y_2\cdots x_k^{m-k}y_k\right] \neq 0.$$

Therefore $x_1^{m-1}y_1x_2^{m-2}y_2\cdots x_k^{m-k}y_k$ is not in $\ker\alpha_{m,k}^*$. Thus we have the following result.

Proposition 2.3. (1) *The monomial* $t_1^{2(m-1)+1}t_2^{2(m-2)+1}\cdots t_k^{2(m-k)+1}$ *does not belong to* $\operatorname{Ind}^{(\boldsymbol{Z}_2)^k}(V_k(\boldsymbol{C}^m);\boldsymbol{Z}_2)$.

In particular, since $\dim V_k(\boldsymbol{C}^m) = 2mk - k^2$, *we can assert*

$$\operatorname{Ind}^{(\boldsymbol{Z}_2)^k}_{\dim V_k(\boldsymbol{C}^m)}(V_k(\boldsymbol{C}^m);\boldsymbol{Z}_2) \neq H^{\dim V_k(\boldsymbol{C}^m)}\left(B(\boldsymbol{Z}_2)^k;\boldsymbol{Z}_2\right).$$

(2) *When* p *is an odd prime, the monomial* $x_1^{m-1}y_1x_2^{m-2}y_2\cdots x_k^{m-k}y_k$ *does not belong to* $\mathrm{Ind}^{(\mathbf{Z}_p)^k}(V_k(\mathbf{C}^m);\mathbf{Z}_p)$.

In particular, since $\dim V_k(\mathbf{C}^m) = 2mk - k^2$, $\dim x_i = 2$ *and* $\dim y_i = 1$, *we can assert*

$$\mathrm{Ind}^{(\mathbf{Z}_p)^k}_{\dim V_k(\mathbf{C}^m)}(V_k(\mathbf{C}^m);\mathbf{Z}_p) \neq H^{\dim V_k(\mathbf{C}^m)}\left(B(\mathbf{Z}_p)^k;\mathbf{Z}_p\right).$$

3. Borsuk-Ulam type theorems on Stiefel manifolds

Let G be a compact Lie group and X be a free G-CW complex. We denote by X/G the orbit space of X. Note that the orbit map $p\colon X \to X/G$ is a fiber bundle with fiber G. Following [4], we define the transfer $p_!\colon H^n(X;\Gamma) \to H^{n-\dim G}(X/G;\Gamma)$ where Γ is a commutative group. Then we have the following.

Lemma 3.1 ([4]). *Let* X, Y *be* G*-CW complexes and* $f\colon X \to Y$ *a* G*-map. Let* $p_X\colon EG\times X \to EG\times_G X$ *and* $p_Y\colon EG\times Y \to EG\times_G Y$ *denote the orbit maps. Then the commutativity holds in the diagram*:

$$\begin{array}{ccc} H^i(Y;\Gamma) & \xrightarrow{f^*} & H^i(X;\Gamma) \\ \downarrow{\scriptstyle (p_Y)_!} & & \downarrow{\scriptstyle (p_X)_!} \\ H_G^{i-\dim G}(Y;\Gamma) & \xrightarrow[\bar{f}^*]{} & H_G^{i-\dim G}(X;\Gamma) \end{array}$$

where $\bar{f} = \mathrm{id}\times_G f\colon EG\times_G X \to EG\times_G Y$ *is the induced map from a* G*-map* $\mathrm{id}\times f\colon EG\times X \to EG\times Y$.

Let M be a smooth closed connected oriented G-manifold of dimension n. Suppose that the G-action on M is free. Note that the orbit space M/G is also a manifold of dimension $n - \dim G$ in this case. Let $p\colon M \to M/G$ be the orbit map. Suppose that M/G is orientable over $\mathbf{K}$. Then the transfer $p_!$ of the p is described as $p_! = D_{M/G}^{-1}\circ p_*\circ D_M$ where D is the Poincaré duality isomorphism. Then $p_!\colon H^n(M;\mathbf{K}) \to H^{n-\dim G}(M/G;\mathbf{K})$ is an isomorphism.

The following theorem has been essentially proved in [4].

Theorem 3.2 ([4]). *Let* G *be a compact Lie group and let* M *and* N *be smooth closed connected* G*-free manifolds of dimension* n *which are orientable over* $\mathbf{K}$. *Assume that the orbit space* M/G *and* N/G *are also orientable. Then we have the following.*

(1) *Suppose* $\mathrm{Ind}^G_{n-\dim G}(M;\mathbf{K})$ *is not equal to* $H^{n-\dim G}(BG;\mathbf{K})$. *Then for any* G*-map* $f\colon M \to N$, $f^*\colon H^n(N;\mathbf{K}) \to H^n(M;\mathbf{K})$ *is non-trivial.*

(2) *Suppose that* $\mathrm{Ind}^G_{n-\dim G}(N;\boldsymbol{K})$ *is not equal to* $\mathrm{Ind}^G_{n-\dim G}(M;\boldsymbol{K})$. *Then for any* G*-map* $f\colon M \to N$, $f^*\colon H^n(N;\boldsymbol{K}) \to H^n(M;\boldsymbol{K})$ *is not injective*.

