Nishimura, Y.
Osaka J. Math.
33 (1996), 775–791

THE K_*-LOCAL TYPE OF THE ORBIT MANIFOLD $(S^{2m+1}\times S^l)/D_q$ BY THE DIHEDRAL GROUP D_q

YASUZO NISHIMURA

(Received September 6, 1995)

Introduction

For a given CW-spectrum E there is an associated E-homology theory $E_*X=\pi_*(E\wedge X)$. A CW-spectrum Y is called E_*-local if any E_*-equivalence $A\to B$ induces an isomorphism $[B,Y]_*\cong[A,Y]_*$. For any CW-spectrum X there exists an E_*-equivalence $\iota_E: X\to X_E$ such that X_E is E_*-local. X_E is called the E_*-localization of X. Let KO and KU be the real and the complex K-spectrum respectively. There is no difference between the KO_*- and KU_*-localizations, and so we denote by S_K the K_*-localization of the sphere spectrem $S=\Sigma^0$. According to the smashing theorem [2, Corollary 4.7] the smash product $S_K\wedge X$ is actually the K_*-localization of X for any CW-spectrum X.

In this note we shall be interested in the K_*-local type of certain orbit manifolds $D(q)^{m,l}$ introduced as a filtration of a classifying space of the dihedral group D_q in [8]. The manifold $D(q)^{m,l}$ is defind as follows: Let $q\geq 3$ be an odd integer, and D_q the dihedral group generated by two elements a and b with relations $a^q=b^2=abab=1$. Consider the unit spheres S^{2m+1} and S^l in the complex $(m+1)$-space C^{m+1} and the real $(l+1)$-space R^{l+1}. Then D_q operates freely on the product space $S^{2m+1}\times S^l$ by

$$a\cdot(z,x)=(z\exp(2\pi\sqrt{-1}/q),x),\qquad b\cdot(z,x)=(\bar{z},-x)$$

where $\bar{z}$ is the conjugate of z. The associted topological quotient spaces

$$D(q)^{2m+1,l}=(S^{2m+1}\times S^l)/D_q=(L(q)^{2m+1}\times S^l)/Z_2,$$
$$D(q)^{2m,l}=(L(q)^{2m}\times S^l)/Z_2\subset D(q)^{2m+1,l}$$

are defined where $L(q)^{2m+1}=L^m(q)$ is the $(2m+1)$-dimensional lens space $\bmod q$ and $L(q)^{2m}=L_0^m(q)$ its $2m$-skeleton.

The group $KU^0D(q)^{m,l}$ is decomposed to a direct sum of KU^0-groups of suspensions of stunted lens spaces $\bmod q$ and $\bmod 2$ (cf. [5, Theorem 3.9]). Moreover KO^0- and J^0-groups of $D(q)^{m,l}$ have a quite similar direct sum decomposition (cf. [10] or [7]). In section 1 we shall show that $D(q)^{m,l}$ itself has such a decomposition as K_*-local spectrum. The K_*-local type of the stunted real projective space $RP^m/RP^n = RP^m_{n+1}$ has been determined explicitly by constructing small cell spectra in [13]. In section 2 we shall study the K_*-local type of the stunted lens space $L(p)^m/L(p)^n = L(p)^m_{n+1}$ for an odd prime p. Consequently we can observe the K_*-local type of $D(q)^{m,l}$ more explicitly in the special case that q is an odd prime p.

1. The K_*-local type of $D(q)^{m,l}$

Let $\mathscr{A}$ be the category of abelian groups with stable Adams operations ψ^k ($k \in Z$) (cf. [4, 5.1]). For an arbitrary set P of primes, let $\mathscr{A}_{(P)}$ be the full subcategory of $Z_{(P)}$-modules of the abelian category $\mathscr{A}$. Then the inclusion functor $\mathscr{A}_{(P)} \subset \mathscr{A}$ has the obvious left adjoint $(\ )\otimes Z_{(P)}$. Assume that P is a finite set of primes. By the Chinese remainder theorem there exists an integer r such that: r generates $(Z/p^2)^*$ for each odd $p \in P$; $r = \pm 3$ mod 8 when $2 \in P$; $|r| \geq 2$ when P is empty. Let $\mathscr{A}^r_{(P)}$ be the category of $Z_{(P)}$-modules with automorphism ψ^r and involution ψ^{-1}. By [4, 6.4] the forgetful functor $\mathscr{A}_{(P)} \to \mathscr{A}^r_{(P)}$ is a categorical isomorphism. Moreover if $2 \notin P$ then we don't need the involution ψ^{-1} in the abelian category $\mathscr{A}^r_{(P)}$ (cf. [3, Proposition 5.7]).

For any prime p let us fix an integer r as above. Denote by $Ad_{(p)}$ the fiber of the $\psi^r_R - 1 : KO_{(p)} \to KO_{(p)}$ where ψ^k_R is the stable real Adams operation. Then we have the following cofiber sequences (cf. [2, section 4]):

$$Ad_{(p)} \xrightarrow{\xi} KO_{(p)} \xrightarrow{\psi^r_R - 1} KO_{(p)} \to \Sigma^1 Ad_{(p)}$$

$$S_{K(p)} \xrightarrow{\iota_A} Ad_{(p)} \to \Sigma^{-1} SQ \to \Sigma^1 S_{K(p)}.$$

For an odd prime p the first sequence can be replaced by

$$Ad_{(p)} \to KU_{(p)} \xrightarrow{\psi^r_C - 1} KU_{(p)} \to \Sigma^1 Ad_{(p)}$$

because $Ad_{(p)}$ also arises as the fiber of $\psi^r_C - 1 : KU_{(p)} \to KU_{(p)}$. Using this fact we can easily verify the following lemma (cf. [3, Theorem 9.1]).

Lemma 1.1. *Let X and Y be CW-spectra such that KU_0X and KU_0Y are odd torsion groups and $KU_1X = KU_1Y = 0$. If KU_0X and KU_0Y are isomorphic in the abelian category $\mathscr{A}$ then X and Y have the same K_*-local type.*

In order to describe the K_*-local type of $D(q)^{m,l}$ we first consider the lens space $L(q)^m$. Recall that

$$KU^0 L(q)^{2m+1} \cong KU^0 L(q)^{2m} \cong Z[\sigma]/(\sigma^{m+1}, (1+\sigma)^q - 1),$$

$$KU^1L(q)^{2m+1} \cong Z, \qquad KU^1L(q)^{2m}=0$$

(cf. [6] or [11]) where $\sigma=[\gamma]-1$ for the canonical line bundle γ over $L(q)^{2m+1}$ (which is induced by the natural surjection $\pi: L(q)^{2m+1} \to CP^m$) or its restriction over $L(q)^{2m}$. Therefore the stable Adams operation ψ_C^k operates on $KU^0L(q)^{2m}$ as

$$\psi_C^k\sigma=(1+\sigma)^k-1.$$

Since $KU^0L(q)^{2m}$ is an odd torsion group, there exist subgroups A^m and B^m on which the conjugation ψ_C^{-1} acts as 1 and -1 respectively (cf. [4, Proposition 3.8]) and a direct sum decomposition $KU^0L(q)^{2m} \cong A^m \oplus B^m$ in $\mathscr{A}$. (In this case A^m and B^m are generated by the elements $\sigma+\psi_C^{-1}\sigma$ and $(\sigma-\psi_C^{-1}\sigma)(\sigma+\psi_C^{-1}\sigma)^{i-1}$ $(i \geq 1)$ respectively (cf. [5, Lemma 3.3]).) From [4, Theorem 10.1](or [3, Proposition 8.7]) and [4, Theorem 11.1] there exist certain finite spectra SA^m and SB^m such that $KU^0SA^m \cong A^m$, $KU^0SB^m \cong B^m$ and $KU^1SA^m=KU^1SB^m=0$ in $\mathscr{A}$. Then the lens space $L(q)^{2m}$ has the same K_*-local type as $SA^m \vee SB^m$ by Lemma 1.1. We obtain the KO_*-groups by the Bott and Anderson cofiber sequences as follows:

$$KO_iSA^m \cong \begin{cases} A^m & \text{for } i \equiv 3 \bmod 4 \\ 0 & \text{otherwise} \end{cases}, \qquad KO_iSB^m \cong \begin{cases} B^m & \text{for } i \equiv 1 \bmod 4 \\ 0 & \text{otherwise} \end{cases}.$$

