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CODAZZI FIELDS ON SURFACES IMMERSED IN EUCLIDEAN 4-SPACE

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Abstract

Consider a Riemannian vector bundle of rank 1 defined by a normal vector field ν on a surface M in $\mathbb{R}^4$. Let II_ν be the second fundamental form with respect to ν which determines a configuration of lines of curvature. In this article, we obtain conditions on ν to isometrically immerse the surface M with II_ν as a second fundamental form into $\mathbb{R}^3$. Geometric restrictions on M are determined by these conditions. As a consequence, we analyze the extension of Loewner's conjecture, on the index of umbilic points of surfaces in $\mathbb{R}^3$, to special configurations on surfaces in $\mathbb{R}^4$.

1. Introduction

Given a surface M immersed in $\mathbb{R}^4$, let NM denote its normal bundle and ν a vector field on M. We consider the following question: *Under what conditions is it possible to isometrically immerse M in $\mathbb{R}^3$, in such a way that there is a vector bundle isomorphism between the normal sub-bundle E_ν determined by ν in NM and the normal bundle of the immersed surface M' in 3-space, taking the second fundamental form II_ν to the second fundamental form on M'?*

It is classically known that any submanifold of $\mathbb{R}^n$ satisfies the Gauss, Codazzi and Ricci equations. Moreover, the fundamental theorem for submanifolds provides us with a kind of converse, in the sense that it asserts that such equations are sufficient to determine uniquely the submanifolds of $\mathbb{R}^n$ ([2]). Starting from this theorem, we are able to determine the necessary and sufficient conditions that a normal field ν over a surface M immersed in $\mathbb{R}^4$ must satisfy in order to get an affirmative answer to the above question. The interpretation of such conditions in terms of the contacts of M with the hyperplanes of $\mathbb{R}^4$, analyzed through the behavior of the height functions family on M, leads in a natural way to the determination of certain geometrical obstructions on M based on the results obtained in [12].

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The article is organized as follows. Section 2 is divided in two parts. We provide in Subsection 2.1 a short review of some well known concepts and results that shall be subsequently used. We obtain in Subsection 2.2 from the structure equations of Gauss and Codazzi, two conditions on the second fundamental form that together are equivalent to the fact that locally the bundle is immersible in $\mathbb{R}^3$ (Proposition 2.2). We include in Section 3 a brief discussion of known results for surfaces in 4-space from the viewpoint of its contacts with hyperplanes, developed in [12] and explore the relation between the Gauss condition and the existence of binormal directions on surfaces in 4-space (defined in [12]). We introduce in Section 4 the concept of *Codazzi fields* on surfaces. This is done as follows: To each non-locally parallel normal field ν on M, we associate a tangent field W_ν, that measures how far ν is from being parallel. In fact, if we consider a moving frame on M whose normal subframe is given by $\{\nu, \nu^\perp\}$, where $\nu^\perp$ denotes a unit normal field orthogonal to ν, we have that critical points of W_ν are those at which the connection form ω_{34} vanishes. We say that a non-locally parallel normal field ν is a Codazzi field provided W_ν belongs to the Kernel of $S_{\nu^\perp}$. This is equivalent to asking that $\overline{\nabla}_{W_\nu}\nu^\perp = 0$ off the zeroes of W_ν (Proposition 4.6). Then, we prove the fundamental result of this paper:

Theorem 4.8. *Let M be a simply connected surface immersed in $\mathbb{R}^4$ and let ν be a unitary normal field on M. Then,*
i) *Assume that M has non flat normal bundle. It admits an isometric immersion in $\mathbb{R}^3$ with prescribed second fundamental form II_ν if and only if ν is a Codazzi field.*
ii) *In case that M has vanishing normal curvature, it admits an isometric immersion in $\mathbb{R}^3$ with prescribed second fundamental form II_ν if and only if $\nu^\perp$ is a binormal field on M, parallel along* $\mathrm{Ker}\, S_{\nu^\perp}$.

We use some basic properties of binormal fields and asymptotic directions to deduce certain geometric properties that the surface must satisfy in order to admit some isometric immersion into $\mathbb{R}^3$ with prescribed second fundamental form. For instance, *a necessary condition for the existence of some normal field ν on a generic surface M which admits an isometric immersion in $\mathbb{R}^3$ with prescribed second fundamental form II_ν is to be locally convex* (Proposition 3.2), *or in other words, it must admit some everywhere defined asymptotic field.* As a consequence, we have that *minimal surfaces in $\mathbb{R}^4$ which are not locally developable never admit isometric immersions into $\mathbb{R}^3$ with a prescribed second fundamental form, for any of their normal fields* (Proposition 3.3). Moreover, *any surface of $\mathbb{R}^4$ may admit at most* 2 *isometric immersions with prescribed second fundamental form in $\mathbb{R}^3$, modulo isometries of $\mathbb{R}^3$* (Proposition 3.1).

Clearly, such isometric immersions must take the ν-principal configuration of the surface M into the (unique) principal configuration of its image M' in 3-space. We use this fact in Section 5 in order to discuss a possible generalization of Loewner's conjecture on the index of an umbilic of surfaces in 3-space to surfaces in 4-space. Section 6 is devoted to the particular case of flat surfaces.

Some results in this article form a part the doctoral thesis of the first author [6].

2. Vector bundles and structure equations

2.1. Vector bundles of rank one defined by normal sections on M. Let M be a smooth oriented surface immersed in $\mathbb{R}^4$ with the Riemannian metric induced by the standard Riemannian metric of $\mathbb{R}^4$. For each $p \in M$ consider the decomposition $T_p\mathbb{R}^4 = T_pM \oplus (T_pM)^\perp$, where $(T_pM)^\perp$ is the orthogonal complement of T_pM in $\mathbb{R}^4$. Let $\chi(M)$ and $\chi(M)^\perp$ be the space of smooth vector fields on M and the space of smooth vector fields normal to M, respectively.

Let $\overline{\nabla}$ be the Riemannian connection of $\mathbb{R}^4$. Given vector fields X, Y in $\chi(M)$, let $\overline{X}, \overline{Y}$ be some local extensions to $\mathbb{R}^4$. The Riemannian connection on M is well defined by the tangent component of the Riemannian connection of $\mathbb{R}^4$: $\nabla_X Y = (\overline{\nabla}_{\overline{X}}\overline{Y})^\top$. On the other hand, given a normal vector field $\xi \in \chi^\perp(M)$ let $\nabla^\perp_X \xi = (\overline{\nabla}_{\overline{X}}\overline{\xi})^\perp$ be the normal component of $\overline{\nabla}_{\overline{X}}\overline{\xi}$, this way we have a compatible connection in $TM^\perp$.