Proof. (1) Assume that there exists a G-map $f\colon M \to N$ such that $f^*\colon H^n(N;\boldsymbol{K}) \to H^n(M;\boldsymbol{K})$ is trivial. By Lemma 3.1, $(p_M)_! \circ f^* = \bar{f}^* \circ (p_N)_!$.

Therefore $\bar{f}^*\colon H_G^{n-\dim G}(N;\boldsymbol{K}) \to H_G^{n-\dim G}(M;\boldsymbol{K})$ is trivial, because $(p_M)_!$ and $(p_N)_!$ are isomorphism and f^* is the trivial homomorphism. Since $c_M = c_N \circ f$,

$$\mathrm{Ind}^G_{n-\dim G}(M;\boldsymbol{K}) = \left(\bar{c}_M^*\right)^{-1}(0) = \left(\bar{c}_N^*\right)^{-1}\left(\left(\bar{f}^*\right)^{-1}(0)\right) = H^{n-\dim G}(M;\boldsymbol{K}).$$

(2) Assume that there exists a G-map $f\colon M \to N$ such that $f^*\colon H^n(N;\boldsymbol{K}) \to H^n(M;\boldsymbol{K})$ is injective. Then $\bar{f}^*\colon H_G^{n-\dim G}(N;\boldsymbol{K}) \to H_G^{n-\dim G}(M;\boldsymbol{K})$ is injective, using Lemma 3.1 again. Hence

$$\begin{aligned}\mathrm{Ind}^G_{n-\dim G}(N;\boldsymbol{K}) &= \ker \bar{c}_N^* = \left(\bar{c}_N^*\right)^{-1}(0) = \left(\bar{c}_N^*\right)^{-1}\left(\left(\bar{f}^*\right)^{-1}(0)\right) = \left(\bar{c}_M^*\right)^{-1}(0)\\ &= \mathrm{Ind}^G_{n-\dim G}(M;\boldsymbol{K})\end{aligned}$$ □

As a consequence of Proposition 2.2 and Theorem 3.2 (1) we get the following theorem.

Theorem 3.3. *Let* $f\colon V_k(\boldsymbol{R}^m) \to V_k(\boldsymbol{R}^m)$ *be a* $(\boldsymbol{Z}_2)^k$*-map. Then the degree of* f *is odd*.

Proof. Set $n = \dim V_k(\boldsymbol{R}^m)$. By Proposition 2.2, $\mathrm{Ind}_n^{(\boldsymbol{Z}_2)^k}(V_k(\boldsymbol{R}^m);\boldsymbol{Z}_2)$ is not equal to $H^n(B(\boldsymbol{Z}_2)^k;\boldsymbol{Z}_2)$. Hence $f^*\colon H^n(N;\boldsymbol{Z}_2) \to H^n(M;\boldsymbol{Z}_2)$ is non-trivial from assertion (1) of Theorem 3.2. □

This theorem implies the following.

Corollary 3.4. *If there exists a* $(\boldsymbol{Z}_2)^k$*-map* $f\colon V_k(\boldsymbol{R}^m) \to V_k(\boldsymbol{R}^n)$, *then* $m \leq n$.

Proof. Let $f\colon V_k(\boldsymbol{R}^m) \to V_k(\boldsymbol{R}^n)$ be a $(\boldsymbol{Z}_2)^k$-map. Assume that $m > n$. The canonical inclusion $i\colon V_k(\boldsymbol{R}^n) \to V_k(\boldsymbol{R}^m)$ is a $(\boldsymbol{Z}_2)^k$-map. Since $i \circ f\colon V_k(\boldsymbol{R}^m) \to V_k(\boldsymbol{R}^m)$ is a $(\boldsymbol{Z}_2)^k$-map, the degree of $i \circ f$ is not even. Otherwise, because $(i \circ f)^* = f^* \circ i^*$ and $H^{\dim V_k(\boldsymbol{R}^m)}(V_k(\boldsymbol{R}^n);\boldsymbol{Z}_2) = 0$, $(i \circ f)^*\colon H^{\dim V_k(\boldsymbol{R}^m)}(V_k(\boldsymbol{R}^m)) \to H^{\dim V_k(\boldsymbol{R}^m)}(V_k(\boldsymbol{R}^m))$ is trivial. This is a contradiction. □

Next if $l < k$, then we regard $(\boldsymbol{Z}_p)^l$ as any subgroup of $(\boldsymbol{Z}_p)^k$. We get a commutative diagram

$$\begin{array}{ccc} E(\boldsymbol{Z}_2)^k \times_{(\boldsymbol{Z}_2)^l} V_k(\boldsymbol{R}^m) & \xrightarrow{\bar{c'}} & B(\boldsymbol{Z}_2)^l \\ \pi \downarrow & & \rho \downarrow \\ E(\boldsymbol{Z}_2)^k \times_{(\boldsymbol{Z}_2)^k} V_k(\boldsymbol{R}^m) & \xrightarrow{\bar{c}} & B(\boldsymbol{Z}_2)^k. \end{array}$$