Let $\bar{f}: \Sigma^{2m} \to L(q)^{2m}$ be the attaching map of the top cell in $L(q)^{2m+1}$. Consider the associated map $f=(f_A, f_B): \Sigma^{2m} \to SA^m \vee SB^m$ such that $l_K \wedge \bar{f}=\varphi f$ where $\varphi: SA^m \vee SB^m \to S_K \wedge L(q)^{2m}$ is a K_*-equivalence. Since $KO_iSA^m=0$ for $i \not\equiv 3 \bmod 4$, $f_A \in [\Sigma^{2m}, S_K \wedge SA^m]=0$ when m is even. Similarly $f_B \in [\Sigma^{2m}, S_K \wedge SB^m]=0$ when m is odd. Therefore $L(q)^{2m+1}$ has the same K_*-local type as the cofiber $C(f)=C(f_A) \vee SB^m$ when m is odd or $C(f)=SA^m \vee C(f_B)$ when m is even. We shall often denote SA^m and SB^m by SA and SB respectively for simplicity.

Lemma 1.2. *Let $\iota_K: S_K \to KO$ denote the K_*-localized map of the unit $\iota: S \to KO$.*

i) *If* $l \equiv 1 \bmod 4$ *then* $[\Sigma^l SA, S_K \wedge SA]=0=[\Sigma^l SB, S_K \wedge SB]$, *and if* $l \equiv 0 \bmod 4$ *then* $\iota_{K_*}: [\Sigma^l SA, S_K \wedge SA] \to [\Sigma^l SA, KO \wedge SA]$ *and* $\iota_{K_*}: [\Sigma^l SB, S_K \wedge SB] \to [\Sigma^l SB, KO \wedge SB]$ *are monomorphisms.*

ii) *If* $l \equiv 3 \bmod 4$ *then* $[\Sigma^l SA, S_K \wedge SB]=0=[\Sigma^l SB, S_K \wedge SA]$, *and if* $l \equiv 2 \bmod 4$ *then* $\iota_{K_*}: [\Sigma^l SA, S_K \wedge SB] \to [\Sigma^l SA, KO \wedge SB]$ *and* $\iota_{K_*}: [\Sigma^l SB, S_K \wedge SA] \to [\Sigma^l SB, KO \wedge SA]$ *are monomorphisms.*

Proof. i) There is an exact sequence

$$[\Sigma^l SA, \Sigma^{-1}KO_{(p)} \wedge SA] \to [\Sigma^l SA, S_{K(p)} \wedge SA] \xrightarrow{\iota_{K_*}} [\Sigma^l SA, KO_{(p)} \wedge SA].$$

It is easily verified that $[\Sigma^l SA, KO \wedge SA]=0$ when $l \equiv 1$ or 2 mod 4 because $KO_iSA=0$ for $i \not\equiv 3 \bmod 4$. Now our result is immediate.

ii) is shown similarly.

Consider the $Z/2$-action on $L(q)^{2m}$ induced by the complex conjugation

$$t: L(q)^{2m} \to L(q)^{2m}, \quad [z] \mapsto [\bar{z}].$$

By definition $t^*\sigma = \psi_C^{-1}\sigma$ and ψ_C^{-1} operates on SA^m and SB^m as 1 and -1 respectively. Therefore we obtain the following commutative diagram after replacing the K_*-equivalence $\varphi: SA^m \vee SB^m \to S_K \wedge L(q)^{2m}$ suitably necessary:

$$\begin{array}{ccc} S_K \wedge L(q)^{2m} & \xrightarrow{t} & S_K \wedge L(q)^{2m} \\ \uparrow \varphi & & \uparrow \varphi \\ SA^m \vee SB^m & \xrightarrow{1 \vee (-1)} & SA^m \vee SB^m. \end{array}$$

This can be also proved by induction on m using Lemma 1.2.

For the orbit manifold $D(q)^{m,l} = (L(q)^m \times S^l)/Z_2$ there is a fibering

$$L(q)^m \xrightarrow{k} D(q)^{m,l} \xrightarrow{p} RP^l.$$

Since the projection p has a right inverse $RP^l = D(q)^{0,l} \subset D(q)^{m,l}$ (cf. [5, Lemma 1.7]) we observe that

$$D(q)^{m,l} = RP^l \vee D(q)^{m,l}_{1,0}$$

where $D(q)^{m,l}_{1,0} = D(q)^{m,l}/RP^l$.

In order to determine the K_*-local type of $D(q)^{2m,l}_{1,0}$ by induction on l we need the following cofiber sequence (cf. [10]):

$$\Sigma^{l-1} L(q)^{2m} \xrightarrow{\pi_{l-1}} D(q)^{2m,l-1}_{1,0} \xrightarrow{k_l} D(q)^{2m,l}_{1,0} \xrightarrow{q_l} \Sigma^l L(q)^{2m}.$$

Note that $q_l \pi_l = \nabla \lambda_l \rho: \Sigma^l L(q)^{2m} \to \Sigma^l L(q)^{2m}$ where $\lambda_l = \mathrm{id} \vee (\tau \wedge t): \Sigma^l L(q)^{2m} \vee \Sigma^l L(q)^{2m} \to \Sigma^l L(q)^{2m} \vee \Sigma^l L(q)^{2m}$ for the antipotal map τ of Σ^l, ρ is the comultiplication of $\Sigma^l L(q)^{2m}$ and ∇ is the folding map (cf. [5, Lemma 1.11]). Therefore we may regard that $q_l \pi_l: \Sigma^l SA^m \vee \Sigma^l SB^m \to \Sigma^l SA^m \vee \Sigma^l SB^m$ is expressed as

$$q_l \pi_l = \begin{cases} 0 \vee 2 & \text{if } l \text{ is even} \\ 2 \vee 0 & \text{if } l \text{ is odd.} \end{cases}$$

The KU-cohomology of $D(q)^{2m,l}_{1,0}$ is given as follows (cf. [5, Theorem 3.9]):

l	even	odd
$KU^0D(q)^{2m,l}_{1,0}$	$A^m\oplus(B^m\otimes KU^0\Sigma^l)$	A^m
$KU^1D(q)^{2m,l}_{1,0}$	0	$A^m\otimes KU^1\Sigma^l$.

The components A^m and $C^m\otimes KU^*\Sigma^l$ (where $C=A$ if l is odd and $C=B$ if l is even) are given via the canonical inclusion $k: L(q)^{2m}=D(q)^{2m,0}_{1,0}\subset D(q)^{2m,l}_{1,0}$ and the natural projection $q_l: D(q)^{2m,l}_{1,0}\to\Sigma^l L(q)^{2m}$ respectively.

Proposition 1.3. *$D(q)^{2m,l}_{1,0}$ has the same K_*-local type as $SA^m\vee\Sigma^l SB^m$ if l is even and $SA^m\vee\Sigma^l SA^m$* if *l is odd.*

Proof. i) The "$l\equiv 0$ mod 4" case: Since the conjugation acts on $KU^0D(q)^{2m,l}_{1,0}$ as $\psi_C^{-1}=1$ on A^m and $\psi_C^{-1}=-1$ on $B^m\otimes KU^0\Sigma^l$, $KU^0D(q)^{2m,l}_{1,0}$ is decomposed to A^m and $B^m\otimes KU^0\Sigma^l$ in the abelian category $\mathscr{A}$. From Lemma 1.1, $D(q)^{2m,l}_{1,0}$ has the same K_*-local type as $SA^m\vee\Sigma^l SB^m$.

ii) The "$l\equiv 1$ mod 4" case: We consider the following cofiber sequence

$$\Sigma^{l-1}L(q)^{2m}\xrightarrow{\pi_{l-1}} D(q)^{2m,l-1}_{1,0}\xrightarrow{k_l} D(q)^{2m,l}_{1,0}\xrightarrow{q_l}\Sigma^l L(q)^{2m}.$$

