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A Selection Method of Observing Positions for Highly Accurate Measurement of Residual Stresses†

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Abstract

The authors have already proposed the general principles in measurement of residual stress. In this paper, the authors developed a new method for selection of observing positions of strains to ensure highly accurate measurement of residual stresses over a whole object by using the theory of inherent strain, which is based on one of these principles.

By this selection method, it is seen that the more number of observations ensures the better average accuracy of estimated values. However, there are many positions where observed values scarcely influence the accuracy of measurement. These positions can be excluded from the measurements for reasons of economy and comparatively small number of observing positions can be selected by the present theory, which still secures a high accuracy of measurement.

KEY WORDS: (Measuring Theory) (Residual Stresses) (Inherent Strains) (Selection of Observing Positions)

1. Introduction

The authors have already proposed the general principles in measurement of residual stresses and shown that there are two measuring theories for the application of the principles, (1) theory of inherent strain in which inherent strains are dealt as parameters of measurement (2) theory of sectioned-force in which sectioned-forces are dealt as parameters¹⁾. These measuring theories have been formulated with the aid of the finite element method, and generalized by a statistic approach in order to investigate the reliability of estimated values. Using these theories, residual stresses over a whole object can be estimated by comparatively less observed strains.

Furthermore, the authors have presented a new measuring method of three dimensional residual stresses induced in a long welded joint, based on theory of inherent strain which simplified by utilizing the characteristics of the distribution of inherent strains. With the support of the rational theory, the distributions of residual stresses and longitudinal inherent strains in a multipass welded joint have been measured for the first time²⁾.

But if very accurate estimate of the stress distribution all over the object is required within a limited number of observing positions, it is necessary to choose appropriate observing positions of strains.

In this paper, a new method for selection of observing positions is proposed to optimize the number of observing positions and ensure highly accurate measurement by

using the theory of inherent strain.

2. Basic Formulae of Measurement of Residual Stresses Based on Theory of Inherent Strain

In a welded joint, residual inherent strains which include dislocations are generally produced at the weld and in its vicinity by thermal elastic plastic strain history due to welding. Some portion of these residual inherent strains results in free expansion-contraction of the welded joint, and does not produce residual stresses. Then, the remaining portion of the residual inherent strains causes residual stresses. These inherent strains are called effective inherent strains (sometimes called inherent strains simply).

If the distribution of effective inherent strains is represented by an equation with q parameters $\{\epsilon^*\}$, the resulting residual elastic strain $\{\epsilon\}$ at any point of the body produced by the inherent strains are obtained in the following form.

$$\{\epsilon\} = [\bar{H}^*]\{\epsilon^*\}, \quad [\bar{H}^*] = (n \times q) \quad (1)$$

where n : the total number of components of elastic strains

q : the total number of parameters of inherent strains

In Eq. (1), the components of the j th row of elastic response matrix $[\bar{H}^*]$ correspond to elastic strains produced in the body when only j th parameter ϵ_j^* of the inherent strain distribution being unit is imposed.

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If m number of elastic strains can be observed, m equations can be taken out of Eq. (1) and the observation equation is constituted as follows.

$$\{m\epsilon\} - [H^*] \{\hat{\epsilon}^*\} = \{v\}, [H^*] = (m \times q) \quad (2)$$

where $\{m\epsilon\}$: observed strains
 $\{v\}$: residuals

In the case of rank $[H^*] = q$, the most probable values $\{\hat{\epsilon}^*\}$ of parameters of the inherent strains are decided so as to minimize the sum of squares of the residuals.

$$\{\hat{\epsilon}^*\} = [A] [H^*]^T \{m\epsilon\} \equiv [G^*] \{m\epsilon\} \quad (3)$$

where $[A] \equiv (a_{ij}) = ([H^*]^T [H^*])^{-1} = (q \times q)$

: variance matrix

$$[G^*] \equiv (g_{ij}) = [A] [H^*]^T = (q \times m)$$

: generalized inverse of matrix $[H^*]$

Then, the elastic strain distribution in the whole object can be calculated by substituting Eq. (3) into Eq. (1).

