Title	Chern characters on compact Lie groups of low rank
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Citation	Osaka Journal of Mathematics. 1985, 22(3), p. 463-488
Version Type	VoR
URL	https://doi.org/10.18910/10102
rights	
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Watanabe, T.
Osaka J. Math.
22 (1985), 463–488

CHERN CHARACTERS ON COMPACT LIE GROUPS OF LOW RANK

Dedicated to Professor Minoru Nakaoka on his sixtieth birthday

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(Received August 8, 1984)

0. Introduction

Let G be a compact, simply connected, simple Lie group of rank l. G has l irreducible representations $\rho_1, \cdots, \rho_l$, whose highest weights are the fundamental weights $\omega_1, \cdots, \omega_l$ respectively (see [19]). Then the representation ring $R(G)$ of G is a polynomial algebra $Z[\rho_1, \cdots, \rho_l]$. By the theorem of Hodgkin [16], the $Z/2$-graded K-theory $K^*(G)$ of G is an exterior algebra $\Lambda_Z(\beta(\rho_1), \cdots, \beta(\rho_l))$, where $\beta\colon R(G)\to K^*(G)$ is the map introduced in [16]. Therefore the Chern character $ch\colon K^*(G)\to H^*(G; Q)$ is injective [5]. We may write

$$H^*(G; Q) = \Lambda_Q(x_{2m_1-1}, x_{2m_2-1}, \cdots, x_{2m_l-1})$$

where $2=m_1\le m_2\le\cdots\le m_l$ and $\deg x_{2m_j-1}=2m_j-1$. If each x_{2m_j-1} is chosen to be integral and not divisible by any other integral classes, we can assign to a representation $\lambda\colon G\to U(n)$ the rational numbers $a(\lambda, 1), \cdots, a(\lambda, l)$ by the equation

$$ch\beta(\lambda) = \sum_{j=1}^{l} a(\lambda, j)x_{2m_j-1}\,.$$

In view of [21] and [23], the $a(\lambda, j)$ are closely related to the *Dynkin coefficients* of λ [14]. On the other hand, as is noted by Atiyah [4, Proposition 1], the determinant of the $l\times l$ matrix $(a(\rho_i, j))$ is equal to 1. We remark that for any system of generators $\{\lambda_1, \cdots, \lambda_l\}$ of the ring $R(G)$, the determinant of $(a(\lambda_i, j))$ is also 1.

In this paper, with a suitable system of generators of $R(G)$, we shall describe the resulting matrix explicitly for the groups G with $l\le 4$ without using the above informations. Indeed, we deal with the following cases:

$$\begin{array}{lllll} l=2\,, & G=\mathrm{SU}(3)\,, & & \mathrm{Sp}(2)\,, & G_2\,.\\ l=3\,, & G=\mathrm{SU}(4)\,, & \mathrm{Spin}\,(7)\,, & \mathrm{Sp}(3)\,. & \\ l=4\,, & G=\mathrm{SU}(5)\,, & \mathrm{Spin}\,(9)\,, & \mathrm{Sp}(4)\,,\ \mathrm{Spin}\,(8)\,, & F_4\,.\end{array}$$

Results are stated in Theorems 2 ($\mathrm{SU}(l+1)$), 3 ($\mathrm{Sp}(l)$), 4 ($\mathrm{Spin}(7)$), 5 ($\mathrm{Spin}(8)$), 6 ($\mathrm{Spin}(9)$), 7 (G_2) and 8 (F_4).

The careful reader should notice that "up to sign" is implicitly added to some of the statements of this paper.

For later use we fix some notations. Let T be a maximal torus of G. The inclusion i: $T\to G$ induces a map of classifying spaces $\rho=Bi$: $BT\to BG$. The action of the normalizer $N_G(T)$ on T induces that of the Weyl group $\Phi(G)=N_G(T)/T$ on BT and hence on $H^*(BT; Z)=Z[\omega_1, \cdots, \omega_l]$ (see [9]). Let $H^*(BT; Z)^{\Phi(G)}$ denote the module of $\Phi(G)$-invariants. For a based space X, let ΩX be its loop space, and let $\sigma^*: H^i(X; Z)\to H^{i-1}(\Omega X; Z)$ be the cohomology suspension. For the rational cohomology, by [8] and [10] we have

$$\begin{array}{l}
\mathrm{Im}\,\rho^* = H^*(BT;\, Q)^{\Phi(G)} = Q[f_{2m_1}, \cdots, f_{2m_l}] \\
\quad \cong \big\downarrow \\
H^*(BG;\, Q) = Q[y_{2m_1}, \cdots, y_{2m_l}] \\
\quad \sigma^* \big\downarrow \\
H^*(G;\, Q) = \Lambda_Q(x_{2m_1-1}, \cdots, x_{2m_l-1}) \\
\quad \sigma^* \big\downarrow \\
H^*(\Omega G;\, Q) = Q[u_{2m_1-2}, \cdots, u_{2m_l-2}]
\end{array}$$

where all the generators, whose degrees are indicated by a subscript, are chosen to be integral and not divisible by any other integral classes.

The paper is organized as follows. The key point of our work is to characterize the generator x_{2m_j-1}. For this purpose we present two methods in Section 1: in the first method we characterize the generator y_{2m_j} and relate it to x_{2m_j-1}; in the second method we characterize the generator u_{2m_j-2} and relate it to x_{2m_j-1}. Moreover in Section 1 we prove a lemma which is very useful if the λ-ring structure of $R(G)$ is known. Subsequent sections are devoted to practical computations. In Section 2 we treat the most elementary cases, i.e., $G=\mathrm{SU}(l+1)$, $\mathrm{Sp}(l)$ ($l=2, 3, 4$) where $H^*(G; Z)$ has no torsion. In Section 3 we consider the cases $G=\mathrm{Spin}(m)$ ($m=7, 8, 9$) where $H^*(G; Z)$ has only 2-torsion. In Section 4 we discuss the cases $G=G_2$ and $G=F_4$.

I would like to thank my colleague H. Minami for showing me a computation of $(a(\rho_i, j))$ for the case $G=G_2$ and many helpful suggestions.

1. Methods

Method I

For any group H let α: $R(H)\to K^*(BH)$ be the homomorphism of [5]. Let σ: $K^i(X)\to K^{i-1}(\Omega X)$ be the suspension map. Then there is a commutative diagram

$$
\begin{tikzcd}
R(T) \arrow[r, "\alpha"] & K^*(BT) \arrow[r, "ch"] & H^*(BT; Q) \\
R(G) \arrow[u, "i^*"] \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & K^*(BG) \arrow[u, "\rho^*"] \arrow[r, "ch"] \arrow[d, "\sigma"] & H^*(BG; Q) \arrow[u, "\rho^*"] \arrow[d, shift right, "\sigma^*"'] \\
 & K^*(G) \arrow[r, "ch"] & H^*(G; Q) \arrow[u, shift right, "\tau"'] \arrow[uu, bend right=60, "\tau'"']
\end{tikzcd}
$$

where τ(resp. τ') is the cohomology transgression in the Serre spec[illegible]nce of the universal fibration $G \to EG \to BG$ (resp. the fibration $G \to G/T \to BT$). For $j=1, \cdots, l$ we may set (modulo decomposables)

$$\sigma^*(y_{2m_j}) = b(m_j)x_{2m_j-1} \qquad \text{for some} \quad b(m_j) \in Z$$

and

$$\rho^*(y_{2m_j}) = c(m_j)f_{2m_j} \qquad \text{for some} \quad c(m_j) \in Z\,.$$

Since σ^* and τ are inverse to each other insofar as they are defined, it follows that

$$\tau'(x_{2m_j-1}) = \frac{c(m_j)}{b(m_j)} f_{2m_j} + \text{decomposables}$$
$$\text{in} \quad H^*(BT; Q)^{\Phi(G)} = Q[f_{2m_1}, \cdots, f_{2m_l}]\,.$$

Let $\lambda\colon G \to U(n)$ be a representation with weights $\mu_1, \cdots, \mu_n$. So

$$ch\alpha i^*(\lambda) = \sum_{i=1}^{n} \exp(\mu_j) = \sum_{m \geq 0} \sum_{i=1}^{n} \mu_i^m / m!$$

where $\mu_i \in H^2(BT; Z)$ (see [9]). Set

$$ch\beta(\lambda) = \sum_{j=1}^{l} a(\lambda, j)x_{2m_j-1} \qquad \textit{where} \quad a(\lambda, j) \in Q\,. \tag{1.1}$$

Apply τ' to this equation. Then the left hand side becomes

$$
\begin{aligned}
\tau' ch\beta(\lambda) &= \rho^* \tau ch \sigma \alpha(\lambda) \\
&= \rho^* \tau \sigma^* ch\alpha(\lambda) \\
&= \rho^* ch\alpha(\lambda) \\
&= ch\alpha i^*(\lambda)
\end{aligned}
$$

and the right hand side becomes

$$
\begin{aligned}
\tau'\Big(\sum_{j=1}^{l} a(\lambda, j)x_{2m_j-1}\Big) &= \sum_{j=1}^{l} a(\lambda, j)\tau'(x_{2m_j-1}) \\
&= \sum_{j=1}^{l} \frac{a(\lambda, j)c(m_j)}{b(m_j)} f_{2m_j} + \text{decomposables.}
\end{aligned}
$$

Hence

$$ch\alpha i^*(\lambda) = \sum_{j=1}^{l} \frac{a(\lambda, j)c(m_j)}{b(m_j)} f_{2m_j} + \text{decomposables}.$$

This argument shows that, in order to compute $a(\lambda, j)$, it suffices to settle f_{2m_j}, determine $b(m_j)$, $c(m_j)$ and find the coefficients of f_{2m_j} in the expression of $ch\alpha i^*(\lambda)$ as a polynomial of the f_{2m_j}. We will use this method in all cases that concern us.

