Ichinose, W.
Osaka J. Math.
39 (2002), 181–208

ON CONVERGENCE OF THE FEYNMAN PATH INTEGRAL IN PHASE SPACE

Dedicated to Professor K. Kajitani on his 60th birthday

WATARU ICHINOSE

(Received June 12, 2000)

1. Introduction

It was an interested and important problem to give the description of quantization, i.e. of passing from classical physical systems to the corresponding quantum ones, from the moment that quantum mechanics came into existence. In the end we succeeded in giving it as follows: Let $\mathcal{L}$ be a Lagrangian function. Then the Hamiltonian function $\mathcal{H}$ is defined through the Legendre transformation of $\mathcal{L}$. The Hamiltonian operator $H(t)$ at time t in quantum mechanics is defined from $\mathcal{H}$. It should be noted that $H(t)$ has ordering ambiguities (cf. [14]). Let f be a probability amplitude at time s. Then its temporal evolution can be given by the solution of the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}u(t) = H(t)u(t), \quad u(s) = f. \tag{1.1}$$

On the other hand Feynman proposed an essentially new description in his famous paper [3] which appeared in 1948. His description is based on the notion of a so-called path integral in configuration space. In 1951 Feynman himself generalized this description, using the notion of a path integral in phase space in [4]. Since then, path integrals in phase space have been discussed by many articles in not only quantum mechanics but also quantum field theory. But it has been pointed out that we have hard difficulties of giving a rigorous meaning to the path integral in phase space. There is even a suggestion that such a path integral can not be defined rigorously. For example see chapter 31 in [16] and section 5 in [2]. It seems to us that only Gawędzki's work [7] succeeded in giving a rigorous meaning to the path integral in phase space. His approach is similar to Ito's one in [11] where the path integral in configuration space was studied. The assumptions put on $H(t)$ in [7] will be mentioned later in this section.

In the present paper we study time-slicing approximation of the Feynman path integral in phase space and prove its convergence under some general assumptions. Paths

*Research partially supported by Grant-in-Aid for Scientific No. 10640176 and No. 13640161, Ministry of Education, Science, and Culture, Japanese Government.

in phase space are approximated by piecewise linear functions of time variable in configuration space and piecewise constant functions in momentum space. This approach, which is different from one in [7], for giving a rigorous meaning to the path integral in phase space is very familiar in physics. For example see [5], [6], [15], [16], and [18].

We consider some charged particles in an electromagnetic field. For the sake of simplicity we suppose charge and mass of every particle to be one and $m > 0$, respectively. Let $x \in R^n$ and $t \in [0, T]$. Electric strength and magnetic strength tensor are denoted by $E(t, x) = (E_1, \dots, E_n) \in R^n$ and $(B_{jk}(t, x))_{1 \le j < k \le n} \in R^{n(n-1)/2}$, respectively. Let $V(t, x) \in R$ and $A(t, x) = (A_1, \dots, A_n) \in R^n$ be electromagnetic potentials. That is,

$$\text{(1.2)} \qquad \begin{cases} E_j = -\dfrac{\partial A_j}{\partial t} - \dfrac{\partial V}{\partial x_j} \quad (j = 1, \dots, n), \\ d\left(\displaystyle\sum_{j=1}^{n} A_j dx_j\right) = \displaystyle\sum_{1 \le j < k \le n} B_{jk} dx_j \wedge dx_k \quad \text{on } R^n. \end{cases}$$

Then the Lagrangian function $\mathcal{L}(t, x, \dot{x})$ is given by

$$\text{(1.3)} \qquad \mathcal{L}(t, x, \dot{x}) = \frac{m}{2}|\dot{x}|^2 + \dot{x} \cdot A - V$$

and the Hamiltonian function $\mathcal{H}(t, x, p)$ is defined through the Legendre transformation of $\mathcal{L}$ by

$$\text{(1.4)} \qquad \mathcal{H}(t, x, p) = \frac{1}{2m}|p - A|^2 + V.$$

Let $T^*R^n = R^n_x \times R^n_p$ be phase space and $(T^*R^n)^{[s,t]}$ denote the space of all paths $\zeta : [s, t] \ni \theta \to \zeta(\theta) \in T^*R^n$. The classical action $S(\zeta)$ for $\zeta = (q, p) \in (T^*R^n)^{[s,t]}$ is given by

$$\text{(1.5)} \qquad S(\zeta) = \int_s^t p(\theta) \cdot \dot{q}(\theta) - \mathcal{H}(\theta, q(\theta), p(\theta)) d\theta, \quad \dot{q}(\theta) = \frac{dq}{dt}(\theta)$$

(cf. [1]).