Consider *the second fundamental form*,

$$\alpha : \chi(M) \times \chi(M) \to (\chi(M))^\perp, \quad \alpha(X, Y) = \overline{\nabla}_{\overline{X}}\overline{Y} - \nabla_X Y.$$

If $p \in M$ and $\nu \in (T_pM)^\perp$, $\nu \neq 0$, define the function

$$l_\nu : T_pM \times T_pM \to \mathbb{R}, \quad l_\nu(X, Y) = \langle \alpha(X, Y), \nu \rangle.$$

The ν-second fundamental form of M at p is the associated quadratic form,

$$\mathrm{II}_\nu : T_pM \to \mathbb{R}, \quad \mathrm{II}_\nu(X) = l_\nu(X, X).$$

Recall *the shape operator*

$$S_\nu : T_pM \to T_pM, \quad S_\nu(X) = -(\overline{\nabla}_{\overline{X}}\overline{\nu})^\top,$$

where $\overline{\nu}$ is a local extension to $\mathbb{R}^4$ of the normal vector field ν at p and $\top$ means the tangent component. This operator is bilinear, self-adjoint and for any $X, Y \in T_pM$ satisfies the following equation: $\langle S_\nu(X), Y \rangle = l_\nu(X, Y)$ [2]. Thus, for each $p \in M$ there exist an orthonormal basis of eigenvectors of S_ν in T_pM, for which the restriction of II_ν to the unitary vectors takes its maximal and minimal values. These eigenvalues are the *ν-principal curvatures*. The point p is a *ν-umbilic* if the ν-principal curvatures coincide. Let $\mathcal{U}_\nu$ be the set of ν-umbilics in M. For any $p \in M \setminus \mathcal{U}_\nu$ there are two ν-principal directions defined by the eigenvectors of S_ν, these fields of directions are smooth and integrable, then they define two families of orthogonal curves, its integrals, which are called the *ν-principal lines of curvature*, one maximal and the other one minimal. The ν-umbilics are the singularities of these families of curves.

Using the normal field ν, we will define a vector bundle of rank 1 on M with first and second fundamental forms as above, determining the family of ν-principal lines of curvature.

Assume that $\{\nu, \nu^{\perp}\}$ is an orthonormal basis of the normal vector bundle $(TM)^{\perp}$. Consider the vector bundle of rank one $\pi\colon \tilde{\nu} \to M$, where $\tilde{\nu}$ is the normal vector bundle on M whose fiber at $p \in M$ is the normal line in the direction $\nu(p)$ and π is the natural projection. Endow this vector bundle with the connection $\tilde{\nabla}$, compatible with the metric, defined as the *projection on* $\tilde{\nu}$ *of the normal connection* $\nabla^{\perp}$ restricted to $\tilde{\nu}$, namely:

$$\tilde{\nabla}_X \eta = \langle \nabla^{\perp}_X \eta, \nu \rangle \nu, \tag{1}$$

for $\eta \in \tilde{\nu}$ and $X \in TM$.

Consider the *Whitney sum* of vector bundles: $E_\nu = TM \oplus_W \tilde{\nu}$, where the metric on E_ν is the orthogonal sum of the metrics on TM and $\tilde{\nu}$. This Riemannian vector bundle E_ν has a connection ∇', compatible with its metric, defined by:

$$\nabla'_X Y = \nabla_X Y + \tilde{\alpha}(X, Y), \quad X, Y \in TM,$$
$$\nabla'_X \xi = -S_\xi X + \tilde{\nabla}_X \xi; \quad X \in TM, \ \xi \in \tilde{\nu},$$

where $\tilde{\alpha}(X, Y) = l_\nu(X, Y)\nu$ is the projection of

$$\alpha(X, Y) = l_\nu(X, Y)\nu + l_{\nu^\perp}(X, Y)\nu^{\perp}, \quad \nu \in \tilde{\nu},$$

on the line determined by ν.

2.2. ν-Gauss and ν-Codazzi equations for M. From the structure equations of the bundle E_ν let us write down Gauss equation:

$$\langle R(X, Y)Z, W\rangle - \langle R_\nu(X, Y)Z, W\rangle$$
$$= \langle \tilde{\alpha}(X, W), \tilde{\alpha}(Y, Z)\rangle - \langle \tilde{\alpha}(X, Z), \tilde{\alpha}(Y, W)\rangle,$$

where R is the curvature tensor with respect to the connection ∇ of M and R_ν is the curvature tensor of the bundle E_ν defined by ∇'.

Observe now that the tangent projection of R_ν along the tangent component vanishes, namely $\langle R_\nu(X, Y)Z, W\rangle = 0$, if and only if

$$\langle R(X, Y)Z, W\rangle = l_\nu(X, W)l_\nu(Y, Z) - l_\nu(X, Z)l_\nu(Y, W), \tag{2}$$

where $X, Y, Z, W \in TM$. Equation (2) is the Gauss equation to be satisfied by the curvature of M to immerse locally isometrically (or globally if it is simply connected) M into $\mathbb{R}^3$. Let us call it ν-*Gauss* equation.

Consider now the Codazzi equation for the bundle E_ν,

$$(R_\nu(X, Y)Z)^{\perp} = (\tilde{\nabla}_X \tilde{\alpha})(Y, Z) - (\tilde{\nabla}_Y \tilde{\alpha})(X, Z),$$

where $(\tilde{\nabla}_X\tilde{\alpha})(Y, Z) = \tilde{\nabla}_X\tilde{\alpha}(Y, Z) - \tilde{\alpha}(\nabla_X Y, Z) - \tilde{\alpha}(Y, \nabla_X Z)$ and ∇ is the connection of M. Analogously, the normal projection of R_ν along the tangent component vanishes, namely $(R_\nu(X, Y)Z)^\perp = 0$, if and only if

$$(\tilde{\nabla}_X\tilde{\alpha})(Y, Z) = (\tilde{\nabla}_Y\tilde{\alpha})(X, Z). \tag{3}$$

This is the Codazzi equation that the Riemannian connection on M has to satisfy in order to ensure that M can be locally isometrically (or globally if it is simply connected) immersed in $\mathbb{R}^3$ with prescribed second fundamental form II_ν. We call it ν-*Codazzi* equation. Considering that $\nabla^\perp_X\nu$ is orthogonal to ν and the following equation holds $\nabla^\perp_X l_\nu(Y, Z)\nu = X(l_\nu(Y, Z))\nu + l_\nu(Y, Z)\nabla X^\perp\nu$, we obtain that the ν-Codazzi equation can be written as follows:

$$\begin{aligned} &X(l_\nu(Y, Z)) - Y(l_\nu(X, Z)) \\ &= l_\nu([X, Y], Z) + l_\nu(Y, \nabla_X Z) - l_\nu(X, \nabla_Y Z). \end{aligned} \tag{4}$$

The fundamental theorem for Riemannian submanifolds [2], applied to the vector bundle $E_\nu \to M$, guarantees that if M is simply connected and equations (2) and (4) hold, there exists a unique (modulo isometries of $\mathbb{R}^3$) isometric immersion $f: M \to \mathbb{R}^3$, and a vector bundle isomorphism $\tilde{f}: E_\nu \to TM^\perp$ along f which transforms the ν-second fundamental form II_ν into the second fundamental form of the immersion.