REMARK. In general we choose the f_{2m_j} as follows. Let $\{f'_{2m_1}, \cdots, f'_{2m_l}\}$ be a system of generators of the ring $H^*(BT; Q)^{\Phi(G)}$. First we take

$$f_{2m_1} = b_1 f'_{2m_1} \in H^{2m_1}(BT; Q)^{\Phi(G)}, \quad b_1 \in Q,$$

so that

(i) f_{2m_1} is integral;

(ii) for any $b \in Q$ with $|b| < |b_1|$, bf'_{2m_1} cannot be integral.

Assume inductively that we have chosen $f_{2m_1}, \cdots, f_{2m_{j-1}}$. Then we take

$$f_{2m_j} = b_j f'_{2m_j} + \text{decomposables} \in H^{2m_j}(BT; Q)^{\Phi(G)}, \quad b_j \in Q,$$

so that

(i) f_{2m_j} is integral;

(ii) for any $b \in Q$ with $|b| < |b_j|$, bf'_{2m_j}+decomposables $\in H^{2m_j}(BT; Q)^{\Phi(G)}$ cannot be integral.

Note that the choice of the f'_{2m_j} has no crucial influence on that of the f_{2m_j}. As will be seen in Sections 3 and 4, this settlement of the f_{2m_j} is not trivial but important.

Method II

There is a commutative diagram

$$\begin{array}{ccccc} R(G) & \xrightarrow{\beta} & K^*(G) & \xrightarrow{ch} & H^*(G; Q) \\ & & \sigma \downarrow & & \sigma^* \downarrow \\ & & K^*(\Omega G) & \xrightarrow{ch} & H^*(\Omega G; Q) \end{array}$$

which is natural with respect to group homomorphisms. For $j=1, \cdots, l$ we may set

$$\sigma^*(x_{2m_j-1}) = d(m_j)u_{2m_j-2} \qquad \text{for some} \quad d(m_j) \in Z.$$

Applying σ^* to (1.1), we have

$$ch\sigma\beta(\lambda) = \sum_{j=1}^{l} a(\lambda, j)d(m_j)u_{2m_j-2}.$$

Let us now consider the case $G=\mathrm{SU}(n+1)$; then $m_j=j+1$ for $j=1, \cdots, n$ and

$$PH^*(\Omega SU(n+1); Z) = Z\{u_{2i} \mid 1 \le i \le n\}$$

where P denotes the primitive module functor. Furthermore, $d(j+1)=1$ for all j (e.g., see [28, Lemma 3]). Let λ_1: $SU(n+1) \to U(n+1)$ be the natural inclusion, and consider the case $\lambda=\lambda_1$. Then it follows from (2.2) of the next secti on that

$$ch\sigma\beta(\lambda_1) = \sum_{i=1}^{n} \frac{(-1)^i}{i!} u_{2i} . \tag{1.2}$$

We return to the general case. Take the inclusion k: $U(n) \to SU(n+1)$ such that $SU(n+1)/U(n)=CP^n$ (see [12, §3]). In [28] it was shown that for the composite

$$PH^*(\Omega SU(n+1); Z) \xrightarrow{(\Omega k)^*} PH^*(\Omega U(n); Z)$$
$$\xrightarrow{(\Omega\lambda)^*} PH^*(\Omega G; Z) = Z\{u_{2m_1-2}, \cdots, u_{2m_l-2}\} ,$$

the following statements are equivalent:

(i) $(\Omega\lambda)^*(\Omega k)^*(u_{2m_j-2})=e(\lambda, j)u_{2m_j-2}$ for some $e(\lambda, j) \in Z$;

(ii) the element $\theta_s(c_{m_j}(\lambda)) \in H^{2m_j-2}(G/C_s; Z)$ is exactly divisible by $e(\lambda, j) \in Z$ (where $H^*(G/C_s; Z)$ has no torsion; for notations and details see [28, §2]).

Applying $(\Omega\lambda^*)(\Omega k)^*$ to (1.2), we have

$$ch\sigma\beta(\lambda) = \sum_{j=1}^{l} \frac{(-1)^{m_j-1}e(\lambda, j)}{(m_j-1)!} u_{2m_j-2} .$$

Hence

$$a(\lambda, j)d(m_j) = \frac{(-1)^{m_j-1}e(\lambda, j)}{(m_j-1)!} .$$

This argument shows that, in order to compute $a(\lambda, j)$, it suffices to determine $d(m_j)$ and $e(\lambda, j)$. In particular, to find $e(\lambda, j)$ one must examine the divisibility of $\theta_s(c_{m_j}(\lambda))$ in $H^{2m_j-2}(G/C_s; Z)$.

Define a map φ: $Z_+ \times Z_+ \times Z_+ \to Z$ by

$$\varphi(n, k, q) = \sum_{i=1}^{k} (-1)^{i-1}\binom{n}{k-i} i^{q-1}$$

where Z_+ denotes the set of positive integers and we use the convention that $\binom{x}{y}=0$ if $y<0$ or $x<y$. Let Λ^k: $R(G) \to R(G)$ be the k-th exterior power operation. Then we have

Lemma 1. *If λ is a representation of G of dimension n, then*

$$a(\Lambda^k\lambda, j) = \varphi(n, k, m_j)a(\lambda, j)$$

for $j=1, \cdots, l$.

Proof. Let ch^q be the $2q$-th component of ch, i.e., $ch(x)=\sum_{q\geq 0} ch^q(x)$ with $ch^q(x)\in H^{2q}(X; Q)$ for any $x\in K^0(X)$. Consider the element $1_n\in R(U(n))$ which comes from the identity $1_{U(n)}\colon U(n)\to U(n)$. Then we assert that

$$(1.3)\qquad \begin{aligned} &ch^q\alpha(\Lambda^k 1_n) = \varphi(n, k, q)ch^q\alpha(1_n)+\textit{decomposables} \\ &\quad \textit{in}\quad H^*(BU(n); Q) = Q[y_2, y_4, \cdots, y_{2n}]\,. \end{aligned}$$

This assertion implies the result. For since $\beta=\sigma\alpha$ and σ^* sends a decomposable element into zero, applying σ^* to (1.3) yields the desired result for the case $G=U(n)$. Then the general case follows from naturality.

To prove (1.3) we proceed by induction on k. The case $k=1$ is clear. Suppose that it is true for $k\leq m-1$, and consider the case $k=m$. Let us recall the following relations:

$$\psi^k(x)+\sum_{i=1}^{k-1}(-1)^i\psi^{k-i}(x)\Lambda^i(x)+(-1)^k k\Lambda^k(x) = 0;$$

$$ch^q(xy) = \sum_{r=0}^{q} ch^r(x)ch^{q-r}(y);$$

$$ch^q\psi^k(x) = k^q ch^q(x)$$

where $x, y\in K^0(X)$ [1]. Since α is a λ-ring homomorphism, we have

$$\begin{aligned}
&ch^q\alpha(m\Lambda^m(1_n)) \\
&= ch^q\alpha((-1)^{m-1}\psi^m(1_n)+\sum_{i=1}^{m-1}(-1)^{m-1-i}\psi^{m-i}(1_n)\Lambda^i(1_n)) \\
&= (-1)^{m-1}ch^q\alpha\psi^m(1_n)+\sum_{i=1}^{m-1}(-1)^{m-1-i}ch^q(\alpha\psi^{m-i}(1_n)\alpha\Lambda^i(1_n)) \\
&= (-1)^{m-1}ch^q\alpha\psi^m(1_n)+\sum_{i=1}^{m-1}(-1)^{m-1-i}[\sum_{r=0}^{q} ch^r\alpha\psi^{m-i}(1_n)ch^{q-r}\alpha\Lambda^i(1_n)] \\
&= (-1)^{m-1}ch^q\alpha\psi^m(1_n)+\sum_{i=1}^{m-1}(-1)^{m-1-i}\left[\binom{n}{i}ch^q\alpha\psi^{m-i}(1_n)+nch^q\alpha\Lambda^i(1_n)\right] \\
&\quad \text{modulo decomposables} \\
&= (-1)^{m-1}ch^q\psi^m\alpha(1_n)+\sum_{i=1}^{m-1}(-1)^{m-1-i}\left[\binom{n}{i}ch^q\psi^{m-i}\alpha(1_n)+nch^q\alpha(\Lambda^i 1_n)\right] \\
&= (-1)^{m-1}m^q ch^q\alpha(1_n)+\sum_{i=1}^{m-1}(-1)^{m-1-i}\left[\binom{n}{i}(m-i)^q ch^q\alpha(1_n)\right. \\
&\quad \left.+n\varphi(n, i, q)ch^q\alpha(1_n)\right] \\
&= [\sum_{i=0}^{m-1}(-1)^{m-1-i}\binom{n}{i}(m-i)^q+\sum_{i=1}^{m-1}(-1)^{m-1-i}n\varphi(n, i, q)]ch^q\alpha(1_n)
\end{aligned}$$

$$=\Big[\sum_{j=1}^{m}(-1)^{j-1}\binom{n}{m-j}j^q+n\sum_{i=1}^{m-1}(-1)^{m-1-i}\varphi(n,i,q)\Big]ch^q\alpha(1_n)\,.$$

