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WEAKLY STABLE IDEALS

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Introduction

Weakly stable ideals generalize the squarefree stable ideals introduced in [3] which are the squarefree analogue of stable ideals first considered by Eliahou and Kervaire [6] whose resolution they constructed explicitly. Their result was used and generalized by many authors. The interest in these ideals is due to the fact that stable ideals include the class of standard Borel-fixed ideals which play an important role in Gröbner basis theory.

Our motivation to study weakly stable ideals was to enlarge the class of squarefree ideals with linear resolution. Indeed, we show that weakly stable ideals which are generated by monomials of the same degree have linear resolutions. There are not so many classes of such squarefree monomial ideals known. Fröberg [8] classified the squarefree monomial ideals generated in degree 2 with linear resolution, Bruns and Hibi [5] studied monomial ideals with pure resolution, Hibi [10] discussed a certain class of monomial ideals with linear resolution, and finally there is the Eliahou–Kervaire resolution [6] for stable ideals and its squarefree analogue; see [3] and [7]. In the particular case that the stable or squarefree stable ideal is generated by monomials of the same degree this resolution is linear. A similar result is obtained by Hulett and Martin [12] for generalized lexsegment ideals.

In the first section we consider general weakly stable ideals, which of course may not have linear resolutions, and compute their graded Betti numbers. The crucial observation is that the generators of a weakly stable ideal can be ordered such that the ideal generated by each partial sequence is again weakly stable.

In the second section we study weakly stable ideals generated by monomials of the same degree. From our main theorem in Section 1 it follows already that they have a linear resolution. However in this section we give a more combinatorial argument for the linearity of the resolution, and describe it in terms of Koszul homology following the method developed in the papers [1] and [2]. To do this we compare the weakly stable ideal I with its stable closure I^{st}, that is, the smallest stable ideal which contains the given ideal I. It turns out that the resolution of I may be viewed as a subcomplex of the resolution of I^{st}.

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In [2] Aramova and Herzog showed how to compute the differentials in the resolution once the cycles of the Koszul complex are known whose homology classes form a basis of the Koszul homology. This computation is particularly simple for stable ideals, since in that case the generating cycles can be chosen to be all monomials of the form $ue_{i_1} \wedge \ldots \wedge e_{i_k}$ where u is a monomial. This is no longer the case for weakly stable ideals. However we succeed to describe the Koszul homology of $K[x_1, \ldots, x_n]/I$ as the kernel of a very explicit map whose entries are only 0 and ± 1. This kernel can easily be computed in each particular case. We demonstrate this by an example.

It would be, of course, of great interest to describe all (squarefree) monomial ideals with linear resolution (even though the linearity may depend on the base field) and to determine the possible Betti numbers of such ideals.

1. The Betti numbers of weakly stable ideals

Let I be an ideal generated by squarefree monomials in the polynomial ring $A = K[x_1, \ldots, x_n]$ over a field K. Denote by $G(I)$ the unique minimal set of monomial generators of I. For a monomial $u \in A$, we write $\mathrm{Supp}(u)$ for the set of all i such that x_i divides u, and set $m(u) = \max\{i : i \in \mathrm{Supp}(u)\}$, $u' = u/x_{m(u)}$.

DEFINITION 1.1. A squarefree monomial ideal I is called *weakly stable* if for every squarefree monomial $u \in I$ the following condition $(\star)$ is satisfied:

$(\star)$ For every integer $\ell \notin \mathrm{Supp}(u)$ such that $\ell < m(u')$, there exists an integer $i \in \mathrm{Supp}(u)$ with $i > \ell$ such that $x_\ell(u/x_i) \in I$.

Note that if I is a squarefree stable ideal [3], then I is weakly stable.

Lemma 1.2. *The ideal I is weakly stable if for every $u \in G(I)$ condition $(\star)$ holds.*

Proof. Let $v \in I$ be a squarefree monomial, and let $\ell \notin \mathrm{Supp}(v)$ be an integer with $\ell < m(v')$. We can assume that $v \notin G(I)$, therefore $w = v/x_i \in I$ for some $i \in \mathrm{Supp}(v)$. If $i \geq m(v')$, then $x_\ell(v/x_i) \in I$ with $i > \ell$ and the lemma is proved. So, let $i < m(v')$. Arguing by induction on $\deg v$, we may assume that w satisfies condition $(\star)$. Since $m(w') = m(v') > \ell$, one has $x_\ell w/x_j \in I$ for some $j > \ell$, $j \in \mathrm{Supp}(w)$. Hence, we obtain $x_\ell v/x_j = (x_\ell w/x_j)x_i \in I$. □

