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ON MAXIMAL QUOTIENT RINGS OF QF-3 1-GORENSTEIN RINGS WITH ZERO SOCLE

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Until now, there appeared several papers which deal with characterizations for a ring to have a QF (quasi-Frobenius) maximal left quotient ring. Among them, Masaike's one [9, Theorem 2.1] seems to us to be the most decisive in a certain direction. As for QF maximal two-sided quotient rings, we shall give another characterization (Theorem 4.2). Its classical version will be given in Theorem 4.4.

On the other hand, Lenagan investigated noetherian rings with Krull dimension one (see [7] and [8]). A ring R is said to satisfy the restricted minimum condition for left ideals if R/I is an artinian module for any dense left ideal I. Applying Lenagan's result and Theorem 4.4, we shall obtain our main theorem that if a noetherian QF-3 ring with zero socle satisfies the restricted minimum condition for left ideals and right ideals, then it has a QF classical (two-sided) quotient ring (Theorem 5.7). As its corollary, we shall see that a QF-3 1-Gorenstein ring with zero socle has a QF classical quotient ring.

Throughout this paper, we assume that all rings have identity elements and all modules are unitary, and we use the Lambek torsion theory except for §2, unless otherwise specified. We denote by $E(M)$ the injective hull of a module M.

When the author was preparing the present paper, T. Sumioka showed him the paper [16] which appears in this volume. The author would like to express his thanks to Dr. Sumioka for his kindness.

1. Preliminaries

We recall some definitions and results which we need in the sequal. A ring is said to be left 1-Gorenstein if it is left and right noetherian and if it has left self-injective dimension at most one. A left and right 1-Gorenstein ring express his thanks to Dr. Sumi- is called 1-Gorenstein in short.

Let M be a left R-module. When M has Krull dimension, we denote its Krull dimension by $K\text{-}dim\ M$ in the present paper. If a ring R is left noetherian, then ${}_RR$ has Krull dimension, and for any ordinal β there exists the largest left ideal $\mathrm{rad}^{\beta}({}_RR)$ with Krull dimension at most β. We call it the left β-radical of R, which is clearly a two-sided ideal. In particular, if R is left and right noetherian, then $\mathrm{rad}^0({}_RR)$ coincides with $\mathrm{rad}^0(R_R)$ which is called the artinian radical of R and is denoted by $A(R)$ (see [3] for Krull dimension and [2] for the artinian radical). Thus, for a left and right noetherian ring R, $\mathrm{Soc}({}_RR)=0$ if and only if $\mathrm{Soc}(R_R)=0$. So we call such a ring a noetherian ring with zero socle. For a 1-Gorenstein ring R, $\mathrm{Soc}(R)=0$ if and only if $E(R)/R$ is an injective cogenerator (see [5]).

A ring R is said to be left QF-3 if every finitely generated submodule of $E({}_RR)$ is torsionless. On the other hand, R is said to be left QF-3′ if every finitely generated submodule of ${}_RQ$ is torsionless where Q is a maximal left quotient ring of R(see [10, Proposition 2]). Clearly left QF-3 implies left QF-3′. But the converse is not true. However, Sumioka has shown in [16] that a left QF-3′ and left 1-Gorenstein ring is QF-3.

REMARK 1. The last statement holds for a commutative n-Gorenstein ring R. For, R has a QF maximal quotient ring by [6, Theorem 5] and R is QF-3 by [12, Theorem 1.5].

REMARK 2. The notion of QF-3′ in the sense of [14] and [15] differs from ours and it is the same as our notion of QF-3.

A finite or infinite chain of submodules of a module M;

$$M_1 \supseteq M_2 \supseteq M_3 \supseteq \cdots$$

is called t-chain if M_i/M_{i+1} is not a torsion module (with respect to the Lambek torsion theory) for any integer i. We define the Lambek dimension of M, which is denoted by *L-dim* M so as not to confuse it with *K-dim* M (the Krull dimension of M), as follows. If M has a maximal t-chain of finite length n, then *L-dim* $M=n$. On the other hand, if M has no finite maximal t-chain, then *L-dim* $M=\infty$. In this case, it is said that the Lambek dimension of M is infinite.

Lemma 1.1. *For any submodule N of a module M, we have L-dim $M=$ L-dim $N+$L-dim M/N.* (See [14].)