3. Selection Method of Observing Positions of Strains

In this section, a new method for selection of observing positions of strains is proposed, in the framework of the theory of inherent strain using the finite element method. According to this method, it is possible to select observing positions which ensure a high accuracy of measurement within a limited number of observing positions. This theory can be applicable to the case where the boundary of distribution region of inherent strains is estimated before measurement of residual stresses.

3.1 Condition for selection of observing positions of strains

Generally, it is considered that errors included in the observed values of strains obeys the normal distribution (Gauss' distribution) $N(0, s^2)$. Then, the most probable value $\hat{\epsilon}_i^*$ is a stochastic variable which also obeys the normal distribution and its variance s_i^{*2} is given by

$$s_i^{*2} = a_{ii} \cdot s^2 \quad (i = 1 \sim q) \quad (4)$$

In the above equation, the coefficients a_{ii} are the diagonal components of the variance matrix $[A]$ and express the degree of propagation of observation errors.

Then, the accuracy of the estimated values over the whole object can be evaluated by the mean value V_m of

the variances of the inherent strains, that is,

$$V_m = \left(\sum_{i=1}^q s_i^{*2} \right) / q = \text{tr} [A] \cdot s^2 / q \quad (5)$$

$$\text{where } \text{tr} [A] = \sum_{i=1}^q a_{ii}$$

The variance s^2 of observation errors in the above equations is related only with the performance of a measuring instrument to observe strains and is not influenced by the observing positions of strains. If the region of the inherent strain distribution can be assumed in advance, and q number of independent parameters of the expression for inherent strains are constant, the value V_m depends only upon the trace of the matrix $[A]$, the sum of the diagonal components, which is related with the components of the observation equation $[H^*]$. These components of the matrix $[H^*]$ may change by selection of m number of observing positions from the elastic response $[\bar{H}^*]$ over the whole object.

If the total number of strains which can be observed is equal to m_t ($< n$), the combination of selection of m observing positions is $m_t C_m$. One of these combinations which minimizes the value V_m , that is $\text{tr}[A]$, is considered the best selection of observing positions. However, when this theory is applied directly for selection of observing positions, it is necessary to compute the inverse matrix $([H^*]^T [H^*])^{-1}$, $m_t C_m$ times and enormous computing time is required. In order to reduce the computing time, a simplified method for selection of observing position is proposed below.

3.2 Increase of the trace $[A]$ with decrease of observing positions

As mentioned above, the average accuracy of estimated values over the object can be evaluated by $\text{tr}[A]$. Here, first of all, changes of $\text{tr}[A]$ will be studied when m_b number of observing positions are eliminated from $(m_a + m_b)$ observing positions.

It is assumed that the observation equation $[H^*]$, the variance matrix $[A]$ and the generalized inverse $[G^*]$ for the original $(m_a + m_b)$ observing positions have been already computed. And the matrices $[H^*]$ and $[G^*]$ may be divided into two parts as follows.

$$\left. \begin{aligned} [H^*] &= \begin{bmatrix} H_a \\ \bar{H}_b \end{bmatrix} & \begin{aligned} [H_a] &= (m_a \times q) \\ [H_b] &= (m_b \times q) \end{aligned} \\ [G^*] &= [G_a \mid G_b] & \begin{aligned} [G_a] &= (q \times m_a) \\ [G_b] &= (q \times m_b) \end{aligned} \end{aligned} \right\} \quad (6)$$

Here, rank $[H_a] = q$ is assumed. This implies that the generalized inverse can be obtained even if the number of observing positions is reduced to m_a .