Let $\Delta : 0 = t_0 < t_1 < \cdots < t_\mu = t$ be a subdivision of an interval $[0, t]$ and set $|\Delta| = \max_{1 \le j \le \mu}(t_j - t_{j-1})$. Let $(x^{(j)}, p^{(j)}) \in T^*R^n$ $(j = 0, 1, \dots, \mu - 1)$. Then $q_\Delta = q_\Delta(x^{(0)}, x^{(1)}, \dots, x^{(\mu-1)}, x) \in (R^n)^{[0,t]}$ denotes the piecewise linear function of $\theta \in [0, t]$ joining $(t_j, x^{(j)})$ $(j = 0, 1, \dots, \mu, x^{(\mu)} = x)$ in the order and $p_\Delta = p_\Delta(p^{(0)}, p^{(1)}, \dots, p^{(\mu-1)}) \in (R^n)^{[0,t]}$ the piecewise constant function taking $p^{(j)}$ for $t_j \le \theta < t_{j+1}$ $(j = 0, 1, \dots, \mu - 1)$. Let $\mathcal{S}$ be the space of all rapidly decreasing functions on R^n with semi-norms $|f|_l = \sum_{|\alpha|+k \le l} \sup |\langle \cdot \rangle^k \partial_x^\alpha f(\cdot)|$ $(l = 0, 1, 2, \dots)$ and take a $\chi \in \mathcal{S}$ such that $\chi(0) = 1$. For $\epsilon_j > 0$ $(j = 0, 1, \dots, \mu - 1)$ and

$\epsilon'_k > 0$ ($k = 1, 2, \dots, \mu - 1$) we set $\overline{\epsilon} = (\epsilon_0, \dots, \epsilon_{\mu-1})$ and $\overline{\epsilon}' = (\epsilon'_1, \dots, \epsilon'_{\mu-1})$. We define for $f \in \mathcal{S}$

$$
(1.6)\quad G_{\overline{\epsilon},\overline{\epsilon}'}(\Delta)f = (2\pi\hbar)^{-n\mu} \int_{R^n} \cdots \int_{R^n} (\exp i\hbar^{-1} S(q_\Delta, p_\Delta)) \times \left\{ \prod_{j=1}^{\mu-1} \chi\left(\epsilon_j \sqrt{\frac{t_{j+1} - t_j}{m\hbar}}\, p^{(j)} \right) \chi(\epsilon'_j x^{(j)}) \right\} \chi\left(\epsilon_0 \sqrt{\frac{t_1}{m\hbar}}\, p^{(0)} \right) \times f(x^{(0)}) dp^{(0)} dx^{(0)} dp^{(1)} \cdots dp^{(\mu-1)} dx^{(\mu-1)}.
$$

Our path integral in phase space is defined by

$$
\lim_{|\Delta| \to 0} \Big(\lim_{|\overline{\epsilon}| + |\overline{\epsilon}'| \to 0} G_{\overline{\epsilon},\overline{\epsilon}'}(\Delta) f \Big)
$$

as will be seen in Theorem below.

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ we write $\partial_x^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$, $|\alpha| = \sum_{j=1}^n \alpha_j$, and $\langle x \rangle = \sqrt{1 + |x|^2}$. Let $L^2 = L^2(R^n)$ be the space of all square integrable functions on R^n with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$. Through the present paper we assume that $\partial_x^\alpha V, \partial_x^\alpha A_j$, and $\partial_x^\alpha \partial_t A_j$ are continuous in $[0, T] \times R^n$ for all j and α. Our purpose in the present paper is to prove the following.