A finite or infinite t-chain of a module M;

$$M_1 \supseteq M_2 \supseteq M_3 \supseteq \cdots$$

is said to be standard if M_{i-1}/M_i is torsion-free and *L-dim* $M_{i-1}/M_i=1$ for any $i \geq 2$. It is clear that if M has finite Lambek dimension, then M has a finite standard t-chain. The following lemma is useful.

Lemma 1.2 (Sumioka [16]). *Let $_RM$ be a torsion-free module with L-dim $M=1$. Then M can be embedded into $_RQ$ where Q is a left maximal quotient ring of R.*

2. Perfect topologies and strongly perfect topologies

In the present section, we recall and some definitions and results concerning topologies so that we can apply them to §4 easily and effectively.

Let R be a ring, and $\mathfrak{G}$ a family of left ideals of R. It is said that $\mathfrak{G}$ is a left topology on R if it satisfies the following properties.

(T1) If $I\in\mathfrak{G}$ and $a\in R$, then $Ia^{-1}=\{r\in R\,|\,ra\in I\}\in\mathfrak{G}$.

(T2) If I is a left ideal of R and there exists $J\in\mathfrak{G}$ such that $Ia^{-1}\in\mathfrak{G}$ for all $a\in J$, then $I\in\mathfrak{G}$.

Let ϕ be a ring homomorphism of R into T. Then it is said that T is a left perfect localization of R if ϕ is a ring epimorphism and T_R is flat. In that case, it is known that the family $\mathfrak{G}=\{_RI\subseteq R\,|\,T(I\phi)=T\}$ is a left topology on R and there exists a ring isomorphism of T onto $R_{\mathfrak{G}}$, the quotient ring of R with respect to $\mathfrak{G}$ (see [13, *XI*, Theorem 2.1]). On the other hand, a left topology $\mathfrak{G}$ is said to be perfect if the canonical ring homomorphism $R\to R_{\mathfrak{G}}$ is a left flat epimorphism. A left perfect topology $\mathfrak{G}$ is said to be strongly perfect if $\mathfrak{G}=\{_RI\subseteq R\,|\,R_{\mathfrak{G}}(I\phi)=R_{\mathfrak{G}}\}$ where ϕ is a canonical ring homomorphism of R into $R_{\mathfrak{G}}$. A left topology $\mathfrak{G}$ on R is strongly perfect if and only if for any R-module M,

$$0\to t_{\mathfrak{G}}(M)\to M\to R_{\mathfrak{G}}\otimes_R M$$

is an exact sequence where $t_{\mathfrak{G}}$ is the torsion radical corresponding to the topology $\mathfrak{G}$ (see [13, *XI*, Proposition 3.4]). We should remark that in Stenström's book [13] topologies and strongly perfect topologies in our sense are called Gabriel topologies and perfect Gabriel topologies respectively.

The following lemmas are used repeatedly in the later sections.

Lemma 2.1. *Let $\mathfrak{G}$ be a strongly perfect topology on a ring R. Then for every left R-module M, M is a $\mathfrak{G}$-torsion module if and only if $T\otimes_R M=0$ where $T=R_{\mathfrak{G}}$.*

Proof. Clear by definition.

Lemma 2.2. *Let ϕ be a left flat epimorphism of R into T. Then for every left T-module E and every left ideal I of R, $Hom_R(R/I, E)\cong Hom_T(T/T(I\phi), E)$.*

Proof. Clear by definition and adjointness.

Lemma 2.3. *Let $\mathfrak{G}$ be a left topology on a ring R, and T a quotient ring of R with respect to $\mathfrak{G}$. Then $\mathfrak{G}$ is strongly perfect if and only if $T(I\phi)=T$ for* *every $I \in \mathfrak{S}$ where ϕ is a canonical ring homomorphism of R into T.*

Proof. It is sufficient to show that $T(I\phi)=T$ implies $I \in \mathfrak{S}$ (see [13, XI, Theorem 2.1]). Let E be a cogenerating injective module for $\mathfrak{S}$. It is well-known that E becomes a T-module in a canonical way. Since $\mathrm{Hom}_R(R/I, E) \cong \mathrm{Hom}_T(T/T(I\phi), E)$ for every left ideal I of R by Lemma 2.2, $T=T(I\phi)$ implies $\mathrm{Hom}_R(R/I, E)=0$, that is, $I \in \mathfrak{S}$.