EXAMPLE 1.3. Let I be the ideal of $A = K[x_1, x_2, x_3, x_4, x_5, x_6]$ which is generated by the squarefree monomials $x_1x_2x_3$, $x_1x_2x_5$, $x_1x_3x_4$, $x_1x_3x_6$, $x_1x_4x_5$, $x_1x_5x_6$, $x_2x_3x_4$, $x_2x_3x_6$, $x_2x_4x_5$, $x_3x_4x_6$, $x_3x_5x_6$ and $x_4x_5x_6$. Then I is a weakly stable ideal. We remark that there exists no permutation τ of $\{1, 2, 3, 4, 5, 6\}$ such that I^τ is a squarefree stable ideal, where I^τ is the ideal generated by the squarefree monomials $x_{\tau(a)}x_{\tau(b)}x_{\tau(c)}$ with $x_ax_bx_c \in I$.

Throughout this paper, we consider the following term order of the squarefree monomials in A: $u < v$ if either $\deg u < \deg v$ or $\deg u = \deg v$ and $i_q = j_q, \dots, i_{s+1} = j_{s+1}$, $i_s < j_s$ for some $1 \le s \le q$, where $u = x_{i_1} \dots x_{i_q}$, $v = x_{j_1} \dots x_{j_q}$.

For a graded A-module M, denote by $H(M)$ the homology of the Koszul complex of M with respect to $x_1, \dots, x_n$. Since $H_i(M) \cong \mathrm{Tor}_i^A(M, K)$, the graded Betti numbers of M are $\beta_{ij}(M) = \dim_K H_i(M)_j$. We consider also the *graded Poincaré series* $P_M(s,t) = \sum_{i,j\ge 0} \beta_{ij}(M)t^is^j$ of M.

Theorem 1.4. *Let I be a weakly stable ideal with $G(I) = \{u_1, \dots, u_m\}$ where $u_1 < u_2 < \dots < u_m$. Set $a_k = \deg u_k$ for $1 \le k \le m$,*

$$\Lambda_k = \{t \notin \mathrm{Supp}(u_k) : x_tu_k \in (u_1, \dots, u_{k-1})\}, \quad R_k = (A/(x_t, t \in \Lambda_k))(-a_k)$$

for $2 \le k \le m$, and $\Lambda_1 = \emptyset$. Then

(a) *For every $i, j \ge 0$, one has $\beta_{ij}(I) = \sum_{k=1}^m \beta_{ij}(R_k)$ with $\beta_{ij}(R_k) = 0$ for $j \ne a_k + i$, and $\beta_{ia_k+i}(R_k) = \binom{|\Lambda_k|}{i}$ for $2 \le k \le m$. In particular, the Betti numbers of I are independent of the base field K.*

(b) $P_I(s,t) = \sum_{k=1}^m s^{a_k}(1+ts)^{|\Lambda_k|}$.

First we show the following

Lemma 1.5. *Let J and $I = (J, v)$ be weakly stable ideals with $G(I) = G(J) \cup \{v\}$ and $\deg v \ge \deg u$ for every $u \in G(J)$. Then $I/J \cong A/(x_t, t \in \Lambda)$ where $\Lambda = \{t \notin \mathrm{Supp}(v) : x_tv \in J\}$.*

Proof. Since $I/J \cong A/(J : v)$, it suffices to prove that $(J : v) = (x_t, t \in \Lambda)$. First we note that if $t < m(v')$ and $t \notin \mathrm{Supp}(v)$, then $t \in \Lambda$. Indeed, the ideal J being weakly stable, one has $x_tv/x_i \in J$ for some $i > t$, $i \in \mathrm{Supp}(v)$, therefore $x_tv \in J$.

Let now y be a monomial generator of the ideal $(J : v)$. Then $yv = wb$ for some squarefree monomial $w \in J$ and some monomial $b \in A$. Since y is a generator, it follows that y and b have no common factor. Therefore y divides w, $v = (w/y)b$ and $\mathrm{Supp}(y) \cap \mathrm{Supp}(v) = \emptyset$. Hence the desired equality follows if we show that $\deg y = 1$.