Here we can replace $\Sigma^{l-1}L(q)^{2m}$ and $D(q)^{2m,l-1}_{1,0}$ by $\Sigma^{l-1}SA\vee\Sigma^{l-1}SB$ and $SA\vee\Sigma^{l-1}SB$ respectively from i). We set:

$$\pi_{l-1}=\begin{pmatrix} x & z\\ y & 2\end{pmatrix},\quad q_{l-1}=\begin{pmatrix} u & w\\ v & 1\end{pmatrix}$$

where all of $x,\cdots,v$ and w become trivial if they are carried from $[X,S_K\wedge Y]$ into $[X,KO\wedge Y]$ via the map $\iota_K: S_K\to KO$. From Lemma 1.2 x and u must be trivial. Since $q_{l-1}\pi_{l-1}=\begin{pmatrix} 0 & 0\\ 0 & 2\end{pmatrix}$, y and w are also trivial. Thus we can express as

$$\pi_{l-1}=\begin{pmatrix} 0 & z\\ 0 & 2\end{pmatrix},\quad q_{l-1}=\begin{pmatrix} 0 & 0\\ v & 1\end{pmatrix}.$$

Consider the following commutative diagram:

$$\begin{array}{ccccccc}
\Sigma^{l-1}SA & \xrightarrow{0} & SA & \rightarrow & SA\vee\Sigma^{l}SA & \rightarrow & \Sigma^{l}SA \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^{l-1}SA\vee\Sigma^{l-1}SB & \xrightarrow{\pi_{l-1}} & SA\vee\Sigma^{l-1}SB & \xrightarrow{k_l} & S_K\wedge D(q)^{2m,l}_{1,0} & \xrightarrow{q_l} & \Sigma^{l}SA\vee\Sigma^{l}SB \\
\downarrow & & \downarrow & & & & \\
\Sigma^{l-1}SB & \overset{2}{\cong} & \Sigma^{l-1}SB & & & &
\end{array}$$

Now we can determine the K_*-local type of $D(q)^{2m,l}_{1,0}$ as desired and we can take

$$k_l=\begin{pmatrix}1 & b\\ 0 & 0\end{pmatrix},\quad q_l=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}.$$

iii) The "$l\equiv 3 \bmod 4$" case: As is shown in ii) we can express as $q_{l+1}=\begin{pmatrix}0 & 0\\ v & 1\end{pmatrix}$. Our result is proved similarly to the case ii).

iv) The "$l\equiv 2 \bmod 4$" case: From Lemma 1.2 we can set $\pi_{l-1}=\begin{pmatrix}0 & x\\ 2 & y\end{pmatrix}$. Since $q_{l-1}=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$ and $q_{l-1}\pi_{l-1}=\begin{pmatrix}2 & 0\\ 0 & 0\end{pmatrix}$, y is trivial. For the canonical inclusion $k: L(q)^m\rightarrow D(q)^{m,l+1}_{1,0}$ we notice that $k\,|\,SA=(1,*): SA\rightarrow SA\vee\Sigma^{l+1}SA$. Then x must be trivial because $k_{l+1}k_l\pi_{l-1}=0$. Now our result is immediate.

REMARK. For the case iv) the subgroup $A^m\subset KU^0D(q)^{2m,l}_{1,0}$ is the image of representation ring of D_q (cf. [5, Section 2]). Therefore $KU^0D(q)^{2m,l}_{1,0}$ is also decomposed to A^m and $B^m\otimes KU^0\Sigma^l$ in $\mathscr{A}$. Then we can prove the case iv) in a similar way to the case i).

Let $RP^{m+l+1}_{m+1}=RP^{m+l+1}/RP^m$ be the stunted real projective space. Consider the following commutaive diagram:

$$\begin{array}{ccccc}
 & & \Sigma^{m+l+1} & = & \Sigma^{m+l+1} \\
 & & \downarrow{\scriptstyle\gamma_0} & & \downarrow{\scriptstyle\gamma} \\
\Sigma^{m+1} & \xrightarrow{\beta_0} & \Sigma^1RP^{m+l}_m & \rightarrow & \Sigma^1RP^{m+l}_{m+1} \\
\| & & \downarrow & & \downarrow \\
\Sigma^{m+1} & \xrightarrow{\beta} & \Sigma^1RP^{m+l+1}_m & \rightarrow & \Sigma^1RP^{m+l+1}_{m+1}
\end{array}$$

where β's are the bottom cell inclusions and γ's are the top cell attaching maps. Recall that K_*-local type of $\Sigma^1RP^{2s+2n}_{2s+1}$ has the same K_*-local type as a certain small cell spectrum $\nabla SZ/2^n$ such that $KU_0\nabla SZ/2^n \cong Z/2^n$ on which $\psi_C^{-1}=1$ and $KU_1\nabla SZ/2^n=0$ (see [13, Theorem 2.7] for details). Then $\Sigma^1 RP_{2s+1}^{2s+2n+1}$, $\Sigma^1 RP_{2s+2}^{2s+2n}$ and $\Sigma^1 RP_{2s+2}^{2s+2n+1}$ have the same K_*-local types as the cofibers of the associated maps $\gamma:\Sigma^{2s+2n+1}\to\nabla SZ/2^n$, $\beta:\Sigma^{2s+2}\to\nabla SZ/2^n$ and $\beta_0\vee\gamma_0$: $\Sigma^{2s+2}\vee\Sigma^{2s+2n+1}\to\nabla SZ/2^n$ respectively, which are given explicitly in [13, Theorems 2.7, 2.9, 3.8]. Using these associated maps we can give the K_*-local type of $D(q)_{1,0}^{2m+1,l}$, as follows.

Theorem 1.4. *$D(q)_{1,0}^{2m+1,l}$ has the same K_*-local type as the spectra tabled below*:

m	l	$D(q)_{1,0}^{2m+1,l}$
even	odd	$SA^m\vee\Sigma^l SA^m\vee\Sigma^m RP_{m+1}^{m+l+1}$
even	even	$SA^m\vee C(\Sigma^l f_B,\Sigma^{m-1}\gamma)$
odd	even	$\Sigma^l SB^m\vee C(f_A,\Sigma^{m-1}\beta)$
odd	odd	$C\begin{pmatrix} f_A & 0 \\ 0 & \Sigma^l f_A \\ \Sigma^{m-1}\beta_0 & \Sigma^{m-1}\gamma_0\end{pmatrix}$

Proof. We have the following cofiber sequence (cf. [5, Lemma 1.12]):

$$\Sigma^{m-1}RP_{m+1}^{m+l+1}\xrightarrow{F} D(q)_{1,0}^{2m,l}\to D(q)_{1,0}^{2m+1,l}.$$

Here we may use $SA^m\vee\Sigma^l SC^m$ instead of $D(q)_{1,0}^{2m,l}$ by virtue of Proposition 1.3. When m is odd we consider the $KZ[1/2]_*$-localization of the following commutative diagram:

$$\begin{array}{ccccc}
\Sigma^{2m} & \xrightarrow{f} & L(q)^{2m} & \to & L(q)^{2m+1}\\
\downarrow{\scriptstyle k_0} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle k}\\
\Sigma^{m-1}RP_{m+1}^{m+l+1} & \xrightarrow{F} & D(q)_{1,0}^{2m,l} & \to & D(q)_{1,0}^{2m+1,l}
\end{array}$$

where k and k_0 are the canonical inclusions. Then we may regard as $k_0=(1,0):\Sigma^{2m}\to\Sigma^{2m}\vee\Sigma^{m-1}RP_m^{m+l+1}$, $f=(f_A,0):\Sigma^{2m}\to SA^m\vee SB^m$ and $k=\begin{pmatrix}1 & b\\ 0 & 0\end{pmatrix}$: $SA^m\vee SB^m\to SA^m\vee\Sigma^l SC^m$. Therefore $F|\Sigma^{2m}$ is expressed as $(f_A,0):\Sigma^{2m}\to SA^m\vee\Sigma^l SC^m$.