3. Topology $\mathfrak{F}_\beta$

Throughout the present section, a ring R is assumed to be left noetherian unless otherwise specified. It is well-known that such a ring R has left Krull dimension. Let $K\text{-}dim_R R=\alpha$. For any ordinal $\beta<\alpha$, we denote by $\mathfrak{F}_\beta$ the family of all left ideals I such that $K\text{-}dim\ R/I \leq \beta$ and by $\mathfrak{F}$ the union of all $\mathfrak{F}_\beta$'s for $\beta<\alpha$. Also we denote by $\mathfrak{D}$ the topology of dense left ideals of R. We clarify the connection of $\mathfrak{D}$ and $\mathfrak{F}$ or $\mathfrak{F}_\beta$ after we give some definitions.

At first, we show

Theorem 3.1. *Let R be a left noetherian ring with $K\text{-}dim_R R=\alpha$. Then for each ordinal $\beta<\alpha$, $\mathfrak{F}_\beta$ is a topology on R.*

Proof. We must show that $\mathfrak{F}_\beta$ satisfies two properties (T1) and (T2) stated in §2. Let $I \in \mathfrak{F}_\beta$ and $a \in R$. Since a homomorphism ϕ of R into R/I defined by $x\phi=xa+I$ for all $x \in R$ induces a monomorphism of R/Ia^{-1} into R/I, we have $K\text{-}dim\ R/Ia^{-1} \leq K\text{-}dim\ R/I \leq \beta$. Hence $Ia^{-1} \in \mathfrak{F}_\beta$. This shows that $\mathfrak{F}_\beta$ satisfies the property (T1). Let I be a left ideal of R, for which there exists $J \in \mathfrak{F}_\beta$ such that $Ia^{-1} \in \mathfrak{F}_\beta$ for all $a \in J$. The above argument shows that $R/Ia^{-1} \cong (Ra+I)/I \subseteq R/I$ for any $a \in J$. Since R is left noetherian, we have $K\text{-}dim\ (J+I)/I \leq \beta$. On the other hand, $K\text{-}dim\ R/(I+J) \leq \beta$ because $J \in \mathfrak{F}_\beta$. Recall $K\text{-}dim\ R/I = \sup\ \{K\text{-}dim(I+J)/I,\ K\text{-}dim\ R/(I+J)\}$. Thus $K\text{-}dim R/I \leq \beta$ and hence $I \in \mathfrak{F}_\beta$. This shows that $\mathfrak{F}_\beta$ satisfies the property (T2).

It is of much interest what condition on R makes $\mathfrak{D}$ contain $\mathfrak{F}_\beta$ or conversely. For this purpose, we give some definitions. Let R be a ring with left Krull dimension. For an ordinal β, we say that R satisfies the β-restricted minimum condition for left ideals if $K\text{-}dim\ R/I \leq \beta$ for each dense left ideal I. If $\beta=0$, we say in short the restricted minimum condition instead of the 0-restricted minimum condition. We should remark that our notion of the restricted minimum condition is slightly different from the notion defined in [1] by Chatters. (In non-singular rings, our notion coincides with his notion.) It seems that his interest lies in non-singular rings and however it seems to us that his notion does not fit rings which are not non-singular.

Proposition 3.2. *Let R be a left noetherian ring. For any ordinal β, the* *following statements hold.*

(1) $\mathfrak{D}$ *contains* $\mathfrak{F}_\beta$ *if and only if* $rad^\beta\ ({}_RR)=0$.

(2) $\mathfrak{F}_\beta$ *contains* $\mathfrak{D}$ *if and only if* R *satisfies the* β*-restricted minimum condition for left ideals.*

Proof. The assertion (2) is the definition of the β-restricted minimum condition. Assume $\mathrm{rad}^\beta({}_RR)=0$. Given $I\in\mathfrak{F}_\beta$ and $f\in\mathrm{Hom}_R(R/I, E(R))$, it is clear that $K\text{-}dim\ (Im(f)\cap R)\leq\beta$ and hence $Im(f)\cap R=0$ because $\mathrm{rad}^\beta({}_RR)=0$. Since ${}_RR$ is essential in $E(R)$, $Im(f)=0$ and hence $I\in\mathfrak{D}$. Conversely assume $\mathfrak{D}\supseteq\mathfrak{F}_\beta$. Since R is left noetherian, $\mathrm{rad}^\beta({}_RR)$ is finitely generated, say $\mathrm{rad}^\beta({}_RR)=\sum_{i=1}^n Rx_i$ for some $x_i\in\mathrm{rad}^\beta({}_RR)$. Denote by $l(x)$ the left annihilator ideal of x in R. Then $K\text{-}dim\ R/l(x_i)=K\text{-}dim\ Rx_i\leq\beta$. Thus $l(x_i)\in\mathfrak{D}$. This implies that all Rx_i's and $\mathrm{rad}^\beta({}_RR)$ are torsion modules. Thus $\mathrm{rad}^\beta({}_RR)=0$.