Observe that since $M \subset \mathbb{R}^4$, it satisfies Gauss and Codazzi structure equations for surfaces immersed in $\mathbb{R}^4$. These can be written respectively as

$$\langle R(X, Y)Z, W\rangle = \langle\alpha(X, W), \alpha(Y, Z)\rangle - \langle\alpha(X, Z), \alpha(Y, W)\rangle, \tag{5}$$

$$(\nabla^\perp_X\alpha)(Y, Z) = (\nabla^\perp_Y\alpha)(X, Z), \tag{6}$$

where α and ∇ are respectively the second fundamental form and the connection of the immersion, and

$$(\nabla^\perp_X\alpha)(Y, Z) = \nabla^\perp_X\alpha(Y, Z) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z).$$

We exploit this fact in order to determine conditions on M and on the normal field ν guaranteeing that the ν-Gauss and ν-Codazzi equations hold.

The following straightforward lemma will allow us to express the ν-Gauss and ν-Codazzi equations in a convenient way, stated in Proposition 2.2, that shall be useful in the sequel.

Lemma 2.1. *The following conditions are equivalent,*

a) $l_{\nu^\perp}(X, Z)\langle\nabla^\perp_Y\nu^\perp, \nu\rangle + l_{\nu^\perp}(Y, Z)\langle\nabla^\perp_X\nu, \nu^\perp\rangle = 0$, *for all* $X, Y, Z \in TM$.

b) $S_{\nu^\perp}(\langle\nabla^\perp_Y\nu^\perp, \nu\rangle X + \langle\nabla^\perp_X\nu, \nu^\perp\rangle Y) = 0$, *for all* $X, Y \in TM$.

Proposition 2.2. *Let M be an oriented surface immersed in $\mathbb{R}^4$. Let ν be an smooth unitary vector field normal to M. Consider the vector bundle defined by $\pi\colon E_\nu \to M$, with Riemannian connection ∇'. Then, ν-Gauss and ν-Codazzi conditions of $\pi\colon E_\nu \to M$ are equivalent respectively to the following two conditions at every point $p \in M$:*

$$l_{\nu^\perp}(X, W)l_{\nu^\perp}(Y, Z) - l_{\nu^\perp}(X, Z)l_{\nu^\perp}(Y, W) = 0, \tag{7}$$

$$\langle \nabla^\perp_Y \nu^\perp, \nu\rangle X + \langle \nabla^\perp_X \nu, \nu^\perp\rangle Y \in \operatorname{Ker} S_{\nu^\perp}, \tag{8}$$

where X, Y, Z, W are vector fields tangent to M in a neighborhood of p, and $\operatorname{Ker} S_{\nu^\perp}$ is the kernel of $S_{\nu^\perp}$.

Proof. Since M is immersed in $\mathbb{R}^4$, Gauss equation holds for this immersion:

$$\begin{aligned}\langle R(X, Y)Z, W\rangle &= \langle \alpha(X, W), \alpha(Y, Z)\rangle - \langle \alpha(X, Z), \alpha(Y, W)\rangle \\ &= l_\nu(X, W)l_\nu(Y, Z) + l_{\nu^\perp}(X, W)l_{\nu^\perp}(Y, Z) \\ &\quad - l_\nu(X, Z)l_\nu(Y, W) - l_{\nu^\perp}(X, Z)l_{\nu^\perp}(Y, W).\end{aligned}$$

Therefore, ν-Gauss equation (2) holds, if and only if equation (7) does it.

On the other hand, consider Codazzi equation in $\mathbb{R}^4$, (6).

$$(\nabla^\perp_X \alpha)(Y, Z) = (\nabla^\perp_Y \alpha)(X, Z),$$

by substituting the values of the image of α in the normal basis and taking the component in the direction of ν, this equation implies the following expression,

$$\begin{aligned}&\langle \nabla^\perp_X(l_\nu(Y, Z)\nu + l_{\nu^\perp}(Y, Z)\nu^\perp), \nu\rangle - l_\nu(\nabla_X Y, Z) - l_\nu(Y, \nabla_X Z) \\ &= \langle \nabla^\perp_Y(l_\nu(X, Z)\nu + l_{\nu^\perp}(X, Z)\nu^\perp), \nu\rangle - l_\nu(\nabla_Y X, Z) - l_\nu(X, \nabla_Y Z).\end{aligned}$$

Thus, by observing that $\nabla^\perp_X \nu$ is orthogonal to ν,

$$\nabla^\perp_X(l_\nu(Y, Z)\nu) = X(l_\nu(Y, Z))\nu + l_\nu(Y, Z)\nabla^\perp_X \nu,$$

and because the normal connection is compatible with the metric, we obtain

$$\begin{aligned}&l_{\nu^\perp}(X, Z)\langle \nabla^\perp_Y \nu^\perp, \nu\rangle + l_{\nu^\perp}(Y, Z)\langle \nabla^\perp_X \nu, \nu^\perp\rangle \\ &= l_{\nu^\perp}(X, Z)\langle \nabla^\perp_Y \nu^\perp, \nu\rangle - l_{\nu^\perp}(Y, Z)\langle \nu, \nabla^\perp_X \nu^\perp\rangle.\end{aligned}$$

So, according to Lemma 2.1, condition b) is equivalent to both members of this equation vanish. Therefore, by substituting it in the previous equation we conclude that it is equivalent to equation (4). □