Then we have

$$\begin{array}{ccc} H^*_{(\boldsymbol{Z}_2)^l}(V_k(\boldsymbol{R}^m)) & \xleftarrow{\bar{c'}^*} & H^*\left(B(\boldsymbol{Z}_2)^l\right) \\ \pi^* \uparrow & & \rho^* \uparrow \\ H^*_{(\boldsymbol{Z}_2)^k}(V_k(\boldsymbol{R}^m)) & \xleftarrow{\bar{c}^*} & H^*\left(B(\boldsymbol{Z}_2)^k\right). \end{array}$$

Theorem 3.5. *If* $\dim V_k(\boldsymbol{R}^m) = \dim V_l(\boldsymbol{R}^n)$, *then for any* $(\boldsymbol{Z}_2)^l$*-map* $f\colon V_k(\boldsymbol{R}^m) \to V_l(\boldsymbol{R}^n)$ *the degree of* f *is even.*

Proof. We set $d = \dim V_k(\boldsymbol{R}^m) = \dim V_l(\boldsymbol{R}^n)$. Then $\pi^*\colon H^d_{(\boldsymbol{Z}_2)^k}(V_k(\boldsymbol{R}^m);\boldsymbol{Z}_2) \to H^d_{(\boldsymbol{Z}_2)^l}(V_k(\boldsymbol{R}^m);\boldsymbol{Z}_2)$ is trivial. Since $\rho^*\colon H^*(B(\boldsymbol{Z}_2)^k;\boldsymbol{Z}_2) \to H^*(B(\boldsymbol{Z}_2)^l;\boldsymbol{Z}_2)$ is surjective, $\bar{c'}^*\colon H^d(B(\boldsymbol{Z}_2)^l;\boldsymbol{Z}_2) \to H^d_{(\boldsymbol{Z}_2)^l}(V_k(\boldsymbol{R}^m);\boldsymbol{Z}_2)$ is also trivial. Therefore we have $\mathrm{Ind}_d^{(\boldsymbol{Z}_2)^l}(V_k(\boldsymbol{R}^m);\boldsymbol{Z}_2) = H^d(B(\boldsymbol{Z}_2)^l;\boldsymbol{Z}_2)$.

Otherwise $\mathrm{Ind}_d^{(\boldsymbol{Z}_2)^l}(V_l(\boldsymbol{R}^n);\boldsymbol{Z}_2) \neq H^d(B(\boldsymbol{Z}_2)^l;\boldsymbol{Z}_2)$ from Proposition 2.2. Therefore it follows from Theorem 3.2 (2) that for any $(\boldsymbol{Z}_2)^l$-map $f\colon V_k(\boldsymbol{R}^m) \to V_l(\boldsymbol{R}^n)$ the degree of f is even. □

Still continuing our complex analogue of the propositions above, we get the following.

Theorem 3.6. *Let* $f\colon V_k(\boldsymbol{C}^m) \to V_k(\boldsymbol{C}^m)$ *be a* $(\boldsymbol{Z}_p)^k$*-map. Then the degree of* f *is not congruent to zero modulo* p.

From this theorem, the following corollary is proved in the same way as Corollary 3.4.

Corollary 3.7. *If there exists a* $(\boldsymbol{Z}_p)^k$*-map* $f\colon V_k(\boldsymbol{C}^m) \to V_k(\boldsymbol{C}^n)$, *then* $m \leq n$.

Next if $l < k$, then we regard $(\boldsymbol{Z}_p)^l$ as any subgroup of $(\boldsymbol{Z}_p)^k$. Hence $V_k(\boldsymbol{C}^m)$ is a free $(\boldsymbol{Z}_p)^l$-manifold. Then we get the following in the same way as Theorem 3.5.

Theorem 3.8. *If* $\dim V_k(\boldsymbol{C}^m) = \dim V_l(\boldsymbol{C}^n)$, *then for any* $(\boldsymbol{Z}_p)^l$*-map* $f: V_k(\boldsymbol{C}^m) \to V_l(\boldsymbol{C}^n)$ *the degree of* f *is congruent to zero modulo* p.

REMARK. If k is even, then $\dim V_k(\boldsymbol{C}^m)$ is even. Hence there does not exist a free $\boldsymbol{Z}_p$-action on $S^{\dim V_k(\boldsymbol{C}^m)}$.

Corollary 3.9. *If* $\dim V_k(\boldsymbol{C}^m) = \dim V_l(\boldsymbol{C}^n)$, *then for any* $(S^1)^l$*-map* $f: V_k(\boldsymbol{C}^m) \to V_l(\boldsymbol{C}^n)$ *the degree of* f *is zero.*

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