Assume $\deg y \ge 2$. Let w be the smallest squarefree monomial in J with $\deg w = \deg v$ such that $yv = wb$ for some monomial $b \in A$. Since y is a generator of $(J : v)$, one has $x_tv \notin J$ for each $t \in \mathrm{Supp}(y)$, therefore, as we noted above, $t > m(v')$ for each $t \in \mathrm{Supp}(y)$. We have $w = (v/b)y$ and $m(b') \le m(v') < m(y') \le m(w')$. Since J is weakly stable, we obtain that $\tilde{w} = x_{m(b')}w/x_i \in J$ for some $i > m(b')$,

$i \in \operatorname{Supp}(w)$. But then $yv = \tilde{w}(x_i b/x_{m(b')})$ with $\tilde{w} < w$, which is a contradiction. Hence $\deg y = 1$, and the lemma is proved. □

Lemma 1.6. *Let I be a weakly stable ideal with $G(I) = \{u_1, \dots, u_m\}$ where $u_1 < u_2 < \cdots < u_m$. Then $I_k = (u_1, \dots, u_k)$ is weakly stable ideal for every k, $1 \leq k \leq m$.*

Proof. By 1.2 it is enough to show that each $u \in G(I_k)$ satisfies condition $(\star)$. Let $\ell < m(u')$, $\ell \notin \operatorname{Supp}(u)$. Since I is weakly stable, we have $v = x_\ell u/x_i \in I$ for some $i > \ell$ with $i \in \operatorname{Supp}(u)$, and $v < u$. Then $v = wb$ for some $w \in G(I)$ with $\deg w \leq \deg v$, so that $w < u$. Therefore $w \in G(I_k)$ and $v \in I_k$. □

Proof of Theorem 1.4. By 1.6, $I_k = (u_1, \dots, u_k)$ is weakly stable ideal for every k, $1 \leq k \leq m$. Assume that (a) is true for I_{k-1}, $2 \leq k \leq m$. We will show that it is true for the ideal I_k.

By 1.5 we have an exact sequence of graded A-modules

$$0 \to I_{k-1} \to I_k \to R_k \to 0$$

which gives a long exact sequence of Koszul homology

$$\dots \to H_{i+1}(R_k) \to H_i(I_{k-1}) \to H_i(I_k) \to H_i(R_k) \to \dots .$$

Set $D_{s,i} = \{a_1 + i, a_2 + i, \dots, a_s + i\}$ for $1 \leq s \leq m$ and $i \geq 0$. By assumption, $\beta_{ij}(I_{k-1}) = \sum_{s=1}^{k-1} \beta_{ij}(R_s)$ and $\beta_{ij}(I_{k-1}) = 0$ if $j \notin D_{k-1,i}$. Therefore, if $j \notin D_{k,i}$, then $\beta_{ij}(I_{k-1}) = 0$ and $\beta_{ij}(R_k) = 0$, so that $\beta_{ij}(I_k) = 0$. Let now $j \in D_{k,i}$. Then $j \leq a_k + i < a_k + i + 1$, hence $\beta_{i+1j}(R_k) = 0$. Moreover, if $\beta_{ij}(R_k) \neq 0$, then $j = a_k + i \geq a_{k-1} + i > a_{k-1} + i - 1$, so that $\beta_{i-1j}(I_{k-1}) = 0$. Thus, for every $j \in D_{k,i}$ we have the exact sequence

$$0 \to H_i(I_{k-1})_j \to H_i(I_k)_j \to H_i(R_k)_j \to 0. \tag{1}$$

Using the induction hypothesis, from this sequence we obtain the required equality for $\beta_{ij}(I_k)$. This completes the proof of (a).

(b) follows immediately from (a). □

2. Weakly stable ideals with linear resolution

In this section we consider weakly stable ideals generated by monomials of the same degree. It follows immediately from 1.4 that these ideals have a linear resolution. Alternatively we present a proof of this fact which is based on Fröberg's result [8] and Hochster's formulas [11]. Further we describe the maps of the resolution in terms of Koszul cycles. We remark that in general we are not able to construct the cycles explicitely using the exact sequences (1) (cf. the proof of 1.4), and in many particular cases it is easier to find the cycles using the map described in 2.2(b) below.

Recall that for a homogeneous ideal $J \subset A = K[x_1, \ldots, x_n]$, the graded minimal free resolution of A/J over A is called q-linear, if for every $i \geq 1$ one has $\beta_{ij}(A/J) = 0$ for every $j \neq q+i-1$. In this case, we say also that J has a linear resolution.