Theorem. *We assume that*

$$
(1.7)\quad \begin{aligned} &|\partial_x^\alpha E_j(t, x)| \le C_\alpha, && |\alpha| \ge 1, \\ &|\partial_x^\alpha B_{jk}(t, x)| \le C_\alpha \langle x \rangle^{-(1+\delta)}, && |\alpha| \ge 1 \end{aligned}
$$

in $[0, T] \times R^n$ *for constants* $\delta > 0$ *and* C_α, *where* δ *is independent of* α. *In addition, we suppose*

$$
(1.8)\quad |\partial_x^\alpha A_j| \le C_\alpha \quad \text{for all } \alpha \quad \text{or} \quad \partial_x^\alpha A_j = 0 \quad \text{for } |\alpha| = 2
$$

for each j *and*

$$
(1.9)\quad \begin{aligned} &|\partial_x^\alpha V| \le C_\alpha \langle x \rangle, && |\alpha| \ge 1, \\ &\sum_{j=1}^n |\partial_x^\alpha \partial_t A_j| \le C_\alpha \langle x \rangle^{a^*}, && |\alpha| \ge 1 \end{aligned}
$$

in $[0, T] \times R^n$ *for a constant* $a^* \ge 0$. *Then we have*:

(1) *Let* $|\Delta|$ *be small. Then* $G_{\overline{\epsilon},\overline{\epsilon}'}(\Delta)$ *on* $\mathcal{S}$ *can be extended uniquely to a bounded operator on* L^2. *In addition, as* $|\overline{\epsilon}| + |\overline{\epsilon}'| \to 0$, $G_{\overline{\epsilon},\overline{\epsilon}'}(\Delta) f$ *for* $f \in L^2$ *converges in* L^2. *We write this limit as* $G(\Delta) f$.

(2) *As* $|\Delta| \to 0$, $G(\Delta) f$ *for* $f \in L^2$ *converges in* L^2 *uniformly in* $t \in [0, T]$. *We call this limit the path integral in phase space.*

(3) *The path integral defined by* (2) *satisfies the Schrödinger equation* (1.1) *where* $s = 0$ *and*

$$H(t) = \frac{1}{2m}\sum_{j=1}^{n}(\hbar D_{x_j} - A_j)^2 + V, \quad D_{x_j} = \frac{1}{i}\frac{\partial}{\partial x_j}. \tag{1.10}$$

As a typical example V and A satisfying the assumptions of Theorem we have the following. Let $V = a(t)|x|^2$, where $a(t) \in R$ is continuous. Let $A_j = A_j(x)$ is a linear function of x or a bounded function on R^n such that $|\partial_x^\alpha A_j(x)| \leq C_\alpha \langle x\rangle^{-(1+\delta)}$, $|\alpha| \geq 2$ for a $\delta > 0$.

In [7] Gawędzki gave a rigorous meaning to the path integral in phase space under the assumption that $A = 0$ and V is the Fourier transform of a complex finite measure on R^n. So our assumptions are much more general than his.

We prove Theorem above, mainly using somewhat delicate modification of the argument in [9] and [10], where convergence of the path integral in configuration space was studied. The outline of the proof of Theorem is as follows. Let $\hat{f}$ be the Fourier transform $\int e^{-ix\cdot\xi} f(x)dx$. Let B^a ($a \geq 0$) denote the weighted Sobolev space $\{f \in L^2; \|f\|_{B^a} \equiv \|\langle\cdot\rangle^a f\| + \|\langle\cdot\rangle^a \hat{f}\| < \infty\}$ and B^{-a} its dual space with norm $\|\cdot\|_{B^{-a}}$. We note that the classical action $S_c(q)$ for $q \in (R^n)^{[s,t]}$ in configuration space is given by

$$\begin{aligned} S_c(q) &= \int_s^t \mathcal{L}(\theta, q(\theta), \dot{q}(\theta))d\theta \\ &= \int_s^t \frac{m}{2}|\dot{q}(\theta)|^2 + \dot{q}(\theta)\cdot A(\theta, q(\theta)) - V(\theta, q(\theta))d\theta \end{aligned} \tag{1.11}$$

(cf. [1]). For x, y, and p in R^n let us define $q^{t,s}_{x,y} \in (R^n)^{[s,t]}$ by

$$q^{t,s}_{x,y}(\theta) = y + \frac{\theta - s}{t - s}(x - y) \quad (s \leq \theta \leq t) \tag{1.12}$$

and $\zeta^{t,s}_{x,y,p} \in (T^*R^n)^{[s,t]}$ by

$$\zeta^{t,s}_{x,y,p}(\theta) = (q^{t,s}_{x,y}(\theta), p) \quad (s \leq \theta \leq t). \tag{1.13}$$