Thus it is sufficient to prove that

$$\varphi(n,m,q+1)+n\sum_{i=1}^{m-1}(-1)^{m-1-i}\varphi(n,i,q)=m\varphi(n,m,q)\,. \tag{1.4}$$

From Pascal's triangle

$$\binom{n}{i}=\binom{n-1}{i}+\binom{n-1}{i-1}$$

we deduce that

$$\sum_{i=0}^{k-1-j}(-1)^i\binom{n}{i}=(-1)^{k-1-j}\binom{n-1}{k-1-j}\,.$$

Using this, we have

$$\begin{aligned}\varphi(n-1,m-1,q)&=\sum_{j=1}^{m-1}(-1)^{j-1}\binom{n-1}{m-1-j}j^{q-1}\\&=\sum_{j=1}^{m-1}\Big[(-1)^m\sum_{i=0}^{m-1-j}(-1)^i\binom{n}{i}\Big]j^{q-1}\\&=\sum_{i=1}^{m-1}(-1)^{m-1-i}\Big[\sum_{j=1}^{i}(-1)^{j-1}\binom{n}{i-j}j^{q-1}\Big]\\&=\sum_{i=1}^{m-1}(-1)^{m-1-i}\varphi(n,i,q)\,.\end{aligned}$$

Therefore

$$\begin{aligned}&n\varphi(n-1,m-1,q)+\varphi(n,m,q+1)\\&=n\sum_{j=1}^{m-1}(-1)^{j-1}\binom{n-1}{m-1-j}j^{q-1}+\sum_{j=1}^{m}(-1)^{j-1}\binom{n}{m-j}j^q\\&=\sum_{j=1}^{m-1}(-1)^{j-1}n\binom{n-1}{m-1-j}j^{q-1}+\sum_{j=1}^{m}(-1)^{j-1}\binom{n}{m-j}j^q\\&=\sum_{j=1}^{m-1}(-1)^{j-1}\binom{n}{m-j}(m-j)j^{q-1}+\sum_{j=1}^{m}(-1)^{j-1}\binom{n}{m-j}j^q\\&=\sum_{j=1}^{m-1}(-1)^{j-1}\binom{n}{m-j}mj^{q-1}-\sum_{j=1}^{m-1}(-1)^{j-1}\binom{n}{m-j}j^q\\&\quad+\sum_{j=1}^{m}(-1)^{j-1}\binom{n}{m-j}j^q\\&=m\sum_{j=1}^{m-1}(-1)^{j-1}\binom{n}{m-j}j^{q-1}+(-1)^{m-1}\binom{n}{0}m^q\\&=m\sum_{j=1}^{m}(-1)^{j-1}\binom{n}{m-j}j^{q-1}\\&=m\varphi(n,m,q)\,.\end{aligned}$$

This proves (1.4) and completes the proof.

2. The special unitary groups and the symplectic groups

Let us first consider the case of $SU(l+1)$. In this case, $m_j=j+1$ for $j=1, \cdots, l$. As is well known we can choose elements $t_1, t_2, \cdots, t_{l+1} \in H^2(BT; Z)$ so that

$$H^*(BT; Z) = Z[t_1, \cdots, t_{l+1}]/(c_1)$$

and

$$H^*(BT; Z)^{\Phi(SU(l+1))} = Z[c_2, \cdots, c_{l+1}]$$

where $c_i=\sigma_i(t_1, \cdots, t_{l+1})$ ($\sigma_i(\ )$ denotes the i-th elementary symmetric function). It is evident that $f_{2j+2}=c_{j+1}$ for $j=1, \cdots, l$. Since $H^*(SU(l+1); Z)$ has no torsion, the theorem of Borel [6] assures us that $b(j+1)=c(j+1)=1$ for all j. Thus we have $\tau'(x_{2j+1})=c_{j+1}$ for $j=1, \cdots, l$.

Let us recall from [17] that

(2.1) $R(SU(l+1)) = Z[\lambda_1, \lambda_2, \cdots, \lambda_l]$ *where*

(a) $\dim \lambda_k = \binom{l+1}{k}$;

(b) *relations* $\Lambda^k\lambda_1=\lambda_k$ *hold*;

(c) *the set of weights of* λ_1 *is given by* $\{t_i \mid 1\leq i\leq l+1\}$.

Put

$$s_m = s_m(t_1, \cdots, t_{l+1}) = \sum_{i=1}^{l+1} t_i^m .$$

From Newton's formula

$$s_m+\sum_{i=1}^{m-1}(-1)^i s_{m-i}c_i+(-1)^m mc_m = 0$$

(where $c_m=0$ if $m>l+1$) it follows that

$$ch\alpha i^*(\lambda_1) = l+1+\sum_{m=1}^{l}\frac{(-1)^m}{m!}c_{m+1}+\text{decomposables}.$$

Therefore

$$ch\beta(\lambda_1) = \sum_{m=1}^{l}\frac{(-1)^m}{m!}x_{2m+1} \tag{2.2}$$

(cf. [20, Theorem 1]). By Lemma 1, if we evaluate $\varphi(l+1, k, j+1)$, $ch\beta(\lambda_k)$ can be calculated. Thus we have

Theorem 2. *The Chern characters on* $SU(l+1)$ *for* $l=2,3,4$ *are given by*:

$l=2$ $\quad ch\beta(\lambda_1) = -x_3+(1/2!)x_5$ 1

$\qquad\quad ch\beta(\lambda_2) = -x_3+(-1/2!)x_5$

$$
\begin{array}{lll}
l=3 & ch\beta(\lambda_1) = -x_3+(1/2!)x_5+(-1/3!)x_7 & \\
 & ch\beta(\lambda_2) = -2x_3 \qquad\qquad +(4/3!)x_7 & -1 \\
 & ch\beta(\lambda_3) = -x_3+(-1/2!)x_5+(-1/3!)x_7 & \\
l=4 & ch\beta(\lambda_1) = -x_3+(1/2!)x_5+(-1/3!)x_7+(1/4!)x_9 & \\
 & ch\beta(\lambda_2) = -3x_3+(1/2!)x_5+(3/3!)x_7+(-11/4!)x_9 & \\
 & ch\beta(\lambda_3) = -3x_3+(-1/2!)x_5+(3/3!)x_7+(11/4!)x_9 & 1 \\
 & ch\beta(\lambda_4) = -x_3+(-1/2!)x_5+(-1/3!)x_7+(-1/4!)x_9 &
\end{array}
$$

where the number on the right hand side indicates the determinant of the corresponding matrix on the left hand side.

Let us consider the case of $\mathrm{Sp}(l)$. In this case, $m_j=2j$ for $j=1, \cdots, l$. We can choose elements $t_1, t_2, \cdots, t_l \in H^2(BT; Z)$ so that

$$H^*(BT; Z) = Z[t_1, \cdots, t_l]$$

and

$$H^*(BT; Z)^{\Phi(Sp(l))} = Z[q_1, \cdots, q_l]$$

where $q_i=\sigma_i(t_1^2, \cdots, t_l^2)$. It is evident that $f_{4j}=q_j$ for $j=1, \cdots, l$. Since $H^*(Sp(l); Z)$ has no torsion, it follows that $b(2j)=c(2j)=1$ for all j. Thus we have $\tau'(x_{4j-1})=q_j$ for $j=1, \cdots, l$.

Let us recall that

(2.3) $R(Sp(l))=Z[\lambda_1, \lambda_2, \cdots, \lambda_l]$ *where*

(a) $\dim \lambda_k=\binom{2l}{k}$;

(b) *relations* $\Lambda^k\lambda_1=\lambda_k$ *hold*;

(c) *the set of weights of* λ_1 *is given by* $\{\pm t_i \mid 1\leq i\leq l\}$.

Put

$$s_{2m}=s_m(t_1^2, \cdots, t_l^2)=\sum_{i=1}^{l} t_i^{2m}.$$

From Newton's formula

$$s_{2m}+\sum_{i=1}^{m-1}(-1)^i s_{2m-2i}q_i+(-1)^m m q_m=0$$

it follows that

$$ch\alpha i^*(\lambda_1)=2l+\sum_{m=1}^{l}\frac{(-1)^{m-1}}{(2m-1)!}q_m+\text{decomposables}.$$

Therefore

$$ch\beta(\lambda_1)=\sum_{m=1}^{l}\frac{(-1)^{m-1}}{(2m-1)!}x_{4m-1}$$

and by Lemma 1 we obtain

Theorem 3. *The Chern characters on $Sp(l)$ for $l=2,3,4$ are given by:*

$$\begin{array}{lll}
l=2 & ch\beta(\lambda_1) = x_3+(-1/3!)x_7 & \\
 & ch\beta(\lambda_2) = 2x_3+(4/3!)x_7 & 1 \\
l=3 & ch\beta(\lambda_1) = x_3+(-1/3!)x_7+(1/5!)x_{11} & \\
 & ch\beta(\lambda_2) = 4x_3+(2/3!)x_7+(-26/5!)x_{11} & 1 \\
 & ch\beta(\lambda_3) = 6x_3+(6/3!)x_7+(66/5!)x_{11} & \\
l=4 & ch\beta(\lambda_1) = x_3+(-1/3!)x_7+(1/5!)x_{11}+(-1/7!)x_{15} & \\
 & ch\beta(\lambda_2) = 6x_3 \qquad\qquad +(-24/5!)x_{11}+(120/7!)x_{15} & \\
 & ch\beta(\lambda_3) = 15x_3+(9/3!)x_7+(15/5!)x_{11}+(-1191/7!)x_{15} & 1 \\
 & ch\beta(\lambda_4) = 20x_3+(16/3!)x_7+(80/5!)x_{11}+(2416/7!)x_{15} &
\end{array}$$

where the number on the right hand side indicates the determinant of the corresponding matrix on the left hand side.