Proposition 3.3. *Let R be a left noetherian ring. For every finitely generated R-module M, the following statements hold.*

(1) *If* $\mathfrak{D}\supseteq\mathfrak{F}_\beta$ *and* $K\text{-}dim\ M\leq\beta$, *then M is a torsion module.*

(2) *If* $\mathfrak{F}_\beta\supseteq\mathfrak{D}$ *and M is a torsion module, then* $K\text{-}dim\ M\leq\beta$.

(3) *If* $\mathfrak{D}=\mathfrak{F}_\beta$, *then M is a torsion module if and only if* $K\text{-}dim\ M\leq\beta$.

Proof. Note that both classes of torsion modules and of modules with Krull dimension at most β are closed under taking submodules, factor modules and extension. Thus we may assume that M is cyclic. Then our assertions are all clear.

Let R be a ring with left Krull dimension α. In the remainder of the present paper, we denote by S the set of all regular elements in R. Let $\sum(S)=\{Ra\mid a\in S\}$. It is well-known that $K\text{-}dim\ R/Ra<\alpha$ for every $a\in S$. This shows, in our notation, that $\sum(S)\subseteq\mathfrak{F}$ where $\mathfrak{F}=\bigcup_{\beta<\alpha}\mathfrak{F}_\beta$. It is obvious that R has a classical left quotient ring if and only if $\sum(S)$ is a cofinal family of some topology $\mathfrak{G}$ on R (see [13, XI, §6]). The latter statement means that any $I\in\mathfrak{G}$ contains a regular element and $I\in\sum(S)$ is a member of $\mathfrak{G}$.

Recall that a ring R is said to be a left Kasch ring if $E({}_RR)$ is a cogenerator.

Proposition 3.4. *The following conditions are equivalent for any ring R.*

(1) $\sum(S)$ *is a cofinal family of* $\mathfrak{D}$.

(2) *R has a classical left quotient ring which is a left Kasch ring.*

Proof. Assume that $\sum(S)$ is cofinal in $\mathfrak{D}$. Then R has a classical left quotient ring Q. First we show that Q is the maximal left quotient ring of R. We give here an elementary proof for it. Let q be any element of the maximal left quotient ring of R. Then a left ideal $Rq^{-1}=\{r\in R\mid rq\in R\}$ is dense. By assumption, it contains a regular element c of R. Put $cq=r$, which is an element of R. Then $q=c^{-1}r$ is an element of Q. This shows that Q is the maximal left quotient ring of R. Thus $\mathfrak{D}$ is a left perfect topology on R. For every dense left ideal I, we have $QI=Q$ by our assumption. It follows from Lemma 2.3 that $\mathfrak{D}$ is a strongly perfect topology and hence from [13, XI, Proposition 5.2] that Q is a left Kasch ring. Assume that R has a classical left quotient ring Q which is a left Kasch ring. The family $\mathfrak{G}=\{_R I\subseteq R \mid QI=Q\}$ is a topology and clearly $\sum(S)$ is cofinal in $\mathfrak{G}$. Hence we have only to show $\mathfrak{G}=\mathfrak{D}$. Let E be the injective hull of $_R R$. Then E is also an injective hull of $_Q Q$. Then for a left ideal I of R, $\mathrm{Hom}_R(R/I, E)\cong\mathrm{Hom}_Q(Q/QI, E)$ by Lemma 2.2. Thus $I\in\mathfrak{G}$ implies $I\in\mathfrak{D}$. Conversely $I\in\mathfrak{D}$ implies that QI is dense left left ideal of Q. Since Q is a left Kasch ring, we have $QI=Q$, that is, $I\in\mathfrak{G}$.