Let $\epsilon > 0$ and set for $f \in \mathcal{S}$

$$G_\epsilon(t,s)f = \begin{cases} (2\pi\hbar)^{-n}\displaystyle\iint (\exp i\hbar^{-1}S(\zeta^{t,s}_{x,y,p}))\chi\left(\epsilon\sqrt{\frac{t-s}{m\hbar}}p\right) f(y)dpdy, & s < t, \\ f, \quad s = t. \end{cases} \tag{1.14}$$

Then we can easily have from (1.6)

$$G_{\overline{\epsilon},\overline{\epsilon}'}(\Delta)f = G_{\epsilon_{\mu-1}}(t, t_{\mu-1})\chi(\epsilon'_{\mu-1}\cdot)\cdots G_{\epsilon_1}(t_2, t_1)\chi(\epsilon'_1\cdot)G_{\epsilon_0}(t_1, 0)f. \tag{1.15}$$

We first prove

$$G_\epsilon(t,s)f = \begin{cases} \left(\sqrt{\dfrac{m}{2\pi i\hbar(t-s)}}\right)^n \displaystyle\int (\exp i\hbar^{-1}\Psi(t,s;x,y))f(y)dy \\ \times\left(\sqrt{\dfrac{i}{2\pi}}\right)^n \displaystyle\int \left(\exp \dfrac{-i|p|^2}{2}\right)\chi\left(\epsilon\left\{p + \sqrt{\dfrac{m}{\hbar(t-s)}}(x-y)\right.\right. \\ \left.\left. \quad + \sqrt{\dfrac{t-s}{m\hbar}}\displaystyle\int_0^1 A(t-\theta\rho, x-\theta(x-y))d\theta\right\}\right)dp, \quad s < t, \\ f, \quad s = t, \end{cases} \tag{1.16}$$

where $\sqrt{i} = e^{i\pi/4}$, $\rho = t - s$, and

$$\begin{aligned} \Psi &= S_c(q^{t,s}_{x,y}) + \frac{1}{2m(t-s)}\left\{\left|\int_s^t A(\theta, q^{t,s}_{x,y}(\theta))d\theta\right|^2 \right. \\ &\qquad\left. - (t-s)\int_s^t |A(\theta, q^{t,s}_{x,y}(\theta))|^2 d\theta\right\} \\ &\equiv S_c(q^{t,s}_{x,y}) + F(t,s;x,y). \end{aligned} \tag{1.17}$$

Set

$$G(t,s)f = \begin{cases} \left(\sqrt{\dfrac{m}{2\pi i\hbar(t-s)}}\right)^n \displaystyle\int (\exp i\hbar^{-1}\Psi(t,s;x,y))f(y)dy, \quad s < t, \\ f, \quad s = t. \end{cases} \tag{1.18}$$

Then we prove: (i) There exist constants $\rho^* > 0$ and $C \geq 0$ so that if $0 \leq t - s \leq \rho^*$, both $G(t,s)$ and $G_\epsilon(t,s)$ ($\epsilon > 0$) can be extended uniquely to bounded operators on L^2 and satisfy

$$\sup_{0\leq\epsilon\leq 1} \|G_\epsilon(t,s)f\| \leq C\|f\|, \quad f \in L^2, \tag{1.19}$$

where $G_0(t,s) = G(t,s)$. (ii) As $\epsilon \to 0$, $G_\epsilon(t,s)f$ for $f \in L^2$ converges to $G(t,s)f$ in L^2. From these results (i), (ii), and (1.15) we can easily prove the first statement (1) in Theorem and

$$G(\Delta) = G(t, t_{\mu-1})G(t_{\mu-1}, t_{\mu-2})\cdots G(t_2, t_1)G(t_1, 0). \tag{1.20}$$

We prove the statements (2) and (3) in Theorem, using (1.20) and showing as in [9] and [10] that $G(t,s)$ is stable and consistent for the initial problem (1.1) in the sense of words used in the theory of difference methods (cf. [13]).

The plan in the present paper is as follows. We prove (1.16) in section 2. In section 3 we show preliminary results. In section 4 it is shown that $G(t,s)$ is stable. In section 5 we show boundedness results for integral operators on B^a. In section 6 we prove (1.19) and (1.20). In section 7 we show that $G(t,s)$ is consistent for (1.1) and then complete the proof of Theorem.