When $m+l$ is even we consider the $KZ[1/2]_*$-localization of the following commutative diagram:

$$\begin{array}{ccccc} \Sigma^{2m+l} & \xrightarrow{f} & \Sigma^l L(q)^{2m} & \rightarrow & \Sigma^l L(q)^{2m+1} \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \pi_l} & & \downarrow{\scriptstyle \pi_l} \\ \Sigma^{m-1} RP^{m+l+1}_{m+1} & \xrightarrow{F} & D(q)^{2m,l}_{1,0} & \rightarrow & D(q)^{2m+1,l}_{1,0} \end{array}$$

where γ is the top cell attaching map and π_l is the natural projection. Then we may regard as $\gamma=(0,1):\Sigma^{2m+l}\rightarrow\Sigma^{m-1}RP^{m+l}_{m+1}\vee\Sigma^{2m+l}$, $f=(f_C,0):\Sigma^{2m+l}\rightarrow\Sigma^l SC^m \vee\Sigma^l SC'^m$ where $C'=B$ if l is odd and $C'=A$ if l is even, and $\pi_l=\begin{pmatrix}0 & * \\ 2 & *\end{pmatrix}:\Sigma^l SC^m \vee\Sigma^l SC'^m\rightarrow SA^m\vee\Sigma^l SC^m$. Therefore $F|\Sigma^{2m+l}$ is expressed as $(0,2f_C):\Sigma^{2m+l} \rightarrow SA^m\vee\Sigma^l SC^m$. Consequently $D(q)^{2m+1,l}_{1,0}$ has the same $KZ[1/2]_*$-local type as $SA^m\vee\Sigma^l SA^m$, $SA^m\vee\Sigma^l C(f_B)$, $C(f_A)\vee\Sigma^l SB^m$ and $C(f_A)\vee\Sigma^l C(f_A)$ according as $(m,l)\equiv(0,1),(0,0),(1,0)$ and $(1,1)$ mod 2 respectively. From the previous observation we can determine the K_*-local type of $D(q)^{2m+1,l}_{1,0}$ as desired.

Let n and k be integers such that $0\leq n\leq m$ and $0\leq k\leq l$. We set:

$$D(q)^{m,l}_{n,k}=D(q)^{m,l}/(D(q)^{m,k-1}\cup D(q)^{n-1,l}).$$

This space is the Thom complex of a canonical bundle over $D(q)^{m-n,l-k}$ when n is even. We shall extend Proposition 1.3 and Theorem 1.4 to the case of $D(q)^{m,l}_{n,k}$. In order to state the extended theorem we express the K_*-local type of the stunted lens space $L(q)^m_{n+1}=L(q)^m/L(q)^n$ as follows: $L(q)^{2m}_{2n+1}$ has the same K_*-local type as $SA^m_n\vee SB^m_n$ where the conjugation acts as $\psi_C^{-1}=1$ on $KU^0SA^m_n\cong A^m_n$ and $\psi_C^{-1}=-1$ on $KU^0SB^m_n\cong B^m_n$. $L(q)^{2m+1}_{2n+1}$, $L(q)^{2m}_{2n+2}$ and $L(q)^{2m+1}_{2n+2}$ have the same K_*-local types as the cofibers of the following maps respectively:

$$f=(f_A,f_B):\Sigma^{2m}\rightarrow SA^m_n\vee SB^m_n;$$

$$g=(g_A,g_B):\Sigma^{2n+1}\rightarrow SA^m_n\vee SB^m_n;$$

$$f\vee g:\Sigma^{2m}\vee\Sigma^{2n+1}\rightarrow SA^m_n\vee SB^m_n.$$

Here $f_A=0$ if m is even and $f_B=0$ if m is odd, and $g_A=0$ if n is even and $g_B=0$ if n is odd.

Let $\langle\Sigma^k\rangle$ be Σ^k if k is odd and $*$ if k is even. Then we can choose the map $\beta\vee\gamma:\Sigma^1\langle\Sigma^k\rangle\vee\langle\Sigma^l\rangle\rightarrow\nabla SZ/2^i$ so that its cofiber $C(\beta\vee\gamma)$ has the same K_*-local type as $\Sigma^1 RP^l_{k+1}$ where i depends on k and l.

Theorem 1.5. i) *$D(q)^{2m,l}_{2n+1,k}$ has the same K_*-local type as $\Sigma^k SE^m_n\vee\Sigma^l SC^m_n$ where $C=A$ if l is odd and $C=B$ if l is even, and $E=A$ if k is even and $E=B$ if k is odd.*

ii) *$D(q)^{2m+1,l}_{2n+1,k}$, $D(q)^{2m,l}_{2n+2,k}$ and $D(q)^{2m+1,l}_{2n+2,k}$ have the same K_*-local types as the* *cofibers of the following maps respectively*:

$$\tilde{F}: X=\Sigma^m\langle\Sigma^{m+k}\rangle\vee\Sigma^{m-1}\langle\Sigma^{m+l+1}\rangle\to\Sigma^k SE_n^m\vee\Sigma^l SC_n^m\vee\Sigma^{m-1}\nabla SZ/2^i,$$
$$\tilde{G}: Y=\Sigma^{n+1}\langle\Sigma^{n+k}\rangle\vee\Sigma^{n}\langle\Sigma^{n+l+1}\rangle\to\Sigma^k SE_n^m\vee\Sigma^l SC_n^m\vee\Sigma^{n}\nabla' SZ/2^j,$$
$$\tilde{H}: X\vee Y\to\Sigma^k SE_n^m\vee\Sigma^l SC_n^m\vee\Sigma^{m-1}\nabla SZ/2^i\vee\Sigma^{n-1}\nabla' SZ/2^j$$

which are expressed as the following matrices:

$$\tilde{F}=\begin{pmatrix} f_E & 0\\ 0 & f_C\\ \beta & \gamma\end{pmatrix},\qquad \tilde{G}=\begin{pmatrix} g_E & 0\\ 0 & g_C\\ \beta' & \gamma'\end{pmatrix},\qquad \tilde{H}=\begin{pmatrix} f_E & 0 & g_E & 0\\ 0 & f_C & 0 & g_C\\ \beta & \gamma & 0 & 0\\ 0 & 0 & \beta' & \gamma'\end{pmatrix}$$

where the maps $\beta\vee\gamma$ and $\beta'\vee\gamma'$ are taken such that the cofibers $C(\beta\vee\gamma)$ and $C(\beta'\vee\gamma')$ have the same K_*-local types as $\Sigma^m RP_{m+k+1}^{m+l+1}$ and $\Sigma^{n+1}RP_{n+k+1}^{n+l+1}$ respectively.

Proof. The case i) is proved similarly to the proof of Proposition 1.3. Consider the following cofiber sequences (cf. [7, Lemma 3.11]):

$$\Sigma^{m-1}RP_{m+k+1}^{m+l+1}\xrightarrow{F} D(q)_{2n+1,k}^{2m,l}\to D(q)_{2n+1,k}^{2m+1,l}$$
$$\Sigma^{n}RP_{n+k+1}^{n+l+1}\xrightarrow{G} D(q)_{2n+1,k}^{2m,l}\to D(q)_{2n+2,k}^{2m,l}.$$

By a similar argument to the proof of Theorem 1.4 we can show that the cofibers $C(F)$ and $C(G)$ have the same K_*-local types as the cofibers $C(\tilde{F})$ and $C(\tilde{G})$ respectively. Moreover the cofiber $C(\tilde{H})$ has the same K_*-local type as $C(F\vee G)=D(q)_{2n+2,k}^{2m+1,l}$.

REMARK. S. Kôno has independently studied the KO^*- and J^*-groups of $D(q)_{n,k}^{m,l}$ in [7]. According to his computations the KO^*- and J^*-groups of $D(q)_{n,k}^{m,l}$ are also decomposed to the KO^*- and J^*-groups of the stunted lens spaces mod q and mod 2 when n is odd; but there is a case the J^*-group doesn't necessarily have such a decomposition when n is even.