4. A criterion for a ring to have a QF two-sided maximal or classical quotient ring

In the present section, we shall give a criterion for a ring to have a QF two-sided maximal or classical quotient ring. Masaike gave a criterion for it, in which he used the notion of "generalized non-singular" (see [9, Theorem 2.1]). We do use nothing but Lambek torsion theoretical notions. Throughout the present section, it is assumed that Q is a maximal left quotient ring of a ring R and we denote by $\mathfrak{D}_l$ the topology of dense left ideals of R, and by $\sum(S)_l$ the family of principal left ideals of R generated by an element in S. Similarly $\mathfrak{D}_r$ and $\sum(S)_r$ are defined.

Lemma 4.1. *Let R be a QF-3 two-sided Kasch ring with finite Lambek dimension. Then R is a QF ring.*

Proof. Let $n=L\text{-}dim_R R$. By [15, Theorem 4], we have $L\text{-}dim\ R_R=n$. Thus, by [14, Proposition 1], R satisfies the ascending chain condition and the descending chain condition on annihilator left ideals and annihilator right ideals. So we prove our statement by showing that R has the double annihilator condition. By left and right symmetry of our assumption, we have only to show $lr(I)=I$ for every left ideal I. Since $r(I)\cong\mathrm{Hom}_R(R/I, R)$, we have $L\text{-}dim\ r(I)=L\text{-}dim\ R/I=n-L\text{-}dim\ I$ by [15, Theorem 4]. Similarly we have $L\text{-}dim\ lr(I)=n-L\text{-}dim\ r(I)$. Thus we have $L\text{-}dim\ lr(I)=L\text{-}dim\ I$. Hence $lr(I)/I$ is a torsion module. Since R is a left Kasch ring, we have $lr(I)=I$.

Theorem 4.2. *For any ring R, the following statements are equivalent.*

(1) *R has a QF two-sided maximal quotient ring.*

(2) *R is a QF-3 ring with finite Lambek dimension, and both $\mathfrak{D}_l$ and $\mathfrak{D}_r$ are strongly perfect topologies.*

Proof. Assume that the statement (1) holds. Then R is a QF-3 ring, and both $\mathfrak{D}_l$ and $\mathfrak{D}_r$ are perfect topologies (see [12, Theorem 1.5]). Let Q be the QF two-sided maximal quotient ring of R. For $I\in\mathfrak{D}_l$, Q/QI is a torsion

Q-module by Lemma 2.2 and hence $Q=QI$ because Q is a QF ring. Thus $\mathfrak{D}_l$ is strongly perfect. Similarly $\mathfrak{D}_r$ is strongly perfect. Assume that ${}_RR$ has a t-chain of length k;

$$ {}_RR = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_k . $$

By Lemma 2.1, $QI_{i-1}/QI_i = Q\otimes_R(I_{i-1}/I_i) \neq 0$ for $1 \leq i \leq k-1$. Thus we have a proper descending chain of left ideals of Q. Since Q is an artinian ring, this shows that $L\text{-}dim_R R$ is finite. Conversely assume that the statement (2) holds. Put $L\text{-}dim_R R = n (<\infty)$. Then $L\text{-}dim\, R_R = n$ by [15, Theorem 4] and R has the QF-3 two-sided maximal quotient ring Q by [10, Proposition 2 and Theorem 1]. Since $\mathfrak{D}_l$ and $\mathfrak{D}_r$ are strongly perfect, Q is a two-sided Kasch ring by [13, XI, Proposition 5.2]. In order to prove that (2) impies (1), we have only to show that $L\text{-}dim_Q Q$ or $L\text{-}dim\, Q_Q$ is finite by Lemma 4.1. Since $E=E({}_RR)=E({}_QQ)$ and Q is a Kasch ring, any Q-module is torsion-free as R-module. Assume that ${}_QQ$ has a t-chain of length k;

$$ {}_QQ = J_0 \supseteq J_1 \supseteq J_2 \supseteq \cdots \supseteq J_k . $$

Then each J_{i-1}/J_i is torsion-free as R-module. Since $L\text{-}dim_R Q = L\text{-}dim_R R = n$, we have $k \leq n$. Thus $L\text{-}dim_Q Q$ is finite and $L\text{-}dim_Q Q \leq n$. This complets the proof of our theorem.