2. Representation in configuration space

Lemma 2.1. *Let x, y, and p in R^n and $s < t$. We consider $q \in (R^n)^{[s,t]}$ such that $q(s) = y$ and $q(t) = x$. Set $\zeta = (q, p) \in (T^*R^n)^{[s,t]}$. Then we have*

$$\begin{aligned} S(\zeta) \;=\; &-\frac{(t-s)}{2m}|p-p^*|^2 + \left\{ \frac{m|x-y|^2}{2(t-s)} + \frac{(x-y)}{(t-s)}\cdot\int_s^t A(\theta,q(\theta))d\theta \right. \\ &\left. - \int_s^t V(\theta,q(\theta))d\theta \right\} \\ &+\frac{1}{2m(t-s)}\left\{ \left| \int_s^t A(\theta,q(\theta))d\theta \right|^2 - (t-s)\int_s^t |A(\theta,q(\theta))|^2 d\theta \right\}, \end{aligned}$$

where

$$p^* = \frac{1}{(t-s)}\int_s^t \frac{\partial\mathcal{L}}{\partial\dot{x}}(\theta,q(\theta),\dot{q}(\theta))d\theta = \frac{1}{(t-s)}\int_s^t m\dot{q}(\theta) + A(\theta,q(\theta))d\theta.$$

Proof. We have from (1.4) and (1.5)

$$S(\zeta) = \int_s^t p\cdot\dot{q}(\theta) - \frac{1}{2m}|p - A(\theta,q(\theta))|^2 - V(\theta,q(\theta))d\theta \tag{2.1}$$

and so together with (1.3)

$$\begin{aligned} \frac{\partial}{\partial p}S(\zeta) \;&=\; \int_s^t \dot{q}(\theta) - \frac{1}{m}\big(p - A(\theta,q(\theta))\big)d\theta \\ &=\; -\frac{1}{m}(t-s)p + \frac{1}{m}\int_s^t \frac{\partial\mathcal{L}}{\partial\dot{x}}(\theta,q(\theta),\dot{q}(\theta))d\theta, \end{aligned}$$

where $\partial S(\zeta)/\partial p = (\partial S(\zeta)/\partial p_1, \ldots, \partial S(\zeta)/\partial p_n)$. So the equation $\partial S(\zeta)/\partial p = 0$ in p is equivalent to $p = p^*$. Consequently, noting that $S(\zeta)$ is a polynomial of degree 2 in p, we have from (2.1)

$$S(\zeta) = -\frac{(t-s)}{2m}|p-p^*|^2 + S(\zeta)|_{p=p^*}. \tag{2.2}$$

We use the assumption $q(s)=y$ and $q(t)=x$. Then we have from (2.1)

$$\begin{aligned} S(\zeta) &= p\cdot(x-y)-\frac{1}{2m}\left\{(t-s)|p|^2-2p\cdot\int_s^t Ad\theta+\int_s^t |A|^2 d\theta\right\}-\int_s^t V d\theta \\ &= \frac{p}{m}\cdot\left\{m(x-y)+\int_s^t Ad\theta\right\}-\frac{(t-s)}{2m}|p|^2-\int_s^t Vd\theta-\frac{1}{2m}\int_s^t |A|^2 d\theta. \end{aligned}$$

Using

$$p^* = \frac{1}{(t-s)}\int_s^t m\dot{q}(\theta)+A(\theta,q(\theta))d\theta = \frac{1}{(t-s)}\left\{m(x-y)+\int_s^t Ad\theta\right\},$$

we have

$$\begin{aligned} S(\zeta)|_{p=p^*} &= \frac{1}{m(t-s)}\left|m(x-y)+\int_s^t Ad\theta\right|^2-\frac{1}{2m(t-s)} \\ &\quad\times\left|m(x-y)+\int_s^t Ad\theta\right|^2-\int_s^t Vd\theta-\frac{1}{2m}\int_s^t |A|^2 d\theta \\ &= \left\{\frac{m|x-y|^2}{2(t-s)}+\frac{(x-y)}{(t-s)}\cdot\int_s^t Ad\theta-\int_s^t Vd\theta\right\} \\ &\quad+\frac{1}{2m(t-s)}\left\{\left|\int_s^t Ad\theta\right|^2-(t-s)\int_s^t |A|^2 d\theta\right\}. \end{aligned}$$

So we can prove Lemma 2.1 from (2.2). □

Proposition 2.2. *Let* $G_\epsilon(t,s)$ *be the operator defined by* (1.14). *Then we have* (1.16).