3. The spinor groups

Let us first consider the case of Spin(7). In this case, $(m_1, m_2, m_3)=(2,4,6)$. We can choose elements $t_1, t_2, t_3, \gamma \in H^2(BT; Z)$ so that

$$H^*(BT; Z) = Z[t_1, t_2, t_3, \gamma]/(c_1-2\gamma)$$

and

$$H^*(BT; Q)^{\Phi(\mathrm{Spin}\,(7))} = Q[p_1, p_2, p_3]$$

where $c_i=\sigma_i(t_1, t_2, t_3)$ and $p_i=\sigma_i(t_1^2, t_2^2, t_3^2)$. In the light of the Remark in Section 1, using the formula

$$p_i = \sum_{j=0}^{2i} (-1)^{i+j} c_{2i-j} c_j\,,$$

we have

$$\begin{aligned}
(3.1) \qquad f_4 &= \frac{1}{2} p_1 = -c_2+2\gamma^2\,, \\
f_8 &= \frac{1}{4} p_2 - \frac{1}{4} f_4^2 = -c_3\gamma+c_2\gamma^2-\gamma^4\,, \\
f_{12} &= p_3 = c_3^2\,.
\end{aligned}$$

Let us determine $b(2)$, $b(4)$, $b(6) \in Z$. To do so we use the Serre spectral sequence $\{E_r(Z)\}$ for the integral cohomology of the universal fibration

$$F = \mathrm{Spin}\,(7) \to E = E\,\mathrm{Spin}\,(7) \to B = B\,\mathrm{Spin}\,(7)\,.$$

Furthermore, to investigate it, we use the Serre spectral sequence $\{E_r(Z/p)\}$ for the mod p cohomology of the same fibration, where p runs over all primes.

Recall that $H^*(\mathrm{Spin}(7); Z)$ has no p-torsion for $p>2$. Let $\Delta_{Z/2}(\ )$ denote a $Z/2$-algebra having a set in parentheses as a simple system of generators. Then it follows from [6] and [7] that

$$H^*(\mathrm{Spin}\,(7);\, Z/p) = \begin{cases} \Delta_{Z/2}(\bar{x}_3,\, \bar{x}_5,\, \bar{x}_6,\, \bar{x}_7) & (p=2) \\ \Lambda_{Z/p}(\bar{x}_3,\, \bar{x}_7,\, \bar{x}_{11}) & (p>2) \end{cases}$$

and

$$H^*(B\,\mathrm{Spin}(7);\, Z/p) = \begin{cases} Z/2[\bar{y}_4,\, \bar{y}_6,\, \bar{y}_7,\, \bar{y}_8] & (p=2) \\ Z/p[\bar{y}_4,\, \bar{y}_8,\, \bar{y}_{12}] & (p>2) \end{cases}$$

where $\bar{x}_i$ transgresses to $\bar{y}_{i+1}$ for all i and $\beta_2(\bar{x}_5)=\bar{x}_6$ (β_p denotes the mod p Bockstein homomorphism). For a based space X, let $\pi_p\colon H^i(X; Z)\to H^i(X; Z/p)$ be the mod p reduction homomorphism. Then if $i=3$ or 7, $\pi_p(x_i)=\bar{x}_i$ and $\pi_p(y_{i+1})=\bar{y}_{i+1}$ for every prime p. Therefore we conclude that $\tau(x_3)=y_4$ and $\tau(x_7)=y_8$. In other words, $b(2)=b(4)=1$.

It remains to determine $b(6)$. Since

$$\pi_p(x_{11}) = \begin{cases} \bar{x}_5\bar{x}_6 & (p=2) \\ \bar{x}_{11} & (p>2) \end{cases} \quad\text{and}\quad \pi_p(y_{12}) = \begin{cases} \bar{y}_6^2 & (p=2) \\ \bar{y}_{12} & (p>2)\,, \end{cases}$$

an analogous argument to the above yields that

(0) if $p>2$, $\nu_p(b(6))=0$

where $\nu_p(m)$ is the power of p in m. To get $\nu_2(b(6))$ we consider $\{E_r(Z/2)\}$, which satisfies

$$E_2^{s,t}(Z/2) \cong H^s(B;\, Z/2)\otimes H^t(F;\, Z/2)$$

and $E_\infty^{s,t}(Z/2)=0$ unless $(s, t)=(0, 0)$. Then it is easy to see that

(i) $d_6(1\otimes\bar{x}_5\bar{x}_6)=\bar{y}_6\otimes\bar{x}_6$.

(ii) $d_6(\bar{y}_6\otimes\bar{x}_5)=\bar{y}_6^2\otimes 1$.

Let

$$\beta_2^F\colon E_1^{s,t}(Z/2) \to E_1^{s,t+1}(Z/2)$$

be the map induced by $\beta_2\colon H^t(F; Z/2)\to H^{t+1}(F; Z/2)$ through the isomorphism

$$E_1^{s,t}(Z/2) \cong C^s(B;\, H^t(F;\, Z/2))\,.$$

Then we have

(iii) $\beta_2^F(\bar{y}_6\otimes\bar{x}_5)=\bar{y}_6\otimes\bar{x}_6$.

Denote again by $\pi_p\colon \{E_r(Z)\}\to\{E_r(Z/p)\}$ the morphism of spectral sequences induced by π_p. By virtue of the isomorphism

$$E_2^{s,t}(Z) \cong H^s(B; H^t(F; Z)),$$

we find that there exist elements $\{x_{11}\} \in E_2^{0,11}(Z)$, $\{v_{12}\} \in E_2^{6,6}(Z)$ and $\{y_{12}\} \in E_2^{12,0}(Z)$ which satisfy $\pi_2(\{x_{11}\}) = 1 \otimes \bar{x}_5 \bar{x}_6$, $\pi_2(\{v_{12}\}) = \bar{y}_6 \otimes \bar{x}_6$ and $\pi_2(\{y_{12}\}) = \bar{y}_6^2 \otimes 1$ respectively. Then the conditions (0), (i), (ii), (iii) imply that in $\{E_r(Z)\}$

(iv) $d_6(\{x_{11}\}) = \{v_{12}\}$.

(v) $d_{12}(\{2x_{11}\}) = \{y_{12}\}$.

In fact, (iv) is an immediate consequence of (i). In what follows we roughly state a proof of (v). Let us begin by recalling the construction of the Serre spectral sequence $\{E_r(R)\}$ in cohomology with R-coefficients of a fibration $F \to E \to B$, where $R = Z$ or Z/p (for details see [24]). There is a cochain complex $\mathrm{Hom}(C_*(E), R)$ which is filtered by its subcomplexes $A^s(R) = \sum_t A^{s,t}(R)$ such that $A^{s,t}(R) \subset A^{s-1,t+1}(R)$ and $\delta(A^{s,t}(R)) \subset A^{s,t+1}(R)$ for all (s, t) (where δ is the differential in $\mathrm{Hom}(C_*(E), R)$). This filtered cochain complex gives rise to $\{E_r(R)\}$, i.e.,

$$\begin{aligned} Z_r^{s,t}(R) &= A^{s,t}(R) \cap \delta^{-1}(A^{s+r,t-r+1}(R)), \\ B_r^{s,t}(R) &= A^{s,t}(R) \cap \delta A^{s-r,t+r-1}(R), \\ E_r^{s,t}(R) &= Z_r^{s,t}(R)/(Z_{r-1}^{s+1,t-1}(R) + B_{r-1}^{s,t}(R)). \end{aligned}$$

Note that there is an exact sequence

$$0 \to A^{s,t}(Z) \xrightarrow{\cdot p} A^{s,t}(Z) \xrightarrow{\pi_p} A^{s,t}(Z/p) \to 0$$

for all (s, t). Since $d_r\colon E_r^{s,t}(R) \to E_r^{s+r,t-r+1}(R)$ is induced by δ, by (iv) we see that there exists a representative $x \in A^{0,11}(Z)$ (resp. $v \in A^{6,6}(Z)$) of $\{x_{11}\}$ (resp. $\{v_{12}\}$) such that

$$\delta(x) = v. \tag{3.2}$$

Let $\bar{u} \in A^{6,5}(Z/2)$ be a representative of $\bar{y}_6 \otimes \bar{x}_5$. Then by (iii) we observe that there exists $u \in A^{6,5}(Z)$ such that $\pi_2(u) = \bar{u}$ and

$$\delta(u) = 2v \tag{3.3}$$

(see [2, Chapter III, §2]). Similarly by (ii) there is a representative $\bar{y} \in A^{12,0}(Z/2)$ of $\bar{y}_6^2 \otimes 1$ such that $\delta(\bar{u}) = \bar{y}$. This implies that there exists a representative $y \in A^{12,0}(Z)$ of $\{y_{12}\}$ such that $\pi_2(y) = \bar{y}$ and

$$\delta(u) = y. \tag{3.4}$$

By (3.2), (3.3) and (3.4), we have

$$\delta(2x) = 2v = \delta(u) = y$$

which gives (v). It is equivalent to $b(6) = 2$.