If I is a weakly stable ideal and all generators of I have the same degree q, then we call I *weakly stable of degree* q.

Theorem 2.1. *Suppose that an ideal I of $A = K[x_1, x_2, \ldots, x_n]$ is weakly stable of degree q. Then, a minimal free resolution of A/I over A is q-linear.*

Proof. We refer the reader to [4], [9], [11] and [13] for algebra and combinatorics on simplicial complexes. Let Δ be a simplicial complex on the vertex set $V = \{x_1, x_2, \ldots, x_n\}$ and I_Δ the ideal of A which is generated by the squarefree monomials $x_{i_1}x_{i_2}\cdots x_{i_r}$, $1 \leq i_1 < i_2 < \cdots < i_r \leq n$, with $\{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\} \notin \Delta$. If $W \subset V$, then $\Delta_W := \{\sigma \in \Delta \ : \ \sigma \subset W\}$. Let $\tilde{H}_i(\Delta_W; K)$ denote the i-th reduced simplicial homology group of Δ with the coefficient field K.

First of all, let an ideal I of A be weakly stable of degree 2 and Δ the simplicial complex on V with $I = I_\Delta$. What we must prove is that the 1-skeleton of Δ is a chordal graph [8]; see also [5, p. 1206]. Suppose that the 1-skeleton of Δ has a cycle Γ of length at least 4 with no chord. Let $1 \leq \ell \leq n$ denote the least integer with $x_\ell \in \Gamma$. If $x_\ell x_i$ and $x_\ell x_j$ belong to Γ, then $\ell < i$, $\ell < j$ and $x_i x_j \notin \Gamma$. Hence $x_i x_j \in I$, while $x_\ell x_i \notin I$ and $x_\ell x_j \notin I$, a contradiction.

Suppose that an ideal I of A is weakly stable of degree q with $q \geq 3$. Again, choose the simplicial complex Δ on V with $I = I_\Delta$, and set $\Delta_1 = \Delta_{V-\{x_1\}}$, $\Delta_2 = \mathrm{star}_\Delta(\{x_1\})$ and $\Delta' = \mathrm{link}_\Delta(\{x_1\})$. Then, the ideal I_{Δ_1} of $A' = K[x_2, x_3, \ldots, x_n]$ is either a weakly stable ideal of degree q or $I_{\Delta_1} = (0)$. Since the ideal $I_{\Delta'}$ of $A' = K[x_2, x_3, \ldots, x_n]$ is generated by the squarefree monomials u in $x_2, x_3, \ldots, x_n$ with $x_1 u \in I$, it follows from $(\star)$ with $t = 0$ and $\ell = 1$ that $I_{\Delta'}$ is generated by the squarefree monomials $x_{i_1}x_{i_2}\cdots x_{i_{q-1}}$, $2 \leq i_1 < i_2 < \cdots < i_{q-1} \leq n$, with $x_1 x_{i_1} x_{i_2} \cdots x_{i_{q-1}} \in I$. Hence, the ideal $I_{\Delta'}$ of A' is weakly stable of degree $q-1$. By virtue of [11, Theorem (5.1)], a minimal free resolution of A/I over A is q-linear if and only if $\tilde{H}_i(\Delta_W; K) = 0$ for every subset W of V and for each $i \neq q-2$. Now, the induction hypothesis enables us to assume that $\tilde{H}_i((\Delta_1)_W; K) = 0$ and $\tilde{H}_{i-1}(\Delta'_W; K) = 0$ for every subset W of $V - \{x_1\}$ and for each $i \neq q-2$. If $x_1 \notin W$, then $\Delta_W = (\Delta_1)_W$. Hence $\tilde{H}_i(\Delta_W; K) = 0$ for every subset W of $V - \{x_1\}$ and for each $i \neq q-2$. If $x_1 \in W$, then $(\Delta_2)_W$ is contractible; in particular, $\tilde{H}_i((\Delta_2)_W; K) = 0$ for every i. Thus, since $\Delta_1 \cup \Delta_2 = \Delta$ and $\Delta_1 \cap \Delta_2 = \Delta'$, the reduced Mayer–Vietoris exact sequence