3. Gauss condition and binormal fields for surfaces in $\mathbb{R}^4$

Suppose that M is a surface embedded by ϕ into $\mathbb{R}^4$. Consider a local isothermic coordinate chart with parameters (u, v) and an orthonormal frame, $\{X_1 = \partial/\partial u,\ X_2 = \partial/\partial v,\ X_3 = \nu,\ X_4 = \nu^\perp\}$ on M. Take the dual 1-forms $\{w_1, w_2, w_3, w_4\}$, given by $w_i = \langle d\phi, X_i\rangle$. Let $\{w_{ij}\}_{i,j=1}^4$ be the corresponding connection forms. These forms have the following expression in terms of the dual 1-forms [10, p.263]:

$$\begin{aligned} w_{13} &= e_{X_3} w_1 + f_{X_3} w_2, \\ w_{23} &= f_{X_3} w_1 + g_{X_3} w_2, \\ w_{14} &= e_{X_4} w_1 + f_{X_4} w_2, \\ w_{24} &= f_{X_4} w_1 + g_{X_4} w_2. \end{aligned} \tag{9}$$

The *Gaussian curvature*, K, is the curvature corresponding to the tangent bundle of M and may be found from the formula: $dw_{12} = -K w_1 \wedge w_2$. Whereas the *normal curvature*, $K^\perp$, of M is obtained from the following expression relative to the curvature form of the normal bundle of M: $dw_{34} = -K^\perp w_1 \wedge w_2$. The function $K^\perp$ is a multiple of the area element on M.

The image of the affine map

$$\eta\colon S^1 \subset T_pM \to N_pM, \quad \eta(\theta) = \alpha_p(\theta, \theta),$$

defines an ellipse possibly degenerate, referred to as the *curvature ellipse* at p [10]. A direct computation shows that

$$\eta(\theta) = H + B\cos(2\theta) + C\sin(2\theta),$$

where in these coordinates $H = (1/2)(e_{X_3} + g_{X_3})X_3 + (1/2)(e_{X_4} + g_{X_4})X_4$, $B = (1/2)(e_{X_3} - g_{X_3})X_3 + (1/2)(e_{X_4} - g_{X_4})X_4$ and $C = f_{X_3}X_3 + f_{X_4}X_4$.

We say that a point $p \in M$ is a *semiumbilic* if and only if the curvature ellipse degenerate into a segment which is equivalent to condition $K^\perp(p) = 0$. Moreover, if this segment is radial the point is called an *inflection point*. These points are important from viewpoint of the extrinsic geometry because the rank of the second fundamental form decreases at them.

A surface M immersed in $\mathbb{R}^4$ is said to be *semiumbilical* provided all its points are semiumbilic. This is equivalent to say that M has vanishing normal curvature and hence that admits a parallel normal field.

There is an invariant function Δ on M defined as follows: Write $e = uX_1 + vX_2$ and consider $\langle de, X_3\rangle \wedge \langle de, X_4\rangle$. Now $de = u\,dX_1 + du X_1 + v\,dX_2 + dv X_2$. Therefore, $\langle de, X_3\rangle = uw_{13} + vw_{23}$ and $\langle de, X_4\rangle = uw_{14} + vw_{24}$. And taking into account that

w_{13}, w_{23}, w_{14} and w_{24} can be put in terms of the basis $\{w_1, w_2\}$ of the dual of T_pM, we obtain

$$\langle de, X_3\rangle \wedge \langle de, X_4\rangle = \delta(u, v)w_1 \wedge w_2,$$

where $\delta(u, v)$ is a quadratic form. The function Δ is defined as the determinant of the matrix associated to $\delta(u, v)$. One may check that if a point p is not a semiumbilic then the origin of N_pM is inside, on, or outside the ellipse according to Δ is respectively, positive, zero, or negative. Accordingly, p is said to be *elliptic*, *parabolic* or *hyperbolic*.

Given any vector $\xi \in \mathbb{R}^4$, the height function on M associated to ξ is defined by $h_\xi(p) = \langle \phi(p), \xi\rangle$. It is easy to see that h_ξ has a singularity at the point p if and only if ξ is normal to M at p. In the case that p is a degenerate singularity (non Morse) of h_ξ, we shall say that ξ defines a *binormal direction* for M at p. It was shown in [12] that according to $\Delta(p) < 0$, $= 0$ or > 0, we may find exactly two, one or none binormal directions respectively.

Given a normal field ξ on M, the Hessian matrix of the height function h_ξ at each point is given by

$$\begin{pmatrix} e_\xi & f_\xi \\ f_\xi & g_\xi \end{pmatrix},$$

where $e_\xi = -\langle \phi_{uu}, \xi\rangle$, $f_\xi = -\langle \phi_{uv}, \xi\rangle$, $g_\xi = -\langle \phi_{vv}, \xi\rangle$. Since this matrix coincides with that of the shape operator S_ξ, it follows that a binormal field is characterized by the fact that its associated shape operator has rank lesser than two. Therefore, if b is a binormal field on M at least one of the principal directions of b has vanishing principal curvature. This direction is said to be *asymptotic*. The umbilical points of the principal configurations associated to the binormal fields are the inflection points. In other words, these points are singularities of the asymptotic foliations on the surface ([12]). Suppose that $\Delta(p) < 0$, so there are exactly two asymptotic directions and two binormal directions at the point p. Then if θ_i, $i = 1, 2$ denote the two asymptotic directions at p and Ω is the angle determined by the two binormals in N_pM, it can be shown (see [10], p.268) that the two following formulae hold

$$\tan^2(\theta_1 - \theta_2) = \frac{\Delta(p)}{K^{\perp 2}},$$
$$\tan^2 \Omega = \frac{\Delta(p)}{K^2}.$$

Hence it follows that

a) $K^\perp(p) = 0$ if and only if the two asymptotic directions at p are orthogonal.
b) $K(p) = 0$ if and only if the two binormal directions at p are orthogonal.

Semiumbilical surfaces are thus characterized by having orthogonal asymptotic fields. A particular case of semiumbilical surfaces is given by the locally developable surfaces.

These were characterized by Little [10] as those for which the functions Δ and $K^{\perp}$ vanish everywhere. Moreover, if the Gaussian curvature also vanishes everywhere, we have that the surface is developable. The fact that Δ is identically zero implies that there is a unique binormal at every point of these surfaces.