We discuss the problem of determining $c(2)$, $c(4)$, $c(6)\in Z$ in a general form. Indeed, we claim that $c(m_j)=1$ for $j=1, \cdots, l$ in all cases that concern us. To prove this we use the integral cohomology spectral sequence $\{E_r\}$ of the fibration

$$G/T \to BT \xrightarrow{\rho} BG .$$

Then the homomorphism $\rho^*: H^m(BG; Z)\to H^m(BT; Z)$ can be regarded as the composite

$$H^m(BG; Z) = E_2^{m,0} \twoheadrightarrow E_\infty^{m,0} = D^{m,0}\subset\cdots\subset D^{0,m} = H^m(BT; Z)$$

where $D^{i,m-i}/D^{i+1,m-i-1}=E_\infty^{i,m-i}$. According to [6], the class $\{y_{2m_j}\}\in E_2^{2m_j,0}$ survives to E_∞. What we have to verify is to observe that no extension problems occur on the class $\{y_{2m_j}\}\in E_\infty^{2m_j,0}$. This is an essentially easy work, because all structures of $H^*(G/T; Z)$, $H^*(BT; Z)$ and $H^*(BG; Z)$ were explicitly described (for $H^*(BG; Z)$ see [7] and [25]; for $H^*(G/T; Z)$ see [27] and also [26]). For example, consider the case $G=\mathrm{Spin}(7)$. Then it is not hard to see that if $m=4$, 8 or 12, $E_\infty^{i,m-i}$ is trivial or torsion free for all i. This assures us that $c(2)=c(4)=c(6)=1$. In the future we omit such checks for the other cases, for our claim (except for the case $G=F_4$) has been proved in a more general setting by [13] and [15].

Let us recall from [17] that

(3.5) $R(\mathrm{Spin}(7)) = Z[\lambda_1', \lambda_2', \Delta_7]$ *where*

(a) $\dim \lambda_k'=\binom{7}{k}$ *and* $\dim \Delta_7=8$;

(b) *relations* $\Lambda^k\lambda_1'=\lambda_k'$ *and* $\Delta_7^2=\lambda_3'+\lambda_2'+\lambda_1'+1$ *hold*;

(c) *the set of weights of* λ_1' *is given by* $\{\pm t_i, 0\,|\,1\leq i\leq 3\}$.

By the same calculation as in the case of $Sp(l)$, we have

$$ch^2\alpha i^*(\lambda_1') = p_1 ,$$

$$ch^4\alpha i^*(\lambda_1') = -\frac{1}{6}p_2+\text{decomposables},$$

$$ch^6\alpha i^*(\lambda_1') = \frac{1}{120}p_3+\text{decomposables}.$$

On the other hand, from (3.1) and the results on $b(m_j)$ and $c(m_j)$ it follows that

$$\tau'(x_3) = f_4 = \frac{1}{2}p_1 ,$$

$$\tau'(x_7) = f_8 = \frac{1}{4}p_2+\text{decomposables},$$

$$\tau'(x_{11}) = \frac{1}{2}f_{12} = \frac{1}{2}p_3 .$$

Combining these, we have

$$ch\beta(\lambda_1') = 2x_3 - \frac{2}{3}x_7 + \frac{1}{60}x_{11} .$$

Therefore by Lemma 1,

$$ch\beta(\lambda_2') = 10x_3 + \frac{2}{3}x_7 - \frac{5}{12}x_{11}$$

and

$$ch\beta(\lambda_3' + \lambda_2' + \lambda_1' + 1) = 32x_3 + \frac{16}{3}x_7 + \frac{4}{15}x_{11} .$$

On the other hand, by the formula (2) of [16, p. 8],

$$\beta(\Delta_7^2) = 8\beta(\Delta_7) + 8\beta(\Delta_7) = 16\beta(\Delta_7) .$$

Thus from the relation $\Delta_7^2 = \lambda_3' + \lambda_2' + \lambda_1' + 1$ we deduce that

$$ch\beta(\Delta_7) = 2x_3 + \frac{1}{3}x_7 + \frac{1}{60}x_{11} .$$

Theorem 4. *The Chern characters on Spin(7) are given by*:

$$\begin{aligned} ch\beta(\lambda_1') &= 2x_3 + (-4/3!)x_7 + (2/5!)x_{11} \\ ch\beta(\lambda_2') &= 10x_3 + (4/3!)x_7 + (-50/5!)x_{11} \\ ch\beta(\Delta_7) &= 2x_3 + (2/3!)x_7 + (2/5!)x_{11} \end{aligned}$$

and the determinant of the corresponding matrix is 1.

Let us next consider the case of Spin(8). In this case, $(m_1, m_2, m_3, m_4) = (2, 4, 4, 6)$. We can choose elements $t_1, t_2, t_3, t_4, \gamma \in H^2(BT; Z)$ so that

$$H^*(BT; Z) = Z[t_1, \cdots, t_4, \gamma]/(c_1 - 2\gamma)$$

and

$$H^*(BT; Q)^{\Phi(\mathrm{Spin}(8))} = Q[p_1, c_4, p_2, p_3]$$

where $c_i = \sigma_i(t_1, \cdots, t_4)$ and $p_i = \sigma_i(t_1^2, \cdots, t_4^2)$. By a similar calculation to the before, we have

$$f_4 = \frac{1}{2}p_1 = -c_2 + 2\gamma^2 ,$$

$$f_8' = c_4 ,$$

$$f_8 = \frac{1}{4}p_2 - \frac{1}{2}f_8' - \frac{1}{4}f_4^2 = -c_3\gamma + c_2\gamma^2 - \gamma^4 ,$$

$$f_{12}=p_3=-2c_4c_2+c_3^2\,.$$

Let us determine $b(2)$, $b(4)'$, $b(4)$, $b(6)\in Z$. But, since $H^*(\mathrm{Spin}(8);Z)$ has no p-torsion for $p>2$ and

$$H^*(\mathrm{Spin}\,(8);\,Z/2)=\Delta_{Z/2}(x_3,\,x_5,\,x_6,\,x_7',\,x_7)$$

where all the x_i are universally transgressive and $\beta_2(x_5)=x_6$ [7], the situation is quite similar to that for $G=\mathrm{Spin}(7)$, and so we get a similar result, i.e., $b(2)=b(4)'=b(4)=1$ and $b(6)=2$. On the other hand, as mentioned earlier, $c(2)=c(4)'=c(4)=c(6)=1$. Thus we have

$$\begin{aligned}
&\tau'(x_3)=f_4=\frac{1}{2}p_1\,,\\
&\tau'(x_7')=f_8'=c_4\,,\\
&\tau'(x_7)=f_8=\frac{1}{4}p_2-\frac{1}{2}c_4+\textit{decomposables},\\
&\tau'(x_{11})=\frac{1}{2}f_{12}=\frac{1}{2}p_3\,.
\end{aligned}\tag{3.6}$$

Let us recall from [17] that

(3.7) $R(\mathrm{Spin}(8))=Z[\lambda_1,\lambda_2,\Delta_8^+,\Delta_8^-]$ *where*

(a) $\dim\lambda_k=\binom{8}{k}$ *and* $\dim\Delta_8^+=\dim\Delta_8^-=8$;

(b) *relations* $\Lambda^k\lambda_1=\lambda_k$ *and* $\Delta_8^+\Delta_8^-=\lambda_3+\lambda_1$ *hold*;

(c) *the set of weights of* λ_1 *is given by* $\{\pm t_i\,|\,1\leq i\leq 4\}$ *and that of* Δ_8^+ *is given by* $\{\pm\gamma,\,\gamma-t_i-t_j\,|\,1\leq i<j\leq 4\}$.