2. The K_*-local type of $L(p)_n^m$

In this section p denotes an odd prime. Recall that the groups $\pi_i S_{K(p)}\cong\pi_i S_K\otimes Z_{(p)}$ are isomorphic to the following: $Z_{(p)}$ for $i=0$; $Q/Z_{(p)}=Z/p^\infty$ for $i\equiv -2$; Z/p^r for $i\equiv -1 \bmod 2(p-1)$ with $i\neq -1$ where $r=v_p(i+1)+1$; and 0 otherwise (cf. [2]). For $t>0$ with $v_p(t)\geq r-1$ there exists an element $\alpha_{t,r}:\Sigma^{2t(p-1)-1}\to\Sigma^0$ of order p^r in the image of J-homomorphism $J:\pi_*SO\to\pi_*\Sigma^0$. Let SZ/p^r be the Moore spectrum of type Z/p^r, and $i_r: \Sigma^0 \to SZ/p^r$ and $j_r: SZ/p^r \to \Sigma^1$ denote the bottom cell inclusion and the top cell projection. Then there exists an Adams' K_*-equivalence

$$A_{t,r}: \Sigma^{2t(p-1)}SZ/p^r \to SZ/p^r$$

such that $j_r A_{t,r} i_r = \alpha_{t,r}$ (see [1, Section 12]). For simplicity we shall often omit the subscript r such as $i=i_r$, $j=j_r$ and $\alpha_t=\alpha_{t,r}$ when $r=v_p(t)+1$.

Let X be a CW-spectrum such that $KU_0X \cong Z/p^r$ and $KU_1X=0$. We fix an integer k such that it generates $(Z/p^2)^*$. Then the Adams operation ψ_C^k on KU_0X is expressed as $\psi_C^k = k^{-t}$ for some integer t because k also gererates $(Z/p^r)^*$. This implies that X has the same K_*-local type as $\Sigma^{2t}SZ/p^r$ for some t $(0 \le t < p^{r-1}(p-1))$ (cf. [4, Proposition 10.5]).

Theorem 2.1. *Let m and n be integers such that $m-n=r(p-1)+s$ $(0 \le s < p-1$, $r \ge 0)$. The function $e(k,j)$ is defined by $e(k,j)=2kp^j-1$ when $j \ge 0$ and $e(k,-1)=2k-1$. Then $L(p)_{2n+1}^{2m}$ has the same K_*-local type as*

$$\bigvee_{i=1}^{p-1} \Sigma^{e(n+i,r(i))}SZ/p^{r(i)+1}$$

where $r(i)=r$ if $i \le s$ and $r(i)=r-1$ if $i>s$.

Proof. If $m=n+1$ then $L(p)_{2n+1}^{2n+2}$ is actually $\Sigma^{2n+1}SZ/p$. Assume that $L(p)_{2n+1}^{2m}$ has the same K_*-local type as the desired wedge sum of Moore spectra. Consider the following cofiber sequence

$$\Sigma^{2m}SZ/p \xrightarrow{g} L(p)_{2n+1}^{2m} \to L(p)_{2n+1}^{2m+2}.$$

It is easily verified that $[\Sigma^{2m}SZ/p,\ S_K \wedge \Sigma^{e(n+i,r(i))}SZ/p^{r(i)+1}]=0$ for $i \ne s+1$. Therefore the K_*-localized map g may be expressed as $g=(0,\cdots,0,g_{s+1},0,\cdots,0)$ where $g_{s+1}: \Sigma^{2m}SZ/p \to S_K \wedge \Sigma^{e(n+s+1,r-1)}SZ/p^r$. Recall that

$$KU_{-1}L(p)_{2n+1}^{2m+2} \cong \bigoplus_{i=1}^{s+1} Z/p^{r+1} \oplus \bigoplus_{i=s+2}^{p-1} Z/p^r$$

(cf. [6] or [11]). Hence $KU_{-1}C(g_{s+1})$ must be Z/p^{r+1} on which $\psi_C^k \equiv 1/k^{n+s+1} \bmod p$ and $\psi_C^{k+p}=\psi_C^k$. This implies that $C(g_{s+1})$ has the same K_*-local type as $\Sigma^{e(n+s+1,r)}SZ/p^{r+1}$.

REMARK. Recall that each $M \in \mathscr{A}_{(p)}$ is a direct sum of its subobject $M^{[i]} \in T^i\mathscr{B}_{(p)}$ for $i=0,1,\cdots,p-2$ (see [3, Proposition 3.7]). We can assert that $KU_{-1}L(p)_{2n+1}^{2m} \cong \bigoplus_{i=1}^{p-1} Z/p^{r(i)+1}$ as an abelian group gives rise to a decomposition in $\mathscr{A}$ because

$\Sigma^1 L(p)^{2m}_{2n+1}$ is mod$p$ decomposable (see [10, Proposition 9.6]) and its Atiyah-Hirzebruch spectral sequence collapses. Using this result we may also obtain the above theorem immediately.

In order to investigate the K_*-local type of $L(p)^{2m+1}_{2n+1}$ we shall describe generators of the group $[\Sigma^{2t(p-1)-1}SZ/p, S_k\wedge SZ/p^r]$. We first assume that $t>0$ and put $q=v_p(t)+1$. For the map $\alpha_t=\alpha_{t,q}:\Sigma^{2t(p-1)-1}\to\Sigma^0$ of order p^q its coextention $\tilde{\alpha}_t=\tilde{\alpha}_{t,q}:\Sigma^{2t(p-1)}\to SZ/p^q$ is given by $A_{t,q}i_q$. Using the obvious map $\pi=\pi_{q,r}:SZ/p^q\to SZ/p^r$ we obtain a generator $\pi\tilde{\alpha}_t$ (denoted simply by $\tilde{\alpha}_{t,r}$) in the group $[\Sigma^{2t(p-1)}, S_K\wedge SZ/p^r]\cong Z/p^{min(r,q)}$ such that $j_r\tilde{\alpha}_{t,r}=\alpha_{t,r}$ if $q\leq r$ and $j_r\tilde{\alpha}_{t,r}=p^{q-r}\alpha_{t,r}$ if $q>r$. The map $i_r\alpha_t$ generates the group $[\Sigma^{2t(p-1)-1}, S_K\wedge SZ/p^r]\cong Z/p^{min(r,q)}$. We may assume that $\alpha_{t,1}=p^{q-1}\alpha_t:\Sigma^{2t(p-1)-1}\to\Sigma^0$. Then its extension $\bar{\alpha}_{t,1}:\Sigma^{2t(p-1)-1}SZ/p\to\Sigma^0$ is given by $j_qA_{t,q}\pi_{1,q}$. Note that $p^{r-1}i_r\alpha_t=(\alpha_t\wedge\pi_{1,r})i_1:\Sigma^{2t(p-1)-1}\to SZ/p^r$. Now we can give two generators of the group

$$[\Sigma^{2t(p-1)-1}SZ/p,\ S_K\wedge SZ/p^r]\cong Z/p\oplus Z/p$$

for $t>0$ as follows (cf. [1, Theorem 12.11]): the first component is generated by $\tilde{\alpha}_{t,r}j_1$; the second component is generated by $i_r\bar{\alpha}_{t,1}$ and $\alpha_t\wedge\pi$ according as $r\geq q$ and $r\leq q$ respectively. Moreover it is easily verifed that these generators have the following relations: $i_r\bar{\alpha}_{t,1}=\tilde{\alpha}_{t,r}j_1$ for $r<q$; $i_r\bar{\alpha}_{t,1}=\tilde{\alpha}_{t,r}j_1+\alpha_t\wedge\pi$ for $r=q$; and $\tilde{\alpha}_{t,r}j_1=\alpha_t\wedge\pi$ for $r>q$.