Proof. Let $t>s$. We can write from (1.11) and (1.12)

$$\begin{aligned} S_c(q^{t,s}_{x,y}) &= \int_s^t \frac{m}{2}|\dot{q}^{t,s}_{x,y}(\theta)|^2+\dot{q}^{t,s}_{x,y}(\theta)\cdot A(\theta,q^{t,s}_{x,y}(\theta))-V(\theta,q^{t,s}_{x,y}(\theta))d\theta \\ &= \frac{m|x-y|^2}{2(t-s)}+\frac{(x-y)}{t-s}\cdot\int_s^t A(\theta,q^{t,s}_{x,y}(\theta))d\theta-\int_s^t Vd\theta. \end{aligned} \tag{2.3}$$

Consequently we have from Lemma 2.1

$$S(\zeta^{t,s}_{x,y,p}) = -\frac{(t-s)}{2m}|p-p^*|^2+\Psi(t,s;x,y), \tag{2.4}$$

$$\begin{aligned} p^* &= \frac{1}{(t-s)}\left\{m(x-y)+\int_s^t A(\theta,q^{t,s}_{x,y}(\theta))d\theta\right\} \\ &= \frac{1}{(t-s)}\left\{m(x-y)+(t-s)\int_0^1 A(t-\theta(t-s),x-\theta(x-y))d\theta\right\}, \end{aligned} \tag{2.5}$$

where Ψ is the function defined by (1.17). Substituting (2.4) into (1.14) and making the change of variables: $R^n \ni p \to p' = \sqrt{(t-s)/(m\hbar)}(p - p^*) \in R^n$, we have

$$G_\epsilon(t,s)f = (2\pi\hbar)^{-n}\left(\sqrt{\frac{m\hbar}{t-s}}\right)^n \int (\exp i\hbar^{-1}\Psi) f(y)dy \times \int \left(\exp \frac{-i|p'|^2}{2}\right) \chi\left(\epsilon\sqrt{\frac{t-s}{m\hbar}}\left(p^* + \sqrt{\frac{m\hbar}{t-s}}p'\right)\right)dp'.$$

Hence we obtain (1.16) from (2.5). □

3. Preliminary results

Let $M \geq 0$ be a constant and $p(x,w)$ an infinitely differentiable function in R^{2n} such that

$$|\partial_w^\alpha \partial_x^\beta p(x,w)| \leq C_{\alpha,\beta}\langle x;w\rangle^M, \quad x, w \in R^n \tag{3.1}$$

for all α and β with constants $C_{\alpha,\beta}$, where $\langle x;w\rangle = \sqrt{1+|x|^2+|w|^2}$. Let $\Psi(t,s;x,y)$ be the function defined by (1.17). For $f \in \mathcal{S}$ we define

(3.2) $P(t,s)f$

$$= \begin{cases} \left(\sqrt{\dfrac{m}{2\pi i\hbar(t-s)}}\right)^n \displaystyle\int (\exp i\hbar^{-1}\Psi(t,s;x,y))p\left(x, \frac{x-y}{\sqrt{t-s}}\right) f(y)dy, & s < t, \\ \left(\sqrt{\dfrac{m}{2\pi i\hbar}}\right)^n \mathrm{Os} - \displaystyle\int \left(\exp \frac{i\hbar^{-1}m|w|^2}{2}\right) p(x,w)dwf(x), & s = t, \end{cases}$$

where $\mathrm{Os} - \int g(w)dw$ means the oscillatory integral $\lim_{\epsilon\to 0}\int \chi(\epsilon w)g(w)dw$. It is easy to see that the formal adjoint operator $P(t,s)^*$ of $P(t,s)$, defined by the relation $(Pu, v) = (u, P^*v)$ for u and v in $\mathcal{S}$, is written as

(3.3) $P(t,s)^* f$

$$= \begin{cases} \left(\sqrt{\dfrac{im}{2\pi\hbar(t-s)}}\right)^n \displaystyle\int (\exp -i\hbar^{-1}\Psi(t,s;y,x))\overline{p\left(y, \frac{y-x}{\sqrt{t-s}}\right)} f(y)dy, & s < t, \\ \left(\sqrt{\dfrac{im}{2\pi\hbar}}\right)^n \mathrm{Os} - \displaystyle\int \left(\exp \frac{-i\hbar^{-1}m|w|^2}{2}\right) \overline{p(x,w)}dwf(x), & s = t, \end{cases}$$

where $\overline{p(x,w)}$ is the complex conjugate of $p(x,w)$.