Proposition 3.1. *ν-Gauss condition is equivalent to asking that $\nu^{\perp}$ be a binormal field on M. Therefore, a surface in $\mathbb{R}^4$ may admit at most two isometric immersions with prescribed second fundamental form, modulo isometries of $\mathbb{R}^3$.*

Proof. Suppose that $U \subset M$ is an open neighborhood with local coordinates (u, v). The coefficients of the $\nu^{\perp}$-second fundamental form are

$$\begin{aligned} e_{\nu^{\perp}} &= \mathrm{II}_{\nu^{\perp}}(\partial_u) = -\langle \alpha(\partial_u, \partial_u), \nu^{\perp} \rangle, \\ f_{\nu^{\perp}} &= -\langle \alpha(\partial_u, \partial_v), \nu^{\perp} \rangle = -\langle \alpha(\partial_v, \partial_u), \nu^{\perp} \rangle, \\ g_{\nu^{\perp}} &= \mathrm{II}_{\nu^{\perp}}(\partial_v) = -\langle \alpha(\partial_v, \partial_v), \nu^{\perp} \rangle, \end{aligned}$$

where $\partial_u = \partial/\partial u$ and $\partial_v = \partial/\partial v$. In this coordinate chart, equation (7) of Proposition 2.2 has the expression: $e_{\nu^{\perp}} g_{\nu^{\perp}} - f_{\nu^{\perp}}^2 = 0$, and the left side of this equation is the determinant of the Hessian matrix of the height function $h_{\nu^{\perp}}$. Since the ν-Gauss condition is equivalent to this equation, we obtain the first result. The second assertion follows from the fact that there are at most two binormals over any surface immersed in $\mathbb{R}^4$. □

In general, a surface immersed in $\mathbb{R}^4$ does not need to have globally defined binormal fields. A surface M immersed in $\mathbb{R}^4$ is said to be *locally convex* provided it admits a local support hyperplane at each one of its points. Surfaces contained in the boundary of their convex hull, in particular those contained in a convex hypersurface such as the standard hypersphere S^3, give us examples of locally convex surfaces in $\mathbb{R}^4$. Also, semiumbilical surfaces can be seen to be locally convex. It was shown in [12] that the function Δ never assumes positive values on locally convex surfaces, therefore such surfaces always have globally defined binormal vector fields. Moreover, if M is generically immersed in the sense of Looijenga's Theorem ([11]) (that is, the family $\lambda(f)$: $M \times S^3 \to \mathbb{R}^3$ of height functions on M is topologically stable), we have that a necessary and sufficient condition for the local convexity of M is the global existence of two binormal fields on it (that coincide over isolated inflection points). In other words, $\Delta < 0$ holds all over M except perhaps at isolated inflection points at which $\Delta = 0$, ([12], Corollary 4.3). On the other hand, it also follows that in non-generic cases, the hypothesis that $\Delta \leq 0$ is enough to guarantee the existence of at least one binormal vector locally defined at every point. In the particular case of surfaces at which Δ vanishes identically, we have a unique binormal field. It was shown by Little [10] that this class contains the local developable surfaces and the surfaces with substantial

codimension one, i.e. those contained in a hyperplane. In the last case, the binormal field is constant and coincides with the orthogonal direction to this hyperplane.

In view of these considerations we apply Proposition 3.1 to state:

Proposition 3.2. a) *A necessary condition for the existence of some normal field ν on a generic surface M such that E_ν admits an isometric immersion into $\mathbb{R}^3$ is that M be locally convex.*
b) *Locally developable surfaces may admit at most one isometric immersion with prescribed second fundamental form II_ν in $\mathbb{R}^3$, modulo isometries of $\mathbb{R}^3$.*

Minimal surfaces are characterized by the fact that the mean curvature vector vanishes at every point. This implies that non semiumbilic points of minimal surfaces must be all elliptic (the origin of the normal plane is inside the ellipse). On the other hand, all the semiumbilic points of such surfaces are necessarily inflection points. From the local viewpoint, we may thus have the two following situations over a minimal surface in 4-space:
a) The subset of inflection points has zero measure, in which case there cannot be binormal fields defined over open subsets.
b) All the points are inflection points. In this case, it was shown by Little [10] that the surface is either a local developable surface, or it lies in some hyperplane. Clearly, it admits a binormal field in both cases.

We observe that condition b) is too strong even for a minimal surface, so it looks sensible to expect that "most minimal surfaces" fulfill the first one, so they do not admit isometric immersions into $\mathbb{R}^3$ with a prescribed second fundamental, for any of their normal fields.

We can thus state the following:

Proposition 3.3. *If M is a substantially immersed minimal surface in $\mathbb{R}^4$ that admits some isometric immersions into $\mathbb{R}^3$ with a prescribed second fundamental form in the above sense, then M is locally developable and has vanishing normal curvature.*

4. Codazzi fields on surfaces

We start from the ν-Codazzi condition in the form of equation (8). Given a pair of tangent vector fields X, Y on M, we define a vector field

$$W_\nu(X, Y) = \langle \nabla^\perp_Y \nu^\perp, \nu\rangle X + \langle \nabla^\perp_X \nu, \nu^\perp\rangle Y.$$

Thus, ν-Codazzi condition holds in M if and only if $W_\nu(X, Y)$ belongs to the kernel of $S_{\nu^\perp}$, for any $X, Y \in \chi(M)$.

Lemma 4.1. a) *For any pair of tangent vector fields X, Y, in M the following equation holds*:

$$W_\nu(X, Y) = -W_\nu(Y, X).$$

b) *If there exist a pair of tangent vector fields X, Y, linear independent at p for which the vector $W_\nu(X, Y)_p \neq 0$. Then, for any local tangent frame $\{X_1, X_2\}$ at p, $W_\nu(X_1, X_2)_p \neq 0$. Furthermore, $W_\nu(X, Y)$ and $W_\nu(X_1, X_2)$ are linearly dependent.*

The proof is straightforward.

Notice that statement b) in this lemma implies that, in order to guarantee ν-Codazzi condition, it is enough to ensure the existence of a couple of linear independent vector fields X, Y, for which $W_\nu(X, Y)$ belongs to the kernel of $S_{\nu^\perp}$. So we can state the following:

Lemma 4.2. *ν-Codazzi condition holds in M, if and only if, for any $p \in M$ there exists a couple of locally defined, linearly independent, tangent vector fields X, Y, such that $W_\nu(X, Y)$ belongs to the kernel of $S_{\nu^\perp}$.*

Lemma 4.3. *If $W_\nu(X, Y) \neq 0$ for some tangent fields X, Y, or in other words, ν is not a parallel field, then ν-Codazzi condition implies ν-Gauss condition.*

Proof. Let us consider a local isothermic coordinate chart at each point $p \in M$ with tangent frame $\{X_1, X_2\}$. Then, Lemma 4.1, b) and Lemma 4.2 imply that the vector field $W_\nu(X_1, X_2)$ does not vanish and belongs to the kernel of $S_{\nu^\perp}$. Thus, $\operatorname{Det} S_{\nu^\perp}$ vanishes. The determinant of $S_{\nu^\perp}$, in this coordinate chart has the expression:

$$\operatorname{Det} S_{\nu^\perp} = \frac{1}{E}(e_{\nu^\perp} g_{\nu^\perp} - f_{\nu^\perp}^2),$$

where E is the non-vanishing coefficient of the first fundamental form. Therefore $e_{\nu^\perp} g_{\nu^\perp} - f_{\nu^\perp}^2 = 0$, which according to Proposition 3.1 is equivalent to ν-Gauss condition. □