Consider the group $\pi_{-2t(p-1)-1}S_{K(p)}$ for $t>0$. Since $\tilde{\alpha}_t=A_{t,q}i_q:\Sigma^{2t(p-1)}\to SZ/p^q$ we obtain a K_*-equivalence $e_t:\Sigma^{2t(p-1)+1}\to C(\tilde{\alpha}_t)$ such that $e_tj_q=i_CA_{t,q}$ and $j_Ce_t=p^q$ for the canonical inclusion $i_C:SZ/p^q\to C(\tilde{\alpha}_t)$ and the canonical projection $j_C:C(\tilde{\alpha}_t)\to\Sigma^{2t(p-1)+1}$. Moreover there exists a K_*-equivalence $A_{-t,q}:SZ/p^q\to\Sigma^{-1}C(\tilde{\alpha}_t)\wedge SZ/p^q$ such that $(1\wedge j_q)A_{-t,q}=i_C$. Set $\alpha_{-t}=i_Ci_q:\Sigma^{2t(p-1)-1}\to\Delta_{-t}\Sigma^0=\Sigma^{-2t(p-1)-1}C(\tilde{\alpha}_t)$ which may be regarded as a generator of the group $\pi_{-2t(p-1)-1}S_{K(p)}$. By using α_{-t} instead of α_t in the previous discussion we can give two generators of the group $[\Sigma^{-2t(p-1)-1}SZ/p,\ S_K\wedge SZ/p^r]\cong Z/p\oplus Z/p$ for $t>0$ when SZ/p^r is replaced by $\Delta_{-t}SZ/p^r=\Sigma^{-2t(p-1)-1}C(\tilde{\alpha}_t)\wedge SZ/p^r$.

Denote by $L^t_{r,1}$ $(t\neq0)$ the spectrum constructed as the cofiber of the map $\alpha_t\wedge\pi:\Sigma^{2t(p-1)-1}SZ/p\to\Delta_tSZ/p^r$ where $\Delta_tSZ/p^r=SZ/p^r$ for $t>0$. Recall that $KU_0C(\alpha_t)\cong Z\oplus Z$ and $KU_0C(i_r\alpha_t)\cong Z\oplus Z/p^r$ on which the Adams operations ψ^k_C act as

$$\psi^k_C=\begin{pmatrix}1/k^{t(p-1)} & 0\\ (1-k^{t(p-1)})/p^qk^{t(p-1)} & 1\end{pmatrix}$$

with $q=v_p(t)+1$ and $KU_1C(\alpha_t)=KU_1(i_r\alpha_t)=0$ (cf. [1]). Then the KU_*-group of $L^t_{r,1}$ is given as follows:

$$KU_0L^t_{r,1} \cong Z/p \oplus Z/p^r;\ \psi^k_C = \begin{pmatrix} 1/k^{t(p-1)} & 0 \\ p^{r-1}(1-k^{t(p-1)})/p^q k^{t(p-1)} & 1 \end{pmatrix}$$

$$KU_1L^t_{r,1} = 0.$$

For a given specturm X, we shall denote by ΔX a CW-spectrum having the same K_*-local type as X.

Proposition 2.2. *Assume that* $t \neq 0$ *and put* $q = v_p(t)+1$ *and* $t = xp^{q-1}$. *Let* $\iota : S \to S_K$ *be the unit of* S_K. *For each map* $g : \Sigma^{2t(p-1)-1}\Delta SZ/p \to \Delta SZ/p^r$ *its cofiber* $C(g)$ *has the same* K_**-local type as the following specturm:*
i) *The "*$q \geq r$*" case:* $SZ/p^r \vee \Sigma^{2t(p-1)}SZ/p$ *when* $\iota \wedge g = 0$; $\Sigma^{2t(p-1)}SZ/p^{r+1}$ *when* $\iota \wedge g = \tilde{\alpha}_{t,r}j$; $L^t_{r,1}$ *when* $\iota \wedge g = \alpha_t \wedge \pi$; and $\Sigma^{2(p-1)w}SZ/p^{r+1}$ when $\iota \wedge g = \alpha_t \wedge \pi + u\tilde{\alpha}_{t,r}j$ *for a unit* u *of* Z/p *where* $w = -u^{-1}xp^{r-1}$ *if* $q > r$ *and* $w = (1-u^{-1})xp^{r-1}$ *if* $q = r$,
ii) *The "*$q < r$*" case:* $SZ/p^r \vee \Sigma^{2t(p-1)}SZ/p$ *when* $\iota \wedge g = 0$; SZ/p^{r+1} *when* $\iota \wedge g = i\tilde{\alpha}_{t,1}$; $L^t_{r,1}$ *when* $\iota \wedge g = \tilde{\alpha}_{t,r}j$; *and* $\Sigma^{2(p-1)w}SZ/p^{r+1}$ *when* $\iota \wedge g = i\tilde{\alpha}_{t,1} + u\tilde{\alpha}_{t,r}j$ *for a unit* u *of* Z/p *where* $w = up^{r-1}$.

Proof. Use the following commutative diagram:

$$\begin{array}{ccccccc}
 & & & & \Sigma^{2t(p-1)} & = & \Sigma^{2t(p-1)} \\
 & & & & \downarrow^{\varphi} & & \downarrow^{p} \\
\Sigma^{2t(p-1)-1} & \xrightarrow{gi} & \Delta SZ/p^r & \to & C(gi) & \to & \Sigma^{2t(p-1)} \\
\downarrow^{i} & & \| & & \downarrow^{h} & & \downarrow \\
\Sigma^{2t(p-1)-1}SZ/p & \xrightarrow{g} & \Delta SZ/p^r & \to & C(g) & \to & \Sigma^{2t(p-1)}SZ/p.
\end{array}$$

i) It is sufficient to show the case $g = \alpha_t \wedge \pi + u\tilde{\alpha}_{t,r}j$. Note that $gi = p^{r-1}i\alpha_t$ and $\varphi_* : KU_0\Sigma^{2t(p-1)} \to KU_0C(p^{r-1}i_r\alpha_t)$ is expressed as $\begin{pmatrix} p \\ u \end{pmatrix} : Z \to Z \oplus Z/p^r$. Hence we obtain that

$$KU_0C(g) \cong Z/p^{r+1}; \qquad h_* = (1, -pu^{-1}) : Z \oplus Z/p^r \to Z/p^{r+1},$$

and that ψ^k_C on $KU_0C(g)$ behaves as $\psi^k_C = 1/k^{t(p-1)} - p^r(1-k^{t(p-1)})/p^q uk^{t(p-1)}$. Put $k^{p-1} = 1 + yp$ and $t = xp^{q-1}$. Then $\psi^k_C = 1 - xyp^q + u^{-1}xyp^r = 1 - zyp^r = 1/k^{w(p-1)}$ where $z = -u^{-1}x$ if $q > r$ and $z = (1-u^{-1})x$ if $q = r$.

ii) From the relation $\tilde{\alpha}_{t,r}j = \alpha_t \wedge \pi$ it follows that $C(\tilde{\alpha}_{t,r}j) = L^t_{r,1}$. Since $C(i\tilde{\alpha}_{t,1})$ has the same K_*-local type as SZ/p^{r+1} we can take $\varphi_* = \begin{pmatrix} p \\ 1 \end{pmatrix} : Z \to Z \oplus Z/p^r$ $\cong KU_0C(i\alpha_{t,1})$ when $u = 0$, and generally $\varphi_* = \begin{pmatrix} p \\ 1 + p^{r-q}u \end{pmatrix} : Z \to Z \oplus Z/p^r$. The rest of proof is similar to i).

We shall next describe generators of the group

$$[\Sigma^{-1}SZ/p,\ S_K\wedge SZ/p^r]\cong Z/p\oplus Z/p.$$

Set $\beta_r=(\tilde{\alpha}_{1,r}\wedge 1)i_C:\Sigma^{-1}SZ/p\to\Delta_0SZ/p^r=\Sigma^{-2p+1}SZ/p^r\wedge C(\tilde{\alpha}_1)$ where $i_C:SZ/p\to C(\tilde{\alpha}_1)$ is the canonical inclusion. Using the relations $i_Ci_1=\alpha_{-1}$ and $(\alpha_1\wedge 1)\alpha_{-1}=(j_r\wedge 1)\beta_ri_1$ we obtain that

$$KU_0C(\beta_ri_1)\cong Z\oplus Z/p^r;\qquad \psi_C^k=\begin{pmatrix}1 & 0\\ p^{r-2}(k^{p-1}-1)/k^{p-1} & 1\end{pmatrix}.$$

Therefore β_r is a generator of the group $[\Sigma^{-1}SZ/p,\ S_K\wedge SZ/p^r]$ and another generator is cleary i_rj_1. Note that $\iota_K\wedge\beta_r$ is identified with the element $p^{r-1}i_rj_1$ of the group $[\Sigma^{-1}SZ/p,\ KO\wedge SZ/p^r]$ where $\iota_K:S_K\to KO$ is the K_*-localized map of the unit of KO.
So we replace the generator β_1 by $\beta_1-i_1j_1$ when $r=1$. Denote by $L^0_{r,1}$ the spectrum constructed as the cofiber of the map β_r. The KU_*-group of $L^0_{r,1}$ is given as follows:

$$KU_0L^0_{r,1}\cong Z/p\oplus Z/p^r;\qquad \psi_C^k=\begin{pmatrix}1 & 0\\ p^{r-2}(k^{p-1}-1)/k^{p-1} & 1\end{pmatrix}$$

$$KU_1L^0_{r,1}=0.$$

Similarly to Proposition 2.2 we can show the following proposition.