In connection with Eqs. (6), there exist some relations between $[H^*]$ and $[G^*]$ which will be expressed in the following.

$$[G^*] = [A] [H^*]^T = [A] [H_a^T | H_b^T] = [G_a | G_b]$$

$$\therefore [H_a] = [G_a]^T [A]^{-1}, [H_b] = [G_b]^T [A]^{-1} \quad (7)$$

$$[G^*] [H^*] = [G_a] [H_a] + [G_b] [H_b] = [I] \quad (8)$$

where $[I]$: identity matrix

Using these relations, the generalized inverse for only m_a number of observing positions is given by

$$\begin{aligned} & ([H_a]^T [H_a])^{-1} [H_a]^T \\ &= ([A]^{-1} [G_a] [H_a])^{-1} [A]^{-1} [G_a] \\ &= ([A] [A]^{-1} [G_a] [H_a])^{-1} [G_a] \\ &= ([G_a] [H_a])^{-1} [G_a] \\ &= ([I] - [G_b] [H_b])^{-1} [G_a] \end{aligned} \quad (9)$$

On the other hand, there exists the next relation.

$$\begin{aligned} & ([I] - [G_b] [H_b]) ([I] + [G_b] [I - H_b G_b]^{-1} [H_b]) \\ &= [I] - [G_b] [H_b] + [G_b] [I - H_b G_b]^{-1} [H_b] \\ &\quad - [G_b] [H_b] [G_b] [I - H_b G_b]^{-1} [H_b] \\ &= [I] - [G_b] [H_b] + [G_b] ([I - H_b G_b]^{-1} \\ &\quad - [H_b] [G_b] [I - H_b G_b]^{-1}) [H_b] \\ &= [I] - [G_b] [H_b] + [G_b] ([I - H_b G_b] \\ &\quad [I - H_b G_b]^{-1}) [H_b] \\ &= [I] \end{aligned}$$

Then, the inverse matrix of the term on the right-hand side of Eq. (9) can be evaluated as follows.

$$[I - G_b H_b]^{-1} = [I] + [G_b] [I - H_b G_b]^{-1} [H_b] \quad (10)$$

And Eq. (9) can be rewritten in the following form by using Eq. (10).

$$\begin{aligned} & ([H_a]^T [H_a])^{-1} [H_a]^T \\ &= ([I] + [G_b] [I - H_b G_b]^{-1} [H_b]) [G_a] \end{aligned} \quad (11)$$

By Eqs. (7) and (11), the variance and the trace are calculated for m_a observing positions, after eliminating m_b observing positions from the $(m_a + m_b)$ original ones, as follows,

$$\begin{aligned} & ([H_a]^T [H_a])^{-1} \\ &= [A] + [G_b] ([I] - [H_b] [G_b])^{-1} [G_b]^T \end{aligned} \quad (12)$$

$$\begin{aligned} & \text{tr} ([H_a]^T [H_a])^{-1} \\ &= \text{tr} [A] + \text{tr} [G_b] ([I - H_b G_b])^{-1} [G_b]^T \end{aligned} \quad (13)$$

On the other hand, the following equation may be derived by taking into account of the symmetry of the matrix $[H_b] [G_b]$,

$$\begin{aligned} & [H_b] [G_b] = [G_b]^T [A]^{-1} [G_b] \\ &= [G_b]^T ([H_a]^T [H_a] \\ &\quad + [H_b]^T [H_b]) [G_b] \\ &= ([H_a] [G_b])^T ([H_a] [G_b]) \\ &\quad + ([H_b] [G_b])^T ([H_b] [G_b]) \\ &= ([H_a] [G_b])^T ([H_a] [G_b]) \\ &\quad + ([H_b] [G_b])^2 \end{aligned} \quad (14)$$

As the first term on the right-hand side of Eq. (14) is a positive symmetric matrix, it has non-negative eigenvalues κ_i^2 . When the eigenvalues of the matrix $[H_b] [G_b]$ are denoted by λ_j , the following relationship between κ_i and λ_j may exist³⁾.

$$\begin{aligned} & \lambda_i = \kappa_i^2 + \lambda_i^2 \\ & 0 \leq \lambda_i \leq 1 \quad (i = 1 \sim m_b) \end{aligned} \quad (15)$$

This eigenvalue λ_i must satisfy the following condition,

$$\det |\lambda_i I - H_b G_b| = 0 \quad (16)$$

If the eigenvalues of the matrix $[I - H_b G_b]^{-1}$ are denoted by μ_i , the following relation is derived.