REMARK 4.4. a) Notice that in this case we obtain directly from the proof that $W_\nu(X, Y)$ is a tangent field pointing in the asymptotic direction associated to the binormal $\nu^\perp$. In fact, there are two different unit vector fields tangent to the asymptotic direction. Let us denote

$$W_\nu = W_\nu(X_1, X_2)/|W_\nu(X_1, X_2)|,$$

where $\{X_1, X_2\}$ is the basis frame on this local chart.

b) If ν, and hence $\nu^\perp$, is parallel in an open neighborhood U of p, then W_ν vanishes and the ν-Codazzi condition holds trivially in U.

Lemma 4.5. *Suppose that ν is a non-locally parallel normal vector field on M. Then, the zeroes of W_ν lie in a measure zero set Z. Moreover, if $\nu^\perp$ is a binormal field, the ν-Codazzi condition is equivalent to $\nabla^\perp_{W_\nu}\nu = 0$ in the complement of Z, where W_ν is the vector field defined in* Remark 4.4 a).

Proof. Since ν is a non-parallel field over any open subset of M, then $W_\nu(X, Y)$ provides, according to Remark 4.4 a), a tangent field W_ν which is well defined (locally) over some open and dense subset of M and may vanish over some zero measure subset Z. It follows from Lemma 4.3 that the ν-Gauss condition also holds in $M - Z$, and hence all over M. But then, as observed in Remark 4.4 a), we have that W_ν determines the asymptotic directions field associated to the binormal $\nu^\perp$, and thus $W_\nu \in \operatorname{Ker} S_\nu$. So, the Codazzi condition given in equation (8) is equivalent to asking that $\langle \nabla^\perp_{W_\nu}\nu, \nu^\perp\rangle = 0$ which, being ν a unitary vector field, is equivalent to the requirement $\nabla^\perp_{W_\nu}\nu = 0$. Conversely, if the ν-Codazzi condition holds, we have that $W_\nu(X, Y) \in \operatorname{Ker} S_\nu$, $\forall X, Y \in \chi(M)$. So, either $W_\nu(X, Y) = 0$, $\forall X, Y \in \chi(M)$ in which case, we have that $\nabla^\perp_X \nu = 0$, $\forall X \in \chi(M)$ and thus ν is a parallel field, or W_ν only may vanish over a zero measure subset of M and the condition $\nabla^\perp_{W_\nu}\nu = 0$ holds all over M. □

A non-locally parallel unitary normal field ν on a surface M immersed in R^4 is said to be a *Codazzi field* provided W_ν belongs to $\operatorname{Ker} S_{\nu^\perp}$.

Lemma 4.1 implies that this definition does not depend on the coordinate chart, and only depends on the field ν.

Proposition 4.6. *Suppose that ν is a non-locally parallel normal field on M and denote by Z the set of zeroes of W_ν. Then, in the open and dense subset $\bar{M} = M - Z \subset M$, ν is a Codazzi field if and only if $\bar{\nabla}_{W_\nu}\nu^\perp = 0$.*

Proof. Assume that ν is a Codazzi field at $p \in \bar{M}$. Then, the non-zero vector field W_ν is well defined in a local neighborhood U of p where $\nu^\perp$ is a binormal. Thus $S_{\nu^\perp}(W_\nu) = 0$. Moreover, since ν-Codazzi condition holds, Lemma 4.5 implies that

$$0 = S_{\nu^\perp}(W_\nu) + \nabla^\perp_{W_\nu}\nu = \bar{\nabla}_{W_\nu}\nu^\perp.$$

Conversely, if $\bar{\nabla}_{W_\nu}\nu^\perp = 0$ at $p \in \bar{M}$, then the equation above implies that $\nabla^\perp_{W_\nu}\nu = 0$, and Lemma 4.5 guarantee that ν-Codazzi condition holds. This implies according to Lemma 4.2 that ν is a Codazzi field. □

Assume now that the coordinate chart where this local analysis has been made is also isothermic, with normal frame $\{\nu, \nu^\perp\}$ as in Section 3. The connection form ω_{34} is given by $\bar{\nabla}_X\nu = \omega_{34}(X)X_4$. It is not difficult to check that $W_\nu \equiv 0$ if and only if $\omega_{34} \equiv 0$.

REMARK 4.7. We observe that if M is ν-umbilic, ω_{34} vanishes identically on M if and only if M is contained either in a hypersphere or in a hyperplane [15].

Theorem 4.8. *Let M be a simply connected surface immersed in $\mathbb{R}^4$ and let ν be a unitary normal field on M. Then,*
i) *Assume that M has non flat normal bundle. It admits an isometric immersion in $\mathbb{R}^3$ with prescribed second fundamental form II_ν if and only if ν is a Codazzi field.*
ii) *In case that M has vanishing normal curvature, it admits an isometric immersion in $\mathbb{R}^3$ with prescribed second fundamental form II_ν if and only if $\nu^\perp$ is a binormal field on M, parallel along* $\operatorname{Ker} S_{\nu^\perp}$.

Proof. Consider the frame associated to the normal field ν as in Section 3. In case i) assume that ν is a Codazzi field. The field ν is not a parallel field in a neighborhood V of any point p, and thus the subset of zeroes of ω_{34} has zero measure in V. If $\omega_{34}(p) \neq 0$, then $W_\nu(X, Y) \neq 0$ for any couple of local independent vector fields at p and Lemma 4.3 implies that ν-Gauss condition holds at p. On the other hand, since ν is not locally parallel this is true for a dense set of M and then the continuity of the local expression of the ν-Gauss condition in equation (8) guarantees that it holds all over M. Therefore, ν-Gauss and ν-Codazzi conditions hold at every point of M.

In case ii) if $\nu^\perp$ is parallel along $\operatorname{Ker} S_{\nu^\perp}$ then ν is also parallel. This implies that ν-Codazzi condition holds in M. So, it is enough to ask that $\nu^\perp$ is a binormal in order to satisfy ν-Gauss condition.