By direct calculations we have

$$\begin{aligned}
&ch^2\alpha i^*(\lambda_1)=p_1\,,\\
&ch^4\alpha i^*(\lambda_1)=\frac{1}{12}(-2p_2+p_1^2)\,,\\
&ch^6\alpha i^*(\lambda_1)=\frac{1}{360}(3p_3-3p_2p_1+p_1^3)
\end{aligned}\tag{3.8}$$

and

$$\begin{aligned}
&ch^2\alpha i^*(\Delta_8^+)=p_1\,,\\
&ch^4\alpha i^*(\Delta_8^+)=\frac{1}{48}(4p_2+24c_4+p_1^2)\,.
\end{aligned}\tag{3.9}$$

There are involutive automorphisms κ and $\tilde{\kappa}$ of T and $\mathrm{Spin}(8)$ respectively, which make the diagram

$$\begin{array}{ccc} T & \xrightarrow{\kappa} & T \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ \mathrm{Spin}(8) & \xrightarrow{\tilde{\kappa}} & \mathrm{Spin}(8) \end{array}$$

commute, such that the automorphism $(B\kappa)^*$ of $H^*(BT; Z)$ satisfies

$$(B\kappa)^*(t_i) = \begin{cases} t_i & (1 \leq i \leq 3) \\ -t_4 & (i = 4). \end{cases}$$

Therefore $(B\kappa)^*(p_i) = p_i$, $(B\kappa)^*(c_4) = -c_4$ and the automorphism $\tilde{\kappa}^*$ of $R(\mathrm{Spin}(8))$ satisfies $\tilde{\kappa}^*(\Delta_8^+) = \Delta_8^-$. Applying $(B\kappa)^*$ to (3.9), it follows that

(3.10)
$$ch^2\alpha i^*(\Delta_8^-) = p_1,$$
$$ch^4\alpha i^*(\Delta_8^-) = \frac{1}{48}(4p_2 - 24c_4 + p_1^2).$$

Combining (3.8), (3.9), (3.10) with (3.6), we have

$$ch\beta(\lambda_1) = 2x_3 - \frac{1}{3}x_7' - \frac{2}{3}x_7 + \frac{1}{60}x_{11},$$
$$ch\beta(\Delta_8^+) = 2x_3 + \frac{2}{3}x_7' + \frac{1}{3}x_7 + ax_{11},$$
$$ch\beta(\Delta_8^-) = 2x_3 - \frac{1}{3}x_7' + \frac{1}{3}x_7 + ax_{11}$$

for some $a \in Q$. From Lemma 1 and the relation $\Delta_8^+\Delta_8^- = \lambda_3 + \lambda_1$ we deduce that $a = 1/60$.

Theorem 5. *The Chern characters on Spin(8) are given by:*

$$\begin{aligned} ch\beta(\lambda_1) &= 2x_3 + (-2/3!)x_7' + (-4/3!)x_7 + (2/5!)x_{11} \\ ch\beta(\lambda_2) &= 12x_3 \qquad\qquad\qquad\qquad\qquad + (-48/5!)x_{11} \\ ch\beta(\Delta_8^+) &= 2x_3 + (4/3!)x_7' + (2/3!)x_7 + (2/5!)x_{11} \\ ch\beta(\Delta_8^-) &= 2x_3 + (-2/3!)x_7' + (2/3!)x_7 + (2/5!)x_{11} \end{aligned}$$

and the determinant of the corresponding matrix is -1.

REMARK. The equation $ch\beta(\Delta_8^+ - \Delta_8^-) = x_7'$ confirms the fact that $\mathrm{Spin}(8)/\mathrm{Spin}(7) = S^7$ (see [22, Proposition 6.2]).

Let us lastly consider the case of Spin(9). In this case, $(m_1, m_2, m_3, m_4) = (2, 4, 6, 8)$. We can choose $t_1, t_2, t_3, t_4, \gamma \in H^2(BT; Z)$ so that

$$H^*(BT; Z) = Z[t_1, \cdots, t_4, \gamma]/(c_1 - 2\gamma)$$

and

$$H^*(BT; Q)^{\Phi(\mathrm{Spin}\,(9))} = Q[p_1, p_2, p_3, p_4]$$

where $c_i=\sigma_i(t_1, \cdots, t_4)$ and $p_i=\sigma_i(t_1^2, \cdots, t_4^2)$. By a straightforward calculation we have

$$\begin{aligned} f_4 &= \frac{1}{2}p_1 = -c_2+2\gamma^2, \\ f_8 &= \frac{1}{2}p_2-\frac{1}{2}f_4^2 = c_4+2(-c_3\gamma+c_2\gamma^2-\gamma^4), \\ f_{12} &= p_3 = -2c_4c_2+c_3^2, \\ f_{16} &= \frac{1}{4}p_4-\frac{1}{4}f_8^2 = c_4c_3\gamma-c_4c_2\gamma^2-c_3^2\gamma^2+2c_3c_2\gamma^3 \\ &\qquad +c_4\gamma^4-c_2^2\gamma^4-2c_3\gamma^5+2c_2\gamma^6-\gamma^8. \end{aligned}$$

Since $H^*(\mathrm{Spin}(9); Z)$ has no p-torsion for $p>2$ and

$$H^*(\mathrm{Spin}\,(9); Z/2) = \Delta_{Z/2}(\bar{x}_3, \bar{x}_5, \bar{x}_6, \bar{x}_7, \bar{x}_{15})$$

where all the $\bar{x}_i$ are universally transgressive and $\beta_2(\bar{x}_5)=\bar{x}_6$ [7], as in the case of $\mathrm{Spin}(7)$, it follows that $b(2)=b(4)=1$, $b(6)=2$ and $b(8)=1$. On the other hand, $c(2)=c(4)=c(6)=c(8)=1$. Thus we have

$$\begin{aligned} &(3.11) \qquad \tau'(x_3) = f_4 = \frac{1}{2}p_1, \\ &\qquad\qquad \tau'(x_7) = f_8 = \frac{1}{2}p_2+\textit{decomposables}, \\ &\qquad\qquad \tau'(x_{11}) = \frac{1}{2}f_{12} = \frac{1}{2}p_3, \\ &\qquad\qquad \tau'(x_{15}) = f_{16} = \frac{1}{4}p_4+\textit{decomposables}. \end{aligned}$$

REMARK. Let j: $\mathrm{Spin}(8)\to\mathrm{Spin}(9)$ be the natural inclusion. Then by (3.6) and (3.11) we see that the homomorphism j^*: $H^i(\mathrm{Spin}(9); Z)\to H^i(\mathrm{Spin}(8); Z)$ satisfies

$$j^*(x_i) = \begin{cases} x_i & (i=3,\ 11) \\ x_7'+2x_7 & (i=7) \\ 0 & (i=15). \end{cases}$$

Let us recall that

(3.12) $R(\mathrm{Spin}\,(9)) = Z[\lambda_1', \lambda_2', \lambda_3', \Delta_9]$ *where*

(a) $\dim \lambda_k' = \binom{9}{k}$ *and* $\dim \Delta_9=16$;

(b) *relations* $\Lambda^k\lambda_1'=\lambda_k'$ *and* $\Delta_9^2=\lambda_4'+\lambda_3'+\lambda_2'+\lambda_1'+1$ *hold*;

(c) *the set of weights of* λ_1' *is given by* $\{\pm t_i, 0 \mid 1 \leq i \leq 4\}$.

The rest of the argument is parallel to that for $G=\mathrm{Spin}(7)$. We only exhibit the result.

Theorem 6. *The Chern characters on Spin*(9) *are given by*:

$$\begin{aligned} ch\beta(\lambda_1') &= 2x_3+(-2/3!)x_7+(2/5!)x_{11}+(-4/7!)x_{15} \\ ch\beta(\lambda_2') &= 14x_3+(-2/3!)x_7+(-46/5!)x_{11}+(476/7!)x_{15} \\ ch\beta(\lambda_3') &= 42x_3+(18/3!)x_7+(-18/5!)x_{11}+(-4284/7!)x_{15} \\ ch\beta(\Delta_9) &= 4x_3+(2/3!)x_7+(4/5!)x_{11}+(34/7!)x_{15} \end{aligned}$$

and the determinant of the corresponding matrix is 1.

4. The exceptional Lie groups G_2 and F_4

Let us first consider the case of G_2. In this case, $(m_1, m_2)=(2, 6)$. We use the root system $\{\alpha_1, \alpha_2\}$ of [11]. Let ω_1, ω_2 be the fundamental weights. If we put

$$t_1 = \omega_1,\ t_2 = \omega_1-\omega_2,\ t_3 = -2\omega_1+\omega_2\,,$$

then

$$H^*(BT; Z) = Z[t_1, t_2, t_3]/(c_1)$$

where $c_i=\sigma_i(t_1, t_2, t_3)$, on which $\Phi(G_2)$ acts as follows:

	R_1	R_2
t_1	$-t_2$	t_1
t_2	$-t_1$	t_3
t_3	$-t_3$	t_2

where R_j $(j=1, 2)$ is the reflection to the hyperplane $\alpha_j=0$, and $\{R_1, R_2\}$ generates $\Phi(G_2)$. Therefore

$$H^*(BT; Q)^{\Phi(G_2)} = Q[p_1, p_3]\,.$$

where $p_i=\sigma_i(t_1^2, t_2^2, t_3^2)$, and it follows that

$$f_4 = \frac{1}{2}p_1 = -c_2\,,$$

$$f_{12} = p_3 = c_3^2\,.$$

Since $H^*(G_2; Z)$ has no p-torsion for $p>2$ and

$$H^*(G_2; Z/2) = \Delta_{Z/2}(\bar{x}_3, \bar{x}_5, \bar{x}_6)$$

where all the x_i are universally transgressive and $\beta_2(x_5)=x_6$ [7], as in the case of Spin(7), it follows that $b(2)=1$ and $b(6)=2$. On the other hand, $c(2)=c(6)=1$. Thus we have

$$\tau'(x_3)=f_4=\frac{1}{2}p_1,$$

$$\tau'(x_{11})=\frac{1}{2}f_{12}=\frac{1}{2}p_3.$$

Let us recall that

(4.1) $R(G_2)=Z[\rho_1, \Lambda^2\rho_1]$ *where*

(a) $\dim \Lambda^k\rho_1=\binom{7}{k}$ (*and* $\dim \rho_2=14$);

((b) *a relation* $\Lambda^2\rho_1=\rho_1+\rho_2$ *holds*;)

(c) *the set of weights of* ρ_1 *is given by* $\{\pm t_i (1\leq i\leq 3), 0\}$.