$$\begin{aligned} \det |\mu_i I - [I - H_b G_b]^{-1}| \\ = \det |I - H_b G_b|^{-1} \cdot \det |\mu_i [I - H_b G_b] - I| \\ = 0 \end{aligned}$$

As $\text{rank}[H_a] = q$, Eq. (11) can be calculated and the determinant of the right-hand side in Eq. (11), $\det |I - H_b G_b|^{-1}$, is not zero. Then, the eigenvalue μ_i of the above equation is not equal to zero and the next equation is derived.

$$\begin{aligned} \det |I - H_b G_b|^{-1} \cdot \det |(1 - 1/\mu_i) \\ - H_b G_b| \cdot \mu_i^{m_b} = 0 \\ \therefore \det |(1 - 1/\mu_i) I - H_b G_b| = 0 \quad (17) \end{aligned}$$

By using Eqs. (16), (17) and (15),

$$\begin{aligned} \lambda_i = 1 - \frac{1}{\mu_i}, \mu_i = \frac{1}{1 - \lambda_i} \geq 1 \\ (i = 1 \sim m_b) \end{aligned} \quad (18)$$

The sign of the second term on the right-hand side of Eq. (13) is studied by utilizing Eq. (18). The matrix $[G_b] = (q \times m_b)$ is divided into m_b -dimensional row vector \mathbf{b}_j . That is,

$$[G_b] = \begin{bmatrix} \mathbf{b}_1^T \\ \vdots \\ \mathbf{b}_q^T \end{bmatrix} \quad (19)$$

The next expression is derived below

$$\begin{aligned} [G_b] [I - H_b G_b]^{-1} [G_b]^T \\ = \begin{bmatrix} f_{11} & \cdots & f_{1q} \\ \vdots & \ddots & \vdots \\ f_{q1} & \cdots & f_{qq} \end{bmatrix} \\ \text{where } f_{ij} = \mathbf{b}_i^T [I - H_b G_b]^{-1} \mathbf{b}_j \end{aligned}$$

Then, the second term on the right-hand side of Eq. (13) can be expressed as follows.

$$\begin{aligned} \text{tr} [G_b] [I - H_b G_b]^{-1} [G_b]^T \\ = \sum_{i=1}^q \mathbf{b}_i^T [I - H_b G_b]^{-1} \mathbf{b}_i \end{aligned} \quad (20)$$

In the above equation, each term on the right-hand side $\mathbf{b}_i^T [I - H_b G_b]^{-1} \mathbf{b}_i$ is quadratic form of the matrix $[I - H_b G_b]^{-1}$ of which eigenvalues are all positive, then, the above equation (20) is always positive. That is,

$$\text{tr} [G_b] [I - H_b G_b]^{-1} [G_b]^T > 0 \quad (21)$$

So, the trace increases being accompanied with eliminations of m_b observing positions, because the first term on right-hand side of Eq. (13) expresses the trace for the $(m_a + m_b)$ original observations and the second term is positive. As the result, the accuracy of the measurement as a whole becomes worse if any observing positions are eliminated.

This may be shown in the special case of $m_b = 1$, the matrices $[H_b]$ and $[G_b]$ are reduced to the following vectors.

$$[H_b] = \{\mathbf{h}_b\}^T, [G_b] = \{\mathbf{g}_b\} \quad (22)$$

where $\{\mathbf{h}_b\}, \{\mathbf{g}_b\}$: row vectors

Then, Eqs. (11), (12) and (13) can be rewritten in simpler forms to be shown below.⁴⁾

$$\begin{aligned} ([H_a]^T [H_a])^{-1} [H_a] = [I] + \{\mathbf{g}_b\} \{\mathbf{h}_b\}^T / \\ (1 - \{\mathbf{h}_b\}^T \{\mathbf{g}_b\}) [G_a] \end{aligned} \quad (11)'$$

$$\begin{aligned} ([H_a]^T [H_a])^{-1} = [A] + \{\mathbf{g}_b\} \{\mathbf{g}_b\}^T / \\ (1 - \{\mathbf{h}_b\}^T \{\mathbf{g}_b\}) \end{aligned} \quad (12)'$$

$$\begin{aligned} \text{tr} ([H_a]^T [H_a])^{-1} = \text{tr} [A] + \{\mathbf{g}_b\}^T \{\mathbf{g}_b\} / \\ (1 - \{\mathbf{h}_b\}^T \{\mathbf{g}_b\}) \end{aligned} \quad (13)'$$

In Eq. (13)' the second term on the right-hand side is positive, since the scalar $\{h_b\}^T \{g_b\}$ is such a value between zero and unit as Eq. (15) indicates. So, the trace always increases if any observing position is excluded from the original observations.