Once we have seen, both in cases i) and ii) that the ν-Gauss and ν-Codazzi conditions hold at each point of M, since M is simply connected, the fundamental theorem for Riemannian submanifolds [2] implies that there exists an isometric immersion of M into $\mathbb{R}^3$ with prescribed second fundamental form II_ν. Conversely, if there exists such an isometric immersion of M into $\mathbb{R}^3$, the fundamental theorem implies that ν-Gauss and ν-Codazzi conditions hold at each point of M. Thus, an analysis similar to that one used in the first part of the proof shows that one of the two statements i) or ii) must hold. □

REMARK 4.9. a) Observe that the fundamental theorem for Riemannian submanifolds guarantees that the isometric immersions determined in Theorem 4.8 are unique, modulo isometries of $\mathbb{R}^3$.
b) Observe that the second fundamental form of the isometric immersion of M into $\mathbb{R}^3$ is the prescribed second fundamental form II_ν. Then the ν-principal configurations of M in $\mathbb{R}^4$ are also preserved under this reduction of codimension.

The behavior of the ν-principal foliations was analyzed in [14] and [5]. In fact, the topological types of the classes of ν-principal foliations of surfaces in $\mathbb{R}^4$ are richer than those of the principal foliations of surfaces in $\mathbb{R}^3$ [7]. A consequence of this fact is pointed out in the next section. We highlight that Theorem 4.8 provides a characterization of the class of local ν-principal configurations that can be realized as curvature lines of surfaces in $\mathbb{R}^3$.

5. Codazzi fields and Loewner's conjecture on surfaces

To each isolated umbilic of a surface in $\mathbb{R}^3$ we can attach the index of either one of the two fields. This index has to be of the form $n/2$, with $n \in \mathbb{Z}$. Examples of umbilics of index j are known for all $j \leq 1$. The classical (local) Loewner's conjecture states that every umbilic of a smooth surface immersed in $\mathbb{R}^3$ must have an index less than or equal to one. This conjecture has been asserted to be true for analytic surfaces by several authors among whom are H. Hamburger [8], G. Bol [1], T. Klotz [9], C.J. Titus [17]. On the other hand, it was proven in [7] that, given $n \in \mathbb{Z}$, there exists an analytic surface M immersed in $\mathbb{R}^4$ and an analytic unitary vector field ν normal to M having an isolated ν-umbilic of index $n/2$. This means that Loewner's statement cannot be generalized to aleatory principal configurations on surfaces in $\mathbb{R}^4$. The question is: *Is Loewner's statement true for some special class of principal configurations?* It follows from the analysis made in [4] that the result holds for binormal fields on locally convex surfaces generically immersed in 4-space. More recently, J.J. Nuño [13] has shown that Loewner's statement for surfaces in 3-space is equivalent to Loewner's statement for binormal fields on totally semiumbilic surfaces (with isolated umbilics).

If in the theorem above the immersed surface M and the normal field ν are real analytic, then the isometric immersion of M into $\mathbb{R}^3$ with prescribed second fundamental form is also real analytic. In fact, the real analytic bundle E_ν for which the ν-Gauss and ν-Codazzi conditions hold, is endowed with a real analytic parallel transport from which the immersion of M into $\mathbb{R}^3$ is obtained. Taking into account the results obtained for analytic surfaces in 3-space relative to Loewner's conjecture, we have the following

Corollary 5.1. *Suppose that M is a real analytic surface immersed in $\mathbb{R}^4$. If ν is either a real analytic Codazzi field, or a real analytic field such that $\nu^\perp$ is a binormal field parallel along* $\operatorname{Ker} S_{\nu^\perp}$ *on M, then every isolated ν-umbilic has index lesser or equal to one.*

6. Flat surfaces

We consider now the special case of flat surfaces:

Corollary 6.1. a) *A connected surface M immersed in $\mathbb{R}^4$ with zero Gaussian curvature has at most one Codazzi field.*
b) *If a connected surface M immersed in $\mathbb{R}^4$ with zero Gaussian curvature admits two different isometric immersions with prescribed second fundamental forms in $\mathbb{R}^3$, modulo isometries of $\mathbb{R}^3$, then its normal curvature also vanishes.*
c) *Developable surfaces in $\mathbb{R}^4$ admit at most one isometric immersion with prescribed second fundamental form in $\mathbb{R}^3$, modulo isometries of $\mathbb{R}^3$.*

Proof. a) Suppose that M is a surface with zero Gaussian curvature in $\mathbb{R}^4$ and that ν_1 and ν_2 are different Codazzi fields on M. Then Lemma 4.3 and Proposition 3.1 guarantee that both, $\nu_1^\perp$ and $\nu_2^\perp$, are binormal fields for M. We observe that the function Δ only may vanish over the subset Z of points at which $\nu_1^\perp$ and $\nu_2^\perp$ coincide. But the hypothesis that K is identically zero on M implies that $\nu_1 = \nu_2^\perp$ at every point of $M - Z$. Therefore, since M is connected, we have that either, $Z = M$ or $Z = \emptyset$. In the first case the proof is done and in the second one, since both ν_1 and ν_2 are Codazzi fields we have that $\nabla^\perp_{W_{\nu_1}} \nu_1 = \nabla^\perp_{W_{\nu_2}} \nu_2 = 0$. Moreover, since $\langle \nu_1, \nu_2 \rangle = 0$, we also have, $\nabla^\perp_{W_{\nu_1}} \nu_2 = \nabla^\perp_{W_{\nu_2}} \nu_1 = 0$. Now Remark 4.4 tells us that W_{ν_1} and W_{ν_2} determine the two asymptotic directions on M. Then, since $\Delta < 0$ it follows from comments made in Section 3 that W_{ν_1} and W_{ν_2} are linearly independent. And hence ν_1 and ν_2 are parallel fields, which contradicts the hypothesis that they are Codazzi fields.

b) If M has zero Gaussian curvature and admits two isometric immersions in $\mathbb{R}^3$ with prescribed second fundamental forms II_{ν_1} and II_{ν_2}, then it follows from the arguments in the proof of part a) that since M can only have a Codazzi field, say ν_1, the other one ν_2 must be a parallel normal field. Therefore M has zero normal curvature.

c) This follows from the fact, pointed out in Section 3, that there is a unique binormal field over a developable surface. □

In order to illustrate some of the above conclusions we provide an example of a surface, contained in the standard 3-sphere and thus locally convex, for which there are exactly two normal directions defining local isometric immersions with prescribed second fundamental form into $\mathbb{R}^3$.