By a calculation we have

$$ch^2\alpha i^*(\rho_1)=p_1,$$

$$ch^6\alpha i^*(\rho_1)=\frac{1}{120}p_3+\text{decomposables}.$$

Therefore

$$ch\beta(\rho_1)=2x_3+\frac{1}{60}x_{11}$$

and by Lemma 1 we get

Theorem 7. *The Chern characters on* G_2 *are given by:*

$$ch\beta(\rho_1)=2x_3+(2/5!)x_{11}$$
$$ch\beta(\Lambda^2\rho_1)=10x_3+(-50/5!)x_{11}$$

and the determinant of the corresponding matrix is -1.

REMARK. Consider the following fibration

$$G_2\xrightarrow{k}\mathrm{Spin}\,(7)\rightarrow\mathrm{Spin}\,(7)/G_2=S^7.$$

Then it is easy to see that $k^*\colon H^i(\mathrm{Spin}(7);Z)\rightarrow H^i(G_2;Z)$ satisfies

$$k^*(x_i)=\begin{cases} x_i & (i=3,\ 11)\\ 0 & (i=7)\end{cases}.$$

On the other hand, $k^*\colon R(\mathrm{Spin}(7))\rightarrow R(G_2)$ satisfies

$$k^*(\lambda_i')=\Lambda^i\rho_1 \qquad (i=1,\ 2)$$
$$k^*(\Delta_7)=\rho_1+1$$

(see [31]). Using these facts, we find that Theorem 7 follows from Theorem 4.

$H^*(\Omega G_2; Z)$ (for degrees ≤ 10) was calculated implicitly by Bott [10]. Using it and the cohomology spectral sequence of the path fibration $\Omega G_2 \to PG_2 \to G_2$, we can show that

$$d(2) = 1 \quad \text{and} \quad d(6) = 2$$

(see [12] and [28, p. 474]).

Let us now consider the case of F_4. In this case, $(m_1, m_2, m_3, m_4) = (2, 6, 8, 12)$. We can choose elements $t_1, t_2, t_3, t_4, \gamma \in H^2(BT; Z)$ so that

$$H^*(BT; Z) = Z[t_1, \cdots, t_4, \gamma]/(c_1 - 2\gamma)$$

and the action of $\Phi(F_4)$ on it is as described in [9, §19] (see [18] and [29]). Let $c_i = \sigma_i(t_1, \cdots, t_4)$ and $p_i = \sigma_i(t_1^2, \cdots, t_4^2)$. If we put

$$\begin{aligned} I_4 &= p_1\,, \\ I_{12} &= -6p_3 + p_2 p_1\,, \\ I_{16} &= 12p_4 - 3p_3 p_1 + p_2^2\,, \\ I_{24} &= -72p_4 p_2 + 27p_4 p_1^2 + 27p_3^2 - 9p_3 p_2 p_1 + 2p_2^3\,, \end{aligned}$$

then we have

$$H^*(BT; Q)^{\Phi(F_4)} = Q[I_4, I_{12}, I_{16}, I_{24}]\,.$$

For a proof see [27, Lemma 5.1], however, its main part is accomplished by a pure calculation; see (4.7) and (4.8) below. By a troublesome calculation we obtain

$$\begin{aligned} f_4 &= \frac{1}{2} I_4 = -c_2 + 2\gamma^2\,, \\ f_{12} &= -\frac{1}{2} I_{12} \\ &= -4c_4 c_2 + 3c_3^2 + c_2^3 - 4c_3 c_2 \gamma - 4c_4 \gamma^2 - 2c_2^2 \gamma^2 + 8c_3 \gamma^3\,, \\ f_{16} &= \frac{1}{16}(I_{16} + 2f_{12} f_4 + f_4^4) \\ &= c_4^2 - c_4 c_3 \gamma + c_4 c_2 \gamma^2 + c_3^2 \gamma^2 - 2c_3 c_2 \gamma^3 - c_4 \gamma^4 + c_2^2 \gamma^4 + 2c_3 \gamma^5 - 2c_2 \gamma^6 + \gamma^8\,, \\ f_{24} &= -\frac{1}{64}(I_{24} + 16 f_{16} f_4^2 - 3f_{12}^2 + f_4^6) \\ &= 2c_4^3 - c_4^2 c_2^2 - 3c_4^2 c_3 \gamma + c_4 c_3 c_2^2 \gamma + 7c_4^2 c_2 \gamma^2 - 3c_4 c_3^2 \gamma^2 - c_4 c_2^3 \gamma^2 - c_3^2 c_2^2 \gamma^2 + 2c_4 c_3 c_2 \gamma^3 \\ &\quad + 2c_3^3 \gamma^3 + 2c_3 c_2^3 \gamma^3 - 7c_4^2 \gamma^4 + 2c_4 c_2^2 \gamma^4 - 2c_3^2 c_2 \gamma^4 - c_2^4 \gamma^4 - 2c_4 c_3 \gamma^5 - 4c_3 c_2^2 \gamma^5 \\ &\quad - 2c_4 c_2 \gamma^6 - c_3^2 \gamma^6 + 4c_2^3 \gamma^6 + 4c_3 c_2 \gamma^7 + c_4 \gamma^8 - 7c_2^2 \gamma^8 - 2c_3 \gamma^9 + 6c_2 \gamma^{10} - 2\gamma^{12}\,. \end{aligned}$$

Let us determine $b(2)$, $b(6)$, $b(8)$, $b(12) \in Z$. Recall that $H^*(F_4; Z)$ has no p-torsion for $p>3$. Since

$$H^*(F_4;\ Z/2) = \Delta_{Z/2}(\bar{x}_3, \bar{x}_5, \bar{x}_6, \bar{x}_{15}, \bar{x}_{23})$$

where all the $\bar{x}_i$ are universally transgressive and $\beta_2(\bar{x}_5)=\bar{x}_6$ [7], it follows that $\nu_2(b(2))=0$, $\nu_2(b(6))=1$, $\nu_2(b(8))=0$ and $\nu_2(b(12))=0$. Consider the case $p=3$. Recall from [7] and [25] that

$$\begin{aligned}
&H^*(F_4;\ Z/3) = Z/3[\bar{x}_8]/(\bar{x}_8^3) \otimes \Lambda_{Z/3}(\bar{x}_3, \bar{x}_7, \bar{x}_{11}, \bar{x}_{15})\\
&H^*(BF_4;\ Z/3) = Z/3[\bar{y}_{36}, \bar{y}_{48}] \otimes C\,,\\
&C = Z/3[\bar{y}_4, \bar{y}_8] \otimes \{1, \bar{y}_{20}, \bar{y}_{20}^2\} + \Lambda_{Z/3}(\bar{y}_9) \otimes Z/3[\bar{y}_{26}] \otimes \{1, \bar{y}_{20}, \bar{y}_{21}, \bar{y}_{25}\}
\end{aligned}$$

where $\tau(\bar{x}_i)=\bar{y}_{i+1}$ for $i=3, 7, 8$ and $\beta_3(\bar{x}_7)=\bar{x}_8$. Here we may suppose that

$$\begin{aligned}
\pi_3(x_3) &= \bar{x}_3\,, & \pi_3(y_4) &= \bar{y}_4\,,\\
\pi_3(x_{11}) &= \bar{x}_{11}\,, & \pi_3(y_{12}) &= \bar{y}_4\bar{y}_8\,,\\
\pi_3(x_{15}) &= \bar{x}_{15}\,, & \pi_3(y_{16}) &= \bar{y}_8^2\,,\\
\pi_3(x_{23}) &= \bar{x}_7\bar{x}_8^2\,, & \pi_3(y_{24}) &= \bar{y}_8^3\,.
\end{aligned}$$

In the mod 3 cohomology spectral sequence $\{E_r(Z/3)\}$ of the universal fibration

$$F = F_4 \to E = EF_4 \to B = BF_4\,,$$

if

$$\beta_3^B\colon E_2^{s,t}(Z/3) \to E_2^{s+1,t}(Z/3)$$

is the map induced by $\beta_3\colon H^s(B; Z/3) \to H^{s+1}(B; Z/3)$ through the isomorphism

$$E_2^{s,t}(Z/3) \cong H^s(B;\ H^t(F;\ Z/3))\,,$$

then we have

$$(4.2)\quad \begin{cases}
d_9(1 \otimes \bar{x}_{11}) = \bar{y}_9 \otimes \bar{x}_3 \cdots\cdots\cdots\cdots\cdots\cdots (*)\\
\beta_3^B(\bar{y}_8 \otimes \bar{x}_3) = \bar{y}_9 \otimes \bar{x}_3\\
d_4(\bar{y}_8 \otimes \bar{x}_3) = \bar{y}_4\bar{y}_8 \otimes 1
\end{cases}$$

$$(4.3)\quad \begin{cases}
d_9(1 \otimes \bar{x}_{15}) = \bar{y}_9 \otimes \bar{x}_7 \cdots\cdots\cdots\cdots\cdots\cdots (*)\\
\beta_3^B(\bar{y}_8 \otimes \bar{x}_7) = \bar{y}_9 \otimes \bar{x}_7\\
d_8(\bar{y}_8 \otimes \bar{x}_7) = \bar{y}_8^2 \otimes 1
\end{cases}$$