However, a special attention should be paid to the fact that there are some observing positions which hardly influence the accuracy of estimation as a whole, because an increase of the trace is very small even if these positions are excluded from the original observations. By excluding these observing positions in Eqs. (11)', (12)' and (13)', it is possible to make highly accurate measurement with a small number of observations as to be described in the following.

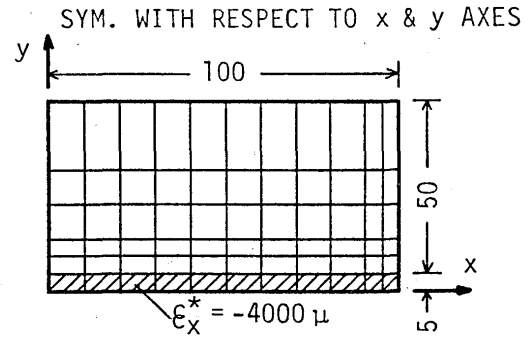
First of all, an acceptable level of the trace or an average variance V_m of parameters in the expression for inherent strain distribution is specified. And the matrices $[H^*]$, $[A]$ and $[G^*]$ for all the original capable observing positions (m_t) are computed. Next, by applying Eq. (13)' to all the m_t observing positions and excluding each position in turn from the original observations, an increase in the trace are computed and the least increase of the trace is inspected. Then, the position which is least influential upon the accuracy could be excluded in the calculation. If the increase of the trace by excluding that position is less than the acceptable level, that position can be eliminated from the observations. For the new set of ($m_t - 1$) observing positions, the general inverse and the variance matrix can be obtained by Eqs. (11)' and (12)'. By repeating this procedure, the number of observations can be decreased step by step until the increase of the trace reaches the acceptable level. Then, the necessary least number of observations and the location of the observations can be determined by the above procedure.

4. Numerical Experiment

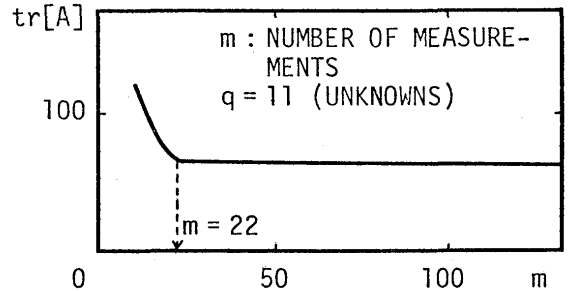
In order to show applicability of the proposed method for selection of observing positions, a numerical experiment is conducted.

In a plate shown in Fig. 1 (a), residual stresses are produced only by independent inherent strains in the direction of x-axis each other which are imposed in 11 elements along the center line of the plate.

It is necessary to calculate the elastic response matrix $[H^*]$ for the selection of observing positions. In this example, the number of unknown independent parameters is $q = 11$ and this matrix can be obtained by an elastic stress analysis 11 times, using the finite element method. If it is assumed that elastic strain ϵ_x and ϵ_y can be observed at the center of every finite element of the plate, the total number of components of elastic strains which can be observed is $m_t = 132$.