$$\cdots \to \tilde{H}_i(\Delta'; K) \to \tilde{H}_i(\Delta_1; K) \oplus \tilde{H}_i(\Delta_2; K) \to \tilde{H}_i(\Delta; K) \to \tilde{H}_{i-1}(\Delta'; K) \to \cdots$$

guarantees $\tilde{H}_i(\Delta_W; K) = 0$ for each $i \neq q-2$ as required. □

For an A-module M, denote by $K(M) = K(\mathbf{x}; M)$ the Koszul complex of M with respect to the sequence $\mathbf{x} = x_1, \ldots, x_n$ of indeterminates and, as before, by $H(M)$ the homology of $K(M)$. Set $e_\sigma = e_{j_1} \wedge \ldots \wedge e_{j_i}$ where $\sigma = \{j_1, \ldots, j_i\}$, $j_1 < \ldots < j_i$, so that the differential ∂ of $K_i(M)$ is given by $\partial(e_\sigma) = \sum_{j \in \sigma} (-1)^{\alpha(\sigma,j)} x_j e_{\sigma \setminus j}$, where $\alpha(\sigma, j) = |\{t \in \sigma : t < j\}|$.

We will denote the image of a monomial $u \in A$ in any quotient ring of A again by u.

Let $I \subset A$ be a weakly stable ideal of degree q. Then I is contained in a squarefree stable ideal. Denote by I^{st} the *stable closure* of I, that is, the smallest squarefree stable ideal in A which contains I. Note that I^{st} is also generated by monomials of degree q.

According to [3], for every $i > 0$, a basis of $H_i(A/I^{st})$ is given by the homology classes of the cycles

$$\text{(2)} \qquad u' e_\tau, \quad u \in G(I^{st}), \quad |\tau| = i, \quad \max(\tau) = m(u), \quad \tau \cap \operatorname{Supp}(u') = \emptyset$$

and the differential d_i of the minimal free resolution $F = A \otimes H(A/I^{st})$ of A/I^{st} is given by (cf. also [1]):

$$d_i(f(\sigma; u)) = \sum_{j \in \sigma} (-1)^{\alpha(\sigma,j)} (-x_j f(\sigma \setminus j; u) + x_{m(u)} f(\sigma \setminus j; x_j u')),$$

where

$$f(\sigma; u) = 1 \otimes [u' e_\sigma \wedge e_{m(u)}],$$
$$u \in G(I^{st}), |\sigma| = i-1, \max(\sigma) < m(u), \sigma \cap \operatorname{Supp}(u) = \emptyset,$$

is the basis of F_i, and where we set $f(\sigma; u) = 0$ if $\max(\sigma) > m(u)$.

Let $\{v_1, \ldots, v_s\} = G(I^{st}) \setminus G(I)$ be all the monomials in the minimal set of monomial generators of I^{st} which do not belong to I. We may assume these monomials are numerated so that $v_1 < v_2 < \cdots < v_s$ with respect to the order used in Section 1. Set $J_0 = I$ and for $1 \leq k \leq s$

$$J_k = (I, v_1, \ldots, v_k); \qquad \Sigma_k = \Sigma_{v_k} = \{t \notin \operatorname{Supp}(v_k) : x_t v_k \in J_{k-1}\}.$$

Theorem 2.2. *Let I be a weakly stable ideal of degree q, and let I^{st} be its stable closure. Then*

(a) *For every* $i \geq 1$, $\beta_i(A/I) = \sum_{k=1}^{m} \binom{|\Lambda_k|}{i-1} = \beta_i(A/I^{st}) - \sum_{k=1}^{s} \binom{|\Sigma_k|}{i-1}$ *where the sets* Λ_k *are defined in* 1.4, *and* $\beta_i(A/I^{st}) = \sum_{u \in G(I^{st})} \binom{m(u)-q}{i-1}$.

(b) *For every* $i \geq 1$, *one has an exact sequence of vector spaces*

$$0 \to H_i(A/I) \to H_i(A/I^{st}) \xrightarrow{\psi} \bigoplus_{k=1}^{s} H_{i-1}((A/(x_t, t \in \Sigma_k))(-q)) \to 0,$$

where the map ψ *is defined on the basis* (2) *of* $H_i(A/I^{st})$ *by*

$$\psi([u'e_\tau]) = \sum_{t \in \rho(\tau;u)} (-1)^{\alpha(\tau,t)}[e_{\tau \setminus t}]$$

with $\rho(\tau;u) = \{t \in \tau : x_t u' \in \{v_1, \ldots, v_s\}, \tau \setminus t \subset \Sigma_{x_t u'}\}$.