Proposition 2.3. *Let $\iota:S\to S_K$ be the unit of S_K. For each map $g:\Sigma^{-1}\Delta SZ/p\to\Delta SZ/p^r$ its cofiber $C(g)$ has the same K_*-local type as the following spectrum $SZ/p^r\vee SZ/p$ when $\iota\wedge g=0$; SZ/p^{r+1} when $\iota\wedge g=ij$; $L^0_{r,1}$ when $\iota\wedge g=\beta_r$; and $\Sigma^{2(p-1)w}SZ/p^{r+1}$ when $\iota\wedge g=\beta_r+uij$ for a unit u of Z/p where $w=u^{-1}p^{r-1}$ if $r>1$ and $w=-u^{-1}$ if $r=1$.*

Set $q=v_p(t)+1$ and $a=\min(r,v_p(t)+1)$ for $t\neq 0$. Denote by M^t_r, N^t_r and P^t_r $(t\neq 0)$ the spectra constructed as the cofibers of the maps $p^{a-1}i_r\alpha_t:\Sigma^{2t(p-1)-1}\to\Delta_tSZ/p^r$, $p^{a-1}\alpha_tj_r:\Sigma^{2t(p-1)-2}SZ/p^r\to\Delta_t\Sigma^0$ and $(1\wedge\pi_{1,r+1})\tilde{\alpha}_{t,1}:\Sigma^{2t(p-1)}\to\Delta_tSZ/p^{r+1}$ respectively. Evidently $N^t_r=\Sigma^{2t(p-1)}DM^t_r$ where DX denotes the Spanier-Whitehead dual of X. For $t>0$ we consider the following commutative diagram:

$$\begin{array}{ccccc} & & \Sigma^{-1}SZ/p^r & = & \Sigma^{-1}SZ/p^r \\ & & \downarrow^{ij} & & \downarrow^{\varphi} \\ \Sigma^{2t(p-1)} & \xrightarrow{\tilde{\alpha}_{t,1}} & SZ/p & \to & C(\tilde{\alpha}_{t,1}) \\ \| & & \downarrow^{\pi} & & \downarrow \\ \Sigma^{2t(p-1)} & \to & SZ/p^{r+1} & \to & P_r^t\,. \end{array}$$

The map φ may be regarded as $\alpha_{-t,1}j_r:\Sigma^{-1}SZ/p^r\to C(\tilde{\alpha}_{t,1})$. Therefore P_r^t has the same K_*-local type as $\Sigma^{2t(p-1)+1}N_r^{-t}$ when $q\le r$ and $\Sigma^{2t(p-1)+1}\vee SZ/p^r$ when $q>r$. This relation still holds in the case of $t<0$ similarly. In the $t=0$ case $M_0^0=\Sigma^0$ and M_r^0 is defind as the cofiber of the map $\beta_r i_1:\Sigma^{-1}\to\Delta_0 SZ/p^r$ when $r\ge 1$. We may also define N_r^0 and P_r^0 by the equalities: $N_r^0=\Sigma^{-1}P_r^0=DM_r^0$.

Theorem 2.4. *Let n and m be integers such that $m-n=r(p-1)+s$ $(0\le s<p-1,$ $r\ge 0)$. Put $t=r-(n+s+1)(p^{r-2}+p^{r-3}+\cdots+1)$ and $l=n$ $(p^{r-2}+p^{r-3}+\cdots+1)$ where we understand $p^{r-2}+p^{r-3}+\cdots+1=0$ when $r\le 1$. The function $e(k,j)$ is defined by $e(k,j)=2kp^j-1$ when $j\ge 0$ and $e(k,-1)=2k-1$. Then*
i) *$L(p)_{2n+1}^{2m+1}$ has the same K_*-local type as the following spectrum:*

$$\Big(\bigvee_{1\le i\le p-1,\, i\ne s+1}\Sigma^{e(n+i,r(i))}SZ/p^{r(i)+1}\Big)\vee\Sigma^{e(n+s+1,r-1)}M_r^t \qquad \textit{when } m+1\not\equiv 0 \bmod p^r,$$

$$L(p)_{2n+1}^{2m}\vee\Sigma^{2m+1} \qquad \textit{when } m+1\equiv 0 \bmod p^r.$$

ii) *$L(p)_{2n}^{2m}$ has the same K_*-local type as*

$$\big(\vee_{i=1}^{p-2}\Sigma^{e(n+i,r(i))}SZ/p^{r(i)+1}\big)\vee\Sigma^{2n}N_r^l \qquad \textit{when } n\not\equiv 0 \bmod p^r,$$

$$L(p)_{2n+1}^{2m}\vee\Sigma^{2n} \qquad \textit{when } n\equiv 0 \bmod p^r.$$

Proof. i) Consider the following commutative diagram:

$$\begin{array}{ccccc} \Sigma^{2m} & \xrightarrow{f} & L(p)_{2n+1}^{2m} & \to & L(p)_{2n+1}^{2m+1} \\ \downarrow^{i} & & \| & & \downarrow \\ \Sigma^{2m}SZ/p & \xrightarrow{g} & L(p)_{2n+1}^{2m} & \to & L(p)_{2n+1}^{2m+2}\,. \end{array}$$

As is shown in the proof of Theorem 2.1 the bottom cofiber sequence is essentially given by the following cofiber sequence:

$$\Sigma^{2t(p-1)-1}SZ/p\xrightarrow{g_{s+1}}\Delta_t SZ/p^r\to\Sigma^{2(p-1)w}\Delta SZ/p^{r+1}$$

where $w=(n+s+1)p^{r-1}$. In the $t\neq 0$ case we set $q=v_p(t)+1$. Note that $m+1\equiv w \bmod p^r$ when $q>r$, $m+1\equiv w-t \bmod p^r$ when $q=r$, and $m+1\not\equiv 0 \bmod p^r$ when $q<r$ because $m+1=t(p-1)+w$. On the other hand, it is immediate that $r=2$, $n+s=1$ and hence $m+1=w=2p$ in the $t=0$ case. Since the cofiber $C(g_{s+1})$ has the same K_*-local type as $\Sigma^{2(p-1)w}SZ/p^{r+1}$, we can determine the form of g_{s+1} uniguely up to K_*-equivalence, by means of Propositions 2.2 and 2.3. In fact the map g_{s+1} is chosen as follows: $\tilde{\alpha}_{t,r}j$ if "$q>r$ and $w\equiv 0 \bmod p^r$" or "$q=r$ and $w\equiv t \bmod p^r$"; $\alpha_t\wedge\pi+u\tilde{\alpha}_{t,r}j$ if "$q>r$ and $w\not\equiv 0 \bmod p^r$" or "$q=r$ and $w\not\equiv t \bmod p^r$"; $i\tilde{\alpha}_1$ if "$q<r$ and $w\equiv 0 \bmod p^r$"; $i\tilde{\alpha}_1+u\tilde{\alpha}_{t,r}j$ if "$q<r$ and $w\not\equiv 0 \bmod p^r$"; and β_2+uij if "$t=0$" where $u\in Z/p$ is a suitable unit. Therefore the cofiber $C(g_{s+1}i)$ has the same K_*-local type as M_r^t when $m+1\not\equiv 0 \bmod p^r$, but it has the same K_*-local type as the wedge sum $SZ/p^r\vee\Sigma^{2t(p-1)}$ when $m+1\equiv 0 \bmod p^r$.

ii) Consider the following cofiber sequence

$$L(p)^{2m}_{2n}\to L(p)^{2m}_{2n+1}\xrightarrow{h}\Sigma^{2n+1}.$$

The dual map Dh has already been given in i), so our result is immediate.