(a) MODEL FOR ANALYSIS



(b) TRACE AGAINST NUMBER OF MEASUREMENTS

Fig. 1 Changes of trace accompanied with decrease of the number of measurements by the present method

When m number of observations are decreased by the present theory, the trace changes as shown in Fig. 1(b). It is seen that the trace scarcely changes if $m \geq 22$, but that the trace rapidly increases and the accuracy of the estimation becomes worse if $m < 22$. In the case of $m = 22$, some possible observing positions are shown in Fig. 2. CASE A in Fig. 2 represents the most accurate set of the observing positions among three cases which were decided intuitively as adopted in Reference 1). CASE B is

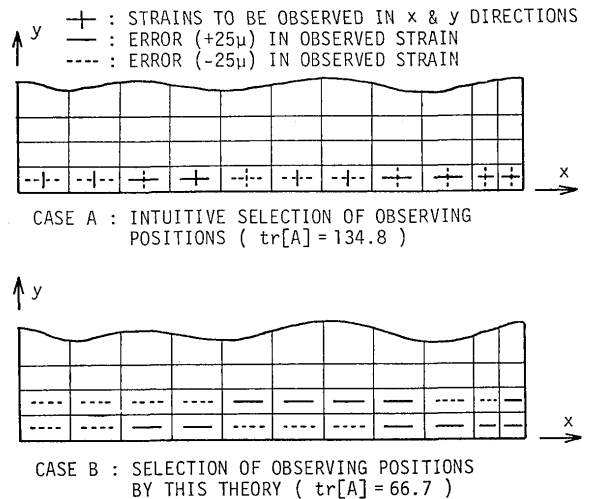


Fig. 2 Selections of observing positions

determined by the present method. It is seen that the trace in CASE B is about a half of that in CASE A and the accuracy of the estimation is extremely high as will be shown later.

Next, the accuracies of the estimated inherent strains in these two cases will be discussed. The true inherent strains contained in 11 elements located along the center line of the plate are assumed to be constant ($\epsilon_x^* = -4,000\mu$) and the residual stress distribution produced by these inherent strains is calculated with the aid of the finite element method, which is called the true residual stress distribution.

In the numerical experiments, strains are assumed to be observed at the above two sets of observing positions and to contain the same observation error of an absolute value 25μ with a positive or negative sign which is determined by random number, as shown in Fig. 2, the stress (strain) distribution induced by the above observation strains are estimated. The deviation $\hat{\epsilon}^*$ of the estimated inherent strains is shown in Fig. 3. It is seen that the accuracy of the estimated values over the object is very high if the

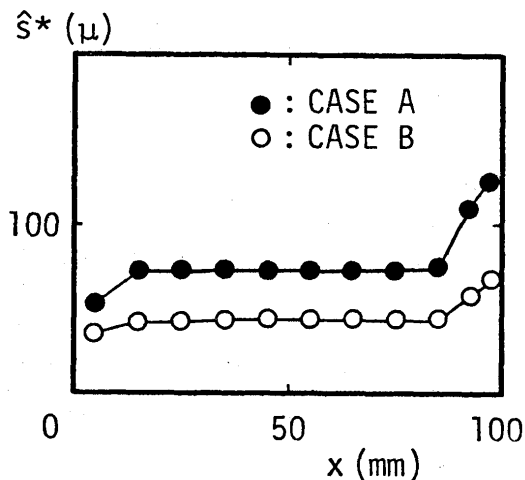


Fig. 3 Estimated deviations of inherent strains

observation positions are selected by the present method. Furthermore, the accuracy of measurement in CASE B is considered to be equivalent to one in the case where a large number of observing strains are used because an increase of the trace is hardly recognized even if the

number of observations is reduced to $m = 22$, as shown in Fig. 1(b).

This implies that the present method is very effective for selection of observing positions.

5. Conclusion

In this paper, based on the theory of inherent strain which is one of the measuring principles of three dimensional residual stresses proposed by the authors, a selection method for observing positions is developed in order to ensure highly accurate measurement within a limited number of observing positions. This method can be applicable in the case where the boundary of inherent strain distribution is estimated in advance.

A summary of the results obtained is shown below.

- 1) Generally, the more number of observations ensures the better average accuracy of estimated values.
- 2) However, there are many positions where observed values scarcely influence the accuracy of measurement. By the proposed selection method, these positions can be excluded from the observations and comparatively small number of observing positions can be selected, which still secures a high accuracy of measurement.

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