(c) *The minimal free resolution* $A \otimes H(A/I)$ *of* A/I *is a subcomplex of the minimal free resolution* $A \otimes H(A/I^{st})$ *of* A/I^{st}.

To show this theorem we need the following:

Lemma 2.3. *Let* $I \subset A$ *be a weakly stable ideal of degree* q. *Let* U *be the set of all monomials of the form* $u'x_l$ *which do not belong to* I, *where* $u \in G(I)$ *and* l *is any integer with* $m(u') < l < m(u)$. *Then* $G(I^{st}) = G(I) \cup U$.

Proof. Since $(G(I), U) \subset I^{st}$, it is enough to show that $(G(I), U)$ is a stable ideal. Take first any generator $u \in G(I)$ and let l be an arbitrary integer such that $l \notin \operatorname{Supp}(u)$ and $l < m(u)$. We have to show $x_l u' \in G(I) \cup U$. If $m(u') < l < m(u)$, then $x_l u' \in G(I) \cup U$, so we can assume $l < m(u')$. Since I is weakly stable, one has $u_1 = x_l u/x_j \in I$ for some $j \in \operatorname{Supp}(u)$ such that $l < j \leq m(u)$. If $j = m(u)$, then $x_l u' = u_1$, so it remains to consider the case $j < m(u)$. Then $m(u_1) = m(u)$, and $x_l u' = x_j u_1'$. If $m(u_1') < j$, then $x_j u_1' \in G(I) \cup U$, otherwise $u_2 = x_j u_1/x_k \in I$ for some $k \in \operatorname{Supp}(u)$ with $j < k \leq m(u)$. Now we may argue for u_2 similarly as for u_1. Proceeding in this way, finally we obtain $x_l u' \in G(I) \cup U$.

Let now $v = u'x_l \in U$. Then $l = m(v)$. Let $k < l$ be any integer such that $k \notin \operatorname{Supp}(v)$. Since $u \in G(I)$ and $k < m(u)$, we have already shown that $x_k u' \in G(I) \cup U$. But $x_k v' = x_k u'$, and this completes the proof. $\square$

Lemma 2.4. *The ideal* J_k *is weakly stable for* $0 \leq k \leq s$.

Proof. It is enough to show that $J = J_1$ is weakly stable. We have only to check that $v = v_1$ has the weakly stable property. By 2.3, $v = u'x_l$ for some $u \in G(I)$ and $m(u') < l < m(u)$. Let $i < m(u')$, $i \notin \operatorname{Supp}(u)$. Since I is weakly stable and $u \in I$, one obtains that $w = x_i u/x_j \in I$ for some $j \in \operatorname{Supp}(u)$ such that $i < j$. If $j = m(u)$, then $x_i v/x_l = x_i u' = w$ and the proof is completed. So, assume $j < m(u)$.

Then $l < m(w)$, and $x_i v/x_j = x_l w' \in I^{st}$. But since $w < u$, we have $x_l w' < x_l u' = v$, therefore we conclude that $x_l w' \in I$. $\square$

Proof of Theorem 2.2. The first equality for $\beta_i(A/I)$ follows immediately from 1.4(a).

For each k, $1 \le k \le s$, setting $A_k = (A/(x_t, t \in \Sigma_k))(-q)$, by 1.5 we have an exact sequence of graded A-modules

$$0 \to J_{k-1} \to J_k \to A_k \to 0$$

which gives a long exact sequence of Koszul homology

$$\cdots \to H_{i+1}(A_k) \to H_i(J_{k-1}) \to H_i(J_k) \to H_i(A_k) \to \cdots.$$

The ideals J_k and J_{k-1} being weakly stable (cf. 2.4), by 2.1 they have linear resolutions. On the other hand, A_k also has a linear resolution over A shifted by q. Therefore, for each $i \ge 0$, considering the above long sequence in degree $q+i$, we obtain that it splits into the exact sequences

$$0 \to H_i(J_{k-1}) \to H_i(J_k) \to H_i(A_k) \to 0.$$

Since for an ideal $J \subset A$ one has $H_i(J) \cong H_{i+1}(A/J)$, for each $i \ge 1$ we get the exact sequence

$$0 \to H_i(A/J_{k-1}) \to H_i(A/J_k) \xrightarrow{\phi_k} H_{i-1}(A_k) \to 0. \tag{3}$$

This proves the second equality for $\beta_i(A/I)$ in (a). The equality for $\beta_i(A/I^{st})$ follows from (2).