As for the Lambek dimension of Q, we have

Proposition 4.3. *Let R be a ring which has a QF maximal two-sided quotient ring Q. Then we have $L\text{-}dim_R R = L\text{-}dim\ R_R = L\text{-}dim_Q Q = L\text{-}dim\ Q_Q$.*

Proof. We have shown in the proof of Theorem 4.2 that all of them are finite and $n = L\text{-}dim_R R = L\text{-}dim\ R_R \geq L\text{-}dim_Q Q = L\text{-}dim\ Q_Q$. So it is sufficient to show $L\text{-}dim_Q Q \geq n$. Since $L\text{-}dim_R R$ is finite, there exists a finite standard t-chain of ${}_RR$;

$$ R = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n . $$

By Lemma 1.2, I_{i-1}/I_i can be embedded into ${}_RQ$. Since $\mathfrak{D}$ is strongly perfect, $0 \neq QI_{i-1}/QI_i = Q\otimes_R(I_{i-1}/I_i) \hookrightarrow Q\otimes_R Q \cong Q$ as left Q-module by Lemma 2.1. Thus we have $L\text{-}dim_Q Q \geq \sum_{i=1}^n L\text{-}dim_Q(QI_{i-1}/QI_i) \geq n$.

Theorem 4.4. *For any ring R, the following statements are equivalent.*

(1) *R has a QF classical two-sided quotient ring.*

(2) *R is a QF-3 ring with finite left Lambek dimension, and $\sum(S)_l$ and $\sum(S)_r$ are cofinal in $\mathfrak{D}_l$ and $\mathfrak{D}_r$ respectively.*

Proof. That (1) implies (2) follows from Theorem 4.2 and Proposition 3.4. Assume the condition (2). By Proposition 3.4, R has a classical two-sided quotient ring Q because R is QF-3. Also clearly $\mathfrak{D}_l$ is a perfect topology.

It is easy to show that for any left ideal I of R, $QI=Q$ if and only if I contains a regular element in R. Since $\sum(S)_l$ is cofinal in $\mathfrak{D}_l$, $QI=Q$ if and only if $I\in\mathfrak{D}_l$. Hence $\mathfrak{D}_l$ is strongly perfect by Lemma 2.3. It follows from Theorem 4.2 that Q is a QF ring.

5. Krull dimension and maximal quotient rings of QF-3′ 1-Gorenstein rings with zero socle

In the present section, we deal with QF-3′ 1-Gorenstein rings with zero socle.

Theorem 5.1. *Let R be a left noetherian ring with finite left Lambek dimension. If R satisfies the β-restricted minimum condition for left ideals, then $K\text{-}dim_R R\leq\beta+1$.*

Proof. Given any descending chain of left ideals of R;

$$R=I_0\supseteq I_1\supseteq I_2\supseteq \cdots ,$$

we have $L\text{-}dim_R R\geq\sum_{i=1}^{\infty} L\text{-}dim\ I_{i-1}/I_i$. Since $L\text{-}dim_R R$ is finite, there exists an integer n such that $L\text{-}dim\ I_{i-1}/I_i=0$ for all $i>n$. Since R is left noetherian, it is immediate from Proposition 3.3 that $K\text{-}dim\ I_{i-1}/I_i\leq\beta$ for all $i>n$. This shows $K\text{-}dim_R R\leq\beta+1$.

The following theorem follows from the proof of [11, Theorem 2.4] (see [11, Added in Proof (2)]).

Theorem 5.2. *Let R be a right* 1*-Gorenstein ring. Then R satisfies the restricted minimum condition for left ideals.*

Thus, by Theorem 5.1 and Theorem 5.2, we have immediately,

Theorem 5.3. *Any right* 1*-Gorenstein ring with finite left Lambek dimension has left Krull dimension at most one.*

Corollary 5.4. *Any QF-*3′ 1*-Gorenstein ring has Krull dimension at most one on both sides.*

Proof. By result of Sumioka in [16], such a ring has finite Lambek dimension. Hence our assertion follows from Theorem 5.3.

Corollary 5.5 (See [4, (3.52)]). *Any noetherian hereditary ring has Krull dimension at most one on both sides.*

Proof. In order to prove the above statement, it is sufficient to show that such a ring R is QF-3. Let Q be a maximal left quotient ring of R. It is well-known that ${}_RQ$ is injective and flat. Thus R is QF-3 (see [12, Theorem 1.1]).

The following lemma is crucial in studying maximal quotient rings of QF-3′ 1-Gorenstein rings with zero socle.