EXAMPLE (the Clifford torus). Consider the coordinate chart:

$$U = \{(u, v) \in \mathbb{R}^2,\ 0 < u < 2\pi,\ 0 < v < 2\pi\},$$

$$\phi : U \to \mathbb{R}^4, \quad \phi(u, v) = \frac{1}{\sqrt{2}}(\cos u, \sin u, \cos v, \sin v).$$

Take the orthonormal frame,

$$X_1 = \sqrt{2}(-\sin u, \cos u, 0, 0), \quad X_2 = \sqrt{2}(0, 0, -\sin v, \cos v),$$

$$X_3 = \frac{1}{\sqrt{2}}(\cos u, \sin u, \cos v, \sin v),$$

$$X_4 = \frac{1}{\sqrt{2}}(-\cos u, -\sin u, \cos v, \sin v).$$

Consider an arbitrary unitary normal vector field $\eta = aX_3 + bX_4$, where $a, b : U \to \mathbb{R}$

are smooth functions such that $a^2 + b^2 = 1$. An straightforward computation gives the coefficients of the η-second fundamental form as follows:

$$e_\eta = -\frac{1}{2}a + \frac{1}{2}b, \quad f_\eta = 0, \quad g_\eta = -\frac{1}{2}a - \frac{1}{2}b.$$

The determinant of the shape operator with respect to $\eta^\perp$ is $\mathrm{Det}\, S_{\eta^\perp} = (b^2 - a^2)/4$, hence the ν-Gauss condition holds for $\nu = \eta$ if and only if $a = \pm b$. Namely, we must take $a^2 = b^2 = 1/2$. This determines the two normal directions $\nu_1 = (X_3 + X_4)/\sqrt{2}$ and $\nu_2 = (X_3 - X_4)/\sqrt{2}$. A direct computation shows that any one of these two normal fields is a Codazzi field on $\phi(U)$. Therefore, these two vector fields (modulo a sign) define the unique vector bundles immersible into $\mathbb{R}^3$. Notice that since the determinant of their shape operators vanishes, these vector fields are binormals. The corresponding asymptotic lines are the ν_i-curvature lines, solutions of the equation:

$$\frac{1}{\sqrt{2}}\, du\, dv = 0.$$

We point out that the Gaussian curvature of the Clifford torus vanishes everywhere. Thus, as expected after Lemma 3.1, these two binormal fields are everywhere orthogonal. Furthermore, $\phi(U)$ is contained in S^3 and hence semiumbilical, so its asymptotic directions $\{W_{\nu_1}, W_{\nu_2}\}$ are everywhere orthogonal, [15]. Naturally, none of the two above considered isometric immersions can be globally extended to the whole torus.

We finally observe that the problem studied in this paper is related to the factorization of an isometric immersion f of a surface M into $\mathbb{R}^4$ as a composition of isometric immersions h and F respectively of M into $\mathbb{R}^3$ and of $\mathbb{R}^3$ into $\mathbb{R}^4$. The obtained results imply that in order to admit such a factorization, an isometric immersion of a generic surface into $\mathbb{R}^4$ must be locally convex. Do Carmo and Dajczer consider in [3] locally flat immersions of the plane into $\mathbb{R}^4$ whose first normal space has constant rank 2. They provide a method to obtain all the immersions that do not factorize and characterize those that are compositions. As a consequence of our results we can say that an isometric immersion of the plane into $\mathbb{R}^4$ may admit at most one factorization, unless it has vanishing normal curvature, in which case it may admit two (as illustrated by any open disc in the Clifford torus). Moreover, if the first normal space has (constant) rank one, then the factorization is unique.

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References

[1] G. Bol: *Über Nabelpunkte auf einer Eifläche*, Math. Z. **49** (1944), 389–410.

[2] M. Dajczer: Submanifolds and Isometric Immersions, Mathematics Lecture Series **13**, Publish or Perish, Houston, TX, 1990.

[3] M. do Carmo and M. Dajczer: *Local isometric immersions of* $\mathbf{R}^2$ *into* $\mathbf{R}^4$, J. Reine Angew. Math. **442** (1993), 205–219.

[4] R.A. Garcia, D.K.H Mochida, M.D.C. Romero Fuster and M.A.S. Ruas: *Inflection points and topology of surfaces in* 4*-space*, Trans. Amer. Math. Soc. **352** (2000), 3029–3043.

[5] R.A. Garcia and F. Sánchez-Bringas: *Closed principal lines of surfaces immersed in the Euclidean* 4*-space*, J. Dynam. Control Systems **8** (2002), 153–166.

[6] J. Gutiérrez: *Rigid immersions of surfaces in* 4*-space*, Doctoral Thesis, Universidad Nacional Autónoma de México (2004).

[7] C. Gutiérrez and F. Sánchez-Bringas: *On a Loewner umbilic-index conjecture for surfaces immersed in* $\mathbf{R}^4$, J. Dynam. Control Systems **4** (1998), 127–136.

[8] H. Hamburguer: *Beweis einer Carathéodoryschen Vermütung*, Ann. Math. **41** (1940), 63–68, II, III, Acta Math. **73** (1941), 174–332.

[9] T. Klotz: *On G. Bol's proof of Carathéodory's conjecture*, Comm. Pure Appl. Math. **12** (1959), 277–311.

[10] J.A. Little: *On singularities of submanifolds of higher dimensional Euclidean spaces*, Ann. Mat. Pura Appl. (4) **83** (1969), 261–335.

[11] E.J.N. Looijenga: *Structural stability of smooth families of* C^∞*-functions*, Tesis, Univ. Amsterdam (1974).

[12] D.K.H. Mochida, M.D.C. Romero Fuster and M.A.S. Ruas: *The geometry of surfaces in* 4*-space from a contact viewpoint*, Geom. Dedicata **54** (1995), 323–332.

[13] J.J. Nuño-Ballesteros: *Submanifolds with a non-degenerate parallel normal vector field in Euclidean spaces*, to appear in Adv. Stud. Pure Math.

[14] F. Sánchez-Bringas and A.I. Ramírez-Galarza: *Lines of curvature near umbilical points on surfaces immersed in* $\mathbf{R}^4$, Ann. Global Anal. Geom. **13** (1995), 129–140.

[15] M.C. Romero-Fuster and F. Sánchez-Bringas: *Umbilicity of surfaces with orthogonal asymptotic lines in* $\mathbb{R}^4$, Differential Geom. Appl. **16** (2002), 213–224.

[16] T. Klotz: *On G. Bol's proof of Carathéodory's conjecture*, Comm. Pure Appl. Math. **12** (1959), 277–311.

[17] C.J. Titus: *A proof of a conjecture of Loewner and of the conjecture of Carathéodory on umbilic points*, Acta Math. **131** (1973), 43–77.

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