(b) Note that for $1 \le k \le s$, $H_{i-1}(A_k)$ has a basis consisting of the homology classes of the cycles e_σ, $\sigma \subset \Sigma_k$, $|\sigma| = i-1$. We will show that $\mathrm{Ker}\psi = H_i(A/I)$. Define a linear map $\psi_k : H_i(A/I^{st}) \to H_{i-1}(A_k)$ on the basis (2) by: $\psi_k([u'e_\tau]) = (-1)^{\alpha(\tau,t)}[e_{\tau\setminus t}]$ if $x_t u' = v_k$ for some $t \in \tau$ and if $\tau \setminus t \subset \Sigma_k$; $\psi_k([u'e_\tau]) = 0$ otherwise. Then $\psi = \sum_{k=1}^{s} \psi_k$, so that $[z] \in \mathrm{Ker}\psi$ if and only if $\psi_k([z]) = 0$ for $1 \le k \le s$. Therefore, the equality $\mathrm{Ker}\psi = H_i(A/I)$ will follow if we show that for $1 \le k \le s$ the restriction of ψ_k on $H_i(A/J_k)$ coincides with the map ϕ_k defined in (3). Let $z = \sum c_{\tau,u} u' e_\tau$ with $c_{\tau,u} \in K$ represent a cycle in $K_i(A/J_k)$, where the $[u'e_\tau]$ belong to the monomial basis (2) of $H_i(A/I^{st})$. Then $\partial(z) = \sum_{\rho,|\rho|=i-1} b_\rho e_\rho$, where $b_\rho = \sum_{t \notin \rho, u \in G(I^{st})} (-1)^{\alpha(\rho,t)} c_{\rho\cup t,u} x_t u' \in J_k$. Let $b_\rho \mathrm{mod} J_{k-1} = \epsilon(b_\rho) v_k$. Then $\phi_k([z]) = [\sum_{\rho,|\rho|=i-1} \epsilon(b_\rho) e_\rho] = \sum_{\rho \subset \Sigma_k} \epsilon(b_\rho)[e_\rho]$, where the second equality follows from the fact that all $\epsilon(b_\rho) \in K$. On the other hand, $\psi([z]) = \sum c_{\tau,u} \psi([u'e_\tau]) = \sum_{\rho\subset\Sigma_k} (\sum_{t\notin\rho} \sum_{u \in G(I^{st}), x_t u' = v_k} (-1)^{\alpha(\rho,t)} c_{\rho\cup t,u})[e_\rho] = \phi_k([z])$.

To show (c) we recall from [1] the method to construct the maps ϑ_i of the resolution of an A-module M from its Koszul cycles in the particular case when M has a linear resolution. For $a \in K_i(M)$ and $1 \le j \le n$ there is a unique decomposition $a = a_j - e_j \wedge \pi_j(a)$ with $a_j, \pi_j(a) \in K(\mathbf{x}_j; M)$, where $\mathbf{x}_j = x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n$. Then, given $1 \otimes [z] \in A \otimes H_i(M)$, we have $\vartheta_i(1 \otimes [z]) = \sum_{j=1}^n x_j \otimes [\pi_j(z)]$.

Let $z = \sum c_{\tau,u} u' e_\tau$ with $c_{\tau,u} \in K$ represent a cycle in $K_i(A/I)$, where the $[u' e_\tau]$ belong to the monomial basis (2) of $H_i(A/I^{st})$. Since the function π_j is linear for each j, one has $\pi_j(z) = \sum c_{\tau,u} \pi_j(u' e_\tau)$, therefore $\sum_{j=1}^n x_j \otimes [\pi_j(z)] = \sum c_{\tau,u} \sum_{j=1}^n x_j \otimes [\pi_j(u' e_\tau)] = d_i([z])$. This shows that the maps of the resolution of A/I are the restrictions of the maps d_i of the resolution of A/I^{st}. □

We conclude with 2 examples.