Lemma 5.6 ([7, Corollary 3.7]). *Let R be a noetherian ring with left Krull dimension one. Further if $Soc(R)=0$, $\sum(S)$ is cofinal family of $\mathfrak{F}$.* (See §3 for the definition of $\mathfrak{F}$.)

We have reached the point to prove our main theorem.

Theorem 5.7. *Let R be a noetherian QF-3 ring with zero socle. Further if R satisfies the restricted minimum condition for left ideals and right ideals, then R has a QF classical two-sided quotient ring.*

Proof. By [14, Proposition 1], both $L\text{-}dim_R R$ and $L\text{-}dim\ R_R$ are finite. Since R satisfies the restricted minimum condition for left ideals, we have $K\text{-}dim_R R \leq 1$ by Theorem 5.1. Since the artinian radical of R is zero, we have $K\text{-}dim_R R=1$. By Lemma 5.6, $\sum(S)_l$ is cofinal in $\mathfrak{F}$. It follows from Proposition 3.2 that $\mathfrak{D}_l$ coincides with $\mathfrak{F}$. Thus $\sum(S)_l$ is cofinal in $\mathfrak{D}_l$. Similarly $\sum(S)_r$ is cofinal in $\mathfrak{D}_r$. Hence it follows from Theorem 4.4 that R has a QF classical two-sided quotient ring.

Corollary 5.8. *Let R be a QF-3′ 1-Gorenstein ring with zero socle. Then R has a QF classical two-sided quotient ring.*

Proof. By Sumioka's result as stated in §1, such a ring is QF-3. Our assertion is immediate from Theorem 5.2 and Theorem 5.7.

6. Noetherian orders in QF rings

In the present section, we shall study noetherian orders in QF rings.

Proposition 6.1. *Let R be a noetherian ring which is a two-sided order in a QF ring. Then R is decomposed into a ring direct sum of a QF ring and a QF-3 ring with zero socle.*

Proof. By Theorem 4.4, R is a QF-3 ring. Let A be the artinian radical of R. By [2, Theorem 10], R is decomposed into a ring direct sum, say $R=A\oplus B$ for some two-sided ideal B of R. Since R is QF-3, so is B. Clearly B has zero artinian radical and hence $Soc(B)=0$. Since R has a QF classical two-sided quotient ring, so does A. Since an artinian ring is its own classical two-sided quotient ring, A is a QF ring.

Lemma 6.2. *Let R be a noetherian ring with zero socle, which is a two-sided order in a QF ring. Then the following statements are equivalent.*

(1) $K\text{-}dim_R R=1$.

(2) *R satisfies the restricted minimum condition for left ideals.*

Proof. Assume the statement (1). It follows from Proposition 3.4 that $\sum(S)$ is cofinal in $\mathfrak{D}$. Since R has zero socle, it follows Proposition 3.2 that $\mathfrak{D}$ contains $\mathfrak{F}$. Since R has left Krull dimension one, $\mathfrak{F}$ contains $\sum(S)$. Hence we have $\mathfrak{D}=\mathfrak{F}$ and hence R satisfies the restricted minimum condition for left ideals. Conversely assume the statement (2). Since R is QF-3, R has finite left Lambek dimension. Thus we have $K\text{-}dim_R R\leq 1$ Theorem 5.1. Thus $K\text{-}dim_R R=1$ because $\mathrm{Soc}(R)=0$.

Proposition 6.3. *Let R be a noetherian ring which is a two-sided order in a QF ring. Then the following statements are equivalent.*

(1) $K\text{-}dim_R R\leq 1$.

(2) *R satisfies the restricted minimum condition for left ideals.*

Proof. It is immediate from Proposition 6.1 and Lemma 6.2.

Finally we show that the converse of Theorem 5.7 holds under a certain condition.

Theorem 6.4. *Let R be a noetherian ring with Krull dimension at most one on both sides. If R is a two-sided order in a QF ring, then R is decomposed into a ring direct sum, say $R=A\oplus B$, where A is a QF ring and B is a noetherian QF-3 ring with zero socle which satisfies the restricted minimum condition for left ideals and right ideals.*

Proof. By Proposition 6.1, we see that R is decomposed into a ring direct sum of a QF ring A and a noetherian QF-3 ring B with zero socle. It is clear that B is a two-sided order in a QF ring and Krull dimension of B is exactly one on both sides. It follows from Proposition 6.3 that B satisfies the restricted minimum condition for left ideals and right ideals.

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