$$(4.4)\quad \begin{cases}
d_8(1 \otimes \bar{x}_7\bar{x}_8^2) = \bar{y}_8 \otimes \bar{x}_8^2\\
\beta_3^F(\bar{y}_8 \otimes \bar{x}_7\bar{x}_8) = \bar{y}_8 \otimes \bar{x}_8^2\\
d_8(\bar{y}_8 \otimes \bar{x}_7\bar{x}_8) = \bar{y}_8^2 \otimes \bar{x}_8\\
\beta_3^F(\bar{y}_8^2 \otimes \bar{x}_7) = \bar{y}_8^2 \otimes \bar{x}_8\\
d_8(\bar{y}_8^2 \otimes \bar{x}_7) = \bar{y}_8^3 \otimes 1
\end{cases}$$

where the asterisks are due to [3]. Generally, with the obvious notation, since $d_1\colon E_1^{s,t}(Z/3)\to E_1^{s+1,t}(Z/3)$ can be identified with the differential $\delta_B\colon C^s(B;Z/3)\to C^{s+1}(B;Z/3)$, if $\beta_3^B(\{\bar{u}\})=\{\bar{v}\}$, then there exist $u, v\in A^{*,*}(Z)$ such that $\pi_3(u)=\bar{u}$, $\pi_3(v)=\bar{v}$ and $\delta(u)=3v$. In this way the same argument as in the case of Spin(7) is valid. Therefore the conditions (4.2), (4.3) and (4.4) imply that $\nu_3(b(6))=1$, $\nu_3(b(8))=1$ and $\nu_3(b(12))=2$ respectively. Summarizing these, we have

$$b(2)=1\,,\quad b(6)=6\,,\quad b(8)=3\quad\text{and}\quad b(12)=9\,.$$

On the other hand, $c(2)=c(6)=c(8)=c(12)=1$. Thus we obtain

$$\text{(4.5)}\quad \begin{aligned} \tau'(x_3) &= f_4=\frac{1}{2}I_4\,,\\ \tau'(x_{11}) &= \frac{1}{6}f_{12}=-\frac{1}{12}I_{12}\,,\\ \tau'(x_{15}) &= \frac{1}{3}f_{16}=\frac{1}{48}I_{16}+\textit{decomposables},\\ \tau'(x_{23}) &= \frac{1}{9}f_{24}=-\frac{1}{576}I_{24}+\textit{decomposables}. \end{aligned}$$

Let us recall from [30] that

(4.6) $R(F_4)=Z[\rho_4, \Lambda^2\rho_4, \Lambda^3\rho_4, \rho_1]$ *where*

(a) $\dim\Lambda^k\rho_4=\binom{26}{k}$ *and* $\dim\rho_1=52$;

(b) *the set of weights of* ρ_4 *is given by*

$$\{\pm t_i(1\le i\le 4),\ \frac{1}{2}(\pm t_1\pm t_2\pm t_3\pm t_4),\ 0,\ 0\}$$

and that of ρ_1 *is given by*

$$\{\pm t_i\pm t_j(1\le i<j\le 4),\ \pm t_i(1\le i\le 4),\ \frac{1}{2}(\pm t_1\pm t_2\pm t_3\pm t_4),\ 0,\ 0,\ 0,\ 0\}\,.$$

We have to calculate $ch\alpha i^*(\rho_4)$ and $ch\alpha i^*(\rho_1)$. Consider the inclusion k: Spin(9) $\to F_4$ such that $F_4/\mathrm{Spin}(9)=\Pi$, the Cayley projective plane (see, e.g., [9, §19]). Then $k^*\colon R(F_4)\to R(\mathrm{Spin}(9))$ satisfies $k^*(\rho_4)=\lambda_1'+\Delta_9+1$ and $k^*(\rho_1)=\lambda_2'+\Delta_9$; see (4.6) (b). Let us calculate $ch\alpha i^*(\Delta_9)$, where the set of weights of Δ_9 is $\{1/2(\pm t_1\pm t_2\pm t_3\pm t_4)\}$. To do so we first calculate $ch\alpha i^*(\Delta_5)$, where the set of weights of Δ_5 is $\{1/2(\pm t_1\pm t_2)\}$; using it, we calculate $ch\alpha i^*(\Delta_7)$; and using it, we calculate $ch\alpha i^*(\Delta_9)$. Our final result is

$$ch^2\alpha i^*(\Delta_9) = 2p_1,$$
$$ch^6\alpha i^*(\Delta_9) = \frac{1}{2880}(48p_3+12p_2p_1+p_1^3),$$
$$ch^8\alpha i^*(\Delta_9) = \frac{1}{645120}(1088p_4+256p_3p_1+16p_2^2+24p_2p_1^2+p_1^4),$$
$$ch^{12}\alpha i^*(\Delta_9) = \frac{1}{122624409600}(31488p_4p_2+42432p_4p_1^2+3072p_3^2+4608p_3p_2p_1 +1920p_3p_1^3+64p_2^3+240p_2^2p_1^2+60p_2p_1^4+p_1^6).$$

By a similar calculation to the before, we have

$$ch^2\alpha i^*(\lambda_1') = p_1,$$
$$ch^6\alpha i^*(\lambda_1') = \frac{1}{360}(3p_3-3p_2p_1+p_1^3),$$
$$ch^8\alpha i^*(\lambda_1') = \frac{1}{20160}(-4p_4+4p_3p_1+2p_2^2-4p_2p_1^2+p_1^4),$$
$$ch^{12}\alpha i^*(\lambda_1') = \frac{1}{239500800}(6p_4p_2-6p_4p_1^2+3p_3^2-12p_3p_2p_1+6p_3p_1^3-2p_2^3 +9p_2^2p_1^2-6p_2p_1^4+p_1^6).$$

Thus we have

$$(4.7)\quad ch^2\alpha i^*(\rho_4) = 3p_1,$$
$$ch^6\alpha i^*(\rho_4) = \frac{1}{960}(24p_3-4p_2p_1+3p_1^3),$$
$$ch^8\alpha i^*(\rho_4) = \frac{1}{645120}(960p_4+384p_3p_1+80p_2^2-104p_2p_1^2+33p_1^4),$$
$$ch^{12}\alpha i^*(\rho_4) = \frac{1}{40874803200}(11520p_4p_2+13120p_4p_1^2+1536p_3^2-512p_3p_2p_1 +1664p_3p_1^3-320p_2^3+1616p_2^2p_1^2-1004p_2p_1^4+171p_1^6).$$

On the other hand, $ch\alpha i^*(\rho_1-\rho_4)$ was calculated in [27, §5] (with certain indeterminacy). Following it, we have

$$(4.8)\quad ch^2\alpha i^*(\rho_1-\rho_4) = 6p_1,$$
$$ch^6\alpha i^*(\rho_1-\rho_4) = \frac{1}{60}(-12p_3+2p_2p_1-p_1^3),$$
$$ch^8\alpha i^*(\rho_1-\rho_4) = \frac{1}{10080}(240p_4-156p_3p_1+20p_2^2+16p_2p_1^2+3p_1^4),$$

$$ch^{12}\alpha i^*(\rho_1-\rho_4)=\frac{1}{39916800}(-720p_4p_2+1270p_4p_1^2+366p_3^2-122p_3p_2p_1$$
$$-346p_3p_1^3+20p_2^3+86p_2^2p_1^2+16p_2p_1^4+p_1^6).$$

Thus we get

$$ch^2\alpha i^*(\rho_1)=9I_4,$$
$$ch^6\alpha i^*(\rho_1)=\frac{7}{240}I_{12}+\text{decomposables},$$
$$ch^8\alpha i^*(\rho_1)=\frac{17}{8064}I_{16}+\text{decomposables},$$
$$ch^{12}\alpha i^*(\rho_1)=\frac{1}{4055040}I_{24}+\text{decomposables}.$$

Combining these with (4.5), it follows that

$$ch\beta(\rho_4)=6x_3+\frac{1}{20}x_{11}+\frac{1}{168}x_{15}+\frac{1}{443520}x_{23},$$
$$ch\beta(\rho_1)=18x_3-\frac{7}{20}x_{11}+\frac{17}{168}x_{15}-\frac{1}{7040}x_{23}$$

and by Lemma 1 we obtain

Theorem 8. *The Chern characters on F_4 are given by*:

$$ch\beta(\rho_4)=6x_3+(6/5!)x_{11}+(30/7!)x_{15}+(90/11!)x_{23}$$
$$ch\beta(\Lambda^2\rho_4)=144x_3+(-36/5!)x_{11}+(-3060/7!)x_{15}+(-181980/11!)x_{23}$$
$$ch\beta(\Lambda^3\rho_4)=1656x_3+(-1584/5!)x_{11}+(-24480/7!)x_{15}+(11180160/11!)x_{23}$$
$$ch\beta(\rho_1)=18x_3+(-42/5!)x_{11}+(510/7!)x_{15}+(-5670/11!)x_{23}$$

and the determinant of the corresponding matrix is 1.

$H^*(\Omega F_4;Z)$ (for degrees ≤ 22) was calculated implicitly in [28]. Using it and the cohomology spectral sequence of the path fibration $\Omega F_4\to PF_4\to F_4$, we can show that

$$d(2)=1,\quad d(6)=2,\quad d(8)=1\quad\text{and}\quad d(12)=3.$$

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