REMARK. In the case ii) we may assert that $L(p)^{2m}_{2n}$ has the same K_*-local type as the wedge sum $\Sigma^{e(n,r-1)}P_r^{-l}\vee\bigvee_{i=1}^{p-2}\Sigma^{e(n+i,r(i))}SZ/p^{r(i)+1}$ in any cases.

Theorem 2.5. *Let $r,s,t,l,e(k,j)$ and $r(i)$ be the integers given in Theorem 2.4 which depend on m and n, and put $\tau=r+1-n(p^{r-1}+\cdots+1)$, $\lambda=n(p^{r-1}+\cdots+1)$. Then $L(p)^{2m+1}_{2n}$ has the same K_*-local type as the following specrum X:*

i) *When $m+1\equiv 0 \bmod p^r$, $X=L(p)^{2m}_{2n}\vee\Sigma^{2m+1}$.*

ii) *When $n\equiv 0 \bmod p^r$, $X=L(p)^{2m+1}_{2n+1}\vee\Sigma^{2n}$.*

iii) *When $m+1$, $n\not\equiv 0 \bmod p^r$ and $m-n+1\not\equiv 0 \bmod p-1$,*

$$X=\Big(\bigvee_{1\le i\le p-1,\, i\neq s+1}\Sigma^{e(n+i,r(i))}SZ/p^{r(i)+1}\Big)\vee\Sigma^{e(n+s+1,r-1)}M_r^t\vee\Sigma^{2n}N_r^l.$$

iv) *When $m+1$, $n\not\equiv 0 \bmod p^r$ and $m-n+1\equiv 0 \bmod p-1$,*

$$\Big(\bigvee_{1\le i\le p-2}\Sigma^{e(n+i,r(i))}SZ/p^{r(i)+1}\Big)\vee\Sigma^{e(n,r)}C(p^{a-1}i_{r+1}\alpha_\tau\vee u\tilde{\alpha}_{-\lambda,1}).$$

where $a=\min(v_p(\tau)+1,\ r+1)$ and $u\in Z/p$ is a suitable unit.

Proof. The cases i), ii) and iii) are immediately shown by use of Theorem 2.4. To show the case iv) we consider the following commutative diagram:

$$\begin{array}{ccccc} & & \Sigma^{2m} & = & \Sigma^{2m} \\ & & \downarrow f & & \downarrow \\ \Sigma^{2n-1} & \xrightarrow{g} & L(p)^{2m}_{2n-1} & \xrightarrow{i_g} & L(p)^{2m}_{2n} \\ \| & & \downarrow i_f & & \downarrow \\ \Sigma^{2n-1} & \rightarrow & L(p)^{2m+1}_{2n-1} & \rightarrow & L(p)^{2m+1}_{2n}. \end{array}$$

By Theorem 2.1 we may decompose $L(p)^{2m}_{2n-1}$ as the wedge sum $\bigvee_{i=1}^{p-2}\Sigma^{e(n+i,r)}SZ/p^{r+1}\vee\Sigma^{e(n,r)}SZ/p^{r+1}$. From Theorem 2.4 i) we can take map $f_{p-1}=p^{a-1}i\alpha_\tau:\Sigma^{2m}\to\Sigma^{2w-1}\Delta_\tau SZ/p^{r+1}$ with $2w-1=e(n,r)=2np^r-1$ because $\tau\neq 0$ in the case iv). Since $\Sigma^{2n}N_r^\lambda$ has the same K_*-local type as $\Sigma^{2w-1}P_r^{-\lambda}$ we may take $g_{p-1}=u(1\wedge\pi)\tilde{\alpha}_{-\lambda,1}$: $\Sigma^{2n-1}\to\Sigma^{2w-1}\Delta_{-\lambda}SZ/p^r$ for some unit $u\in Z/p$. Then the $(p-1)$-th component of $L(p)^{2m+1}_{2n}$ has the same K_*-local type as the cofiber of the map

$$p^{a-1}i\alpha_\tau\vee u(1\wedge\pi)\tilde{\alpha}_{-\lambda,1}:\Sigma^{2m}\vee\Sigma^{2n-1}\to\Sigma^{2w-1}\Delta_v SZ/p^{r+1}$$

after compositing suitable K_*-equivalences $\Delta_\tau\Sigma^0\to\Delta_v\Sigma^0$ and $\Delta_{-\lambda}\Sigma^0\to\Delta_v\Sigma^0$ for some integer v if necessary (cf. [14]).

REMARK. Recall that the J-group is given as the cokernel of ψ^k-1. Note that

$$J^{2t(p-1)}N_r^l\otimes Z_{(p)}\cong\begin{cases} Z/p^q\oplus Z/p^{min(q,r)} & \text{for } q<s \\ Z/p^{q-s+1+min(r,v)}\oplus Z/p^{s-1} & \text{for } q\geq s\end{cases}$$

where $q=v_p(t)+1$, $s=v_p(l)+1$ and $v=v_p(l-t)+1$ ($=s$ when $q>s$) and $J^iN_r^l\otimes Z_{(p)}=0$ for $i\neq 0$ mod $2(p-1)$. Applying Theorems 2.1 and 2.4 ii) we can compute $J^*L(p)^{2m}_n$ and hence $J^*L(p)^{2m+1}_n$ immediately although they have already been calculated in [9]. Note that the K_*-local type of $L(p)^{2m}_n$ is classified by the J-group $J^*L(p)^{2m}_n$ (cf. [3, Lemma 6.7]).

References

[1] J.F. Adams: *On the group $J(X)$-IV*, Topology **5** (1966) 21–71.

[2] A.K. Bousfield: *The localization of spectra with respect to homology*, Topology **18** (1979) 257–281.

[3] A.K. Bousfield: *On the homotopy theory of K-local spectra at an odd prime*, Amer. J. Math. **107** (1985) 895–932.

[4] A.K. Bousfield: *A Classification of K-local spectra*, J. Pure and Applied Algebra **66** (1990) 121–163.

[5] M. Imaoka, M. Sugawara: *On the K-Ring of the Orbit Manifold $(S^{2m+1}\times S^1)/D_n$ by the dihedral group D_n*, Hiroshima Math J. **4** (1974) 53–70.

[6] T. Kambe: *The structure of K_Λ-rings of the lens space and their applications*, J. Math. Soc. Japan **18** (1966) 135–146.

[7] S. Kôno: *Stable homotopy type of Thom spaces of canonical bundles over Orbit Manifolds*

$(S^{2m+1}\times S^l)/D_p$ *by the dihedral group* D_p, preprint.
[8] M. Kamata, H. Minami: *Bordism groups of dihidral groups*, J. Math. Soc. Japan **25** (1973) 334–341.
[9] S. Kôno, A. Tamamura: *J-groups of the suspensions of the stunted lens spaces* mod p, Osaka J. Math **24** (1987) 481–498.
[10] S. Kôno, A. Tamamura, M. Fujii: *J-group of the Orbit Manifolds* $(S^{2m+1}\times S^l)/D_n$ *by the dihedral group* D_n, Math J. Okayama Univ. **22** (1980) 205–221.
[11] N. Mahammed: *A propos de la K-théorie des espaces lenticularies*, C.R. Acad. Sc. Paris **271** (1970) 639–642.
[12] M. Mimura, G. Nishida, H.Toda: *Localization of CW-complexes and its applications*, J. Math. Soc. Japan **23** (1971) 593–624,
[13] Z. Yosimura: *The* K_**-localizations of the stunted real projective spaces*, J. Math. Kyoto Univ. **33** (1993) 523–541.
[14] Z. Yosimura: K_**-localizations of spectra with simple K-homology*, II, preprint.

Department of Mathematics
Osaka City University
Sugimoto, Sumiyoshi-ku
Osaka 558, Japan