EXAMPLE 2.5. Consider the ideal $I \subset K[x_1, \ldots, x_6]$ generated by $x_1x_2x_3$, $x_1x_2x_5$, $x_1x_3x_4$, $x_1x_3x_6$, $x_1x_4x_5$, $x_2x_3x_5$, $x_3x_4x_5$ and $x_3x_5x_6$. Then I is weakly stable, and $G(I^{st}) = G(I) \cup \{v_1, v_2, v_3\}$ where $v_1 = x_1x_2x_4$, $v_2 = x_1x_3x_5$ and $v_3 = x_2x_3x_4$. It follows that $\Sigma_1 = \{3,5\}$, $\Sigma_2 = \{2,4,6\}$ and $\Sigma_3 = \{1,5\}$. Therefore, using formula 2.2(a) we obtain

$$\beta_2(A/I) = 19 - 7 = 12, \quad \beta_3(A/I) = 11 - 5 = 6, \quad \beta_4(A/I) = 2 - 1 = 1.$$

Next we give the matrix of the map

$$H_3(A/I^{st}) \xrightarrow{\psi} H_2(A_1) \bigoplus H_2(A_2) \bigoplus H_2(A_3)$$

with respect to the basis of $H_3(A/I^{st})$ consisting of the homology classes of the cycles $z_1 = x_1x_2e_3e_4e_5$, $z_2 = x_1x_3e_2e_4e_5$, $z_3 = x_1x_3e_2e_5e_6$, $z_4 = x_1x_3e_4e_5e_6$, $z_5 = x_2x_3e_1e_4e_5$, $z_6 = x_1x_4e_2e_3e_5$, $z_7 = x_3x_4e_1e_2e_5$, $z_8 = x_3x_5e_1e_2e_6$, $z_9 = x_3x_5e_1e_4e_6$, $z_{10} = x_1x_3e_2e_4e_6$ and $z_{11} = x_3x_5e_2e_4e_6$, and the basis of $H_2(A_1) \bigoplus H_2(A_2) \bigoplus H_2(A_3)$ given by the homology classes of the cycles e_3e_5, e_2e_4, e_2e_6, e_4e_6 and e_1e_5:

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore a basis of $H_3(A/I)$ is given by the homology classes of the cycles

$$z_1 + z_6, \quad z_3 + z_8, \quad z_4 + z_9, \quad z_5 - z_7, \quad z_{10}, \quad z_{11}.$$

Note that $H_3(A/I)$ has no basis consisting only of the homology classes of monomial cycles.

Similarly one computes $H_2(A/I)$ and $H_4(A/I)$. The cycles for $H_2(A/I)$ which are not of the form $u'e_je_{m(u)}$, $u \in G(I)$, are $x_1x_2e_4e_5 - x_1x_4e_2e_5$, $x_1x_3e_5e_6 - x_3x_5e_1e_6$ and $x_2x_3e_4e_5 - x_3x_4e_2e_5$.

Finally, $H_4(A/I)$ is generated by $[x_1x_3e_2e_4e_5e_6 - x_3x_5e_1e_2e_4e_6]$, and also has no monomial basis.

We could consider the stable closure of a weakly stable ideal which is generated by monomials of different degrees. However, the following simple example shows that in general $H_i(A/I)$ cannot be considered as a subspace of $H_i(A/I^{st})$.

EXAMPLE 2.6. Consider the ideal $I \subset K[x_1, \ldots, x_6]$ generated by $u_1 = x_1x_2x_4$, $u_2 = x_1x_3x_4$, $u_3 = x_1x_2x_3x_5$, $u_4 = x_2x_3x_4x_5$ and $u_5 = x_2x_3x_4x_6$. Then I is weakly stable and $I^{st} = (I, x_1x_2x_3)$. Here $\Lambda_2 = \{2\}$, $\Lambda_3 = \{4\}$, $\Lambda_4 = \{1\}$ and $\Lambda_5 = \{1,5\}$, so that by 1.4 $\beta_{24}(A/I) = 1$ and $\beta_{25}(A/I) = 4$. On the other hand, $H_2(A/I^{st})$ has a basis consisting of the homology classes of $z_1 = x_1x_2e_3e_4$, $z_2 = x_1x_3e_2e_4$, $z_3 = x_2x_3x_4e_1e_5$, $z_4 = x_2x_3x_4e_1e_6$ and $z_5 = x_2x_3x_4e_5e_6$. Using the exact sequence (1), we obtain that the homology classes of $z_1 - z_2$, z_3, z_4, z_5 and $z = x_1x_2x_3e_4e_5$ form a basis of $H_2(A/I)$. Note that $[z]$ maps to 0 in $H_2(A/I^{st})$.

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