REMARK 3.1. Set $p(x,w) = 1$ in (3.2). Then we have $P(t,s) = G(t,s)$ from (1.18) and

$$\mathrm{Os} - \int \left(\exp \frac{i\hbar^{-1}m|w|^2}{2}\right)dw = \left(\sqrt{\frac{2\pi i\hbar}{m}}\right)^n. \tag{3.4}$$

Lemma 3.1. *Let $p(x, w)$ be a function satisfying* (3.1). *We assume that there exist constants $M' \geq 0$ and C_α satisfying*

$$|\partial_x^\alpha V| + \sum_{j=1}^n |\partial_x^\alpha A_j| \leq C_\alpha \langle x \rangle^{M'}, \quad (t, x) \in [0, T] \times R^n \tag{3.5}$$

for all α. Let $f \in \mathcal{S}$. Then $\partial_x^\alpha (P(t, s) f)$ are continuous in $0 \leq s \leq t \leq T$ and $x \in R^n$ for all α.

Proof. We can write $F(t, s; x, y)$ defined by (1.17) as

$$\begin{aligned} F(t, s; x, y) = \frac{\rho}{2m} \Bigg\{ \left| \int_0^1 A(t - \theta\rho, x - \theta(x - y)) d\theta \right|^2 \\ - \int_0^1 |A(t - \theta\rho, x - \theta(x - y))|^2 d\theta \Bigg\}, \quad \rho = t - s. \end{aligned} \tag{3.6}$$

We can also write from (2.3)

$$\begin{aligned} S_c(q_{x,y}^{t,s}) = \frac{m|x - y|^2}{2(t - s)} + (x - y) \cdot \int_0^1 A(t - \theta\rho, x - \theta(x - y)) d\theta \\ -\rho \int_0^1 V(t - \theta\rho, x - \theta(x - y)) d\theta. \end{aligned} \tag{3.7}$$

Make the change of variables: $R^n \ni y \to w = (x - y)/\sqrt{t - s} \in R^n$ in (3.2). Then, using (1.17) and (3.7), we have

$$P(t, s) f = \left(\sqrt{\frac{m}{2\pi i \hbar}} \right)^n \text{Os} - \int e^{i\hbar^{-1}\psi(t,s;x,w)} p(x, w) f(x - \sqrt{\rho} w) dw, \quad s \leq t, \tag{3.8}$$

$$\begin{aligned} \psi(t, s; x, w) &= \frac{m}{2}|w|^2 + \sqrt{\rho} w \cdot \int_0^1 A(t - \theta\rho, x - \theta\sqrt{\rho} w) d\theta \\ &\quad - \rho \int_0^1 V(t - \theta\rho, x - \theta\sqrt{\rho} w) d\theta + F(t, s; x, x - \sqrt{\rho} w) \\ &\equiv \frac{m}{2}|w|^2 + \phi(t, s; x, \sqrt{\rho} w), \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} \phi(t, s; x, \xi) = \xi \cdot \int_0^1 A(t - \theta\rho, x - \theta\xi) d\theta - \rho \int_0^1 V(t - \theta\rho, x - \theta\xi) d\theta \\ + F(t, s; x, x - \xi). \end{aligned}$$

We have from the assumption (3.5) together with (3.6)

$$|\partial_\xi^\alpha \partial_x^\beta \phi| \leq C_{\alpha,\beta} \langle x; \xi \rangle^{2M'+1}, \quad 0 \leq s \leq t \leq T, \ x, \ \xi \in R^n \tag{3.10}$$

for all α and β.

Let $L = \langle w\rangle^{-2}(1 - i\hbar m^{-1}\sum_{j=1}^{n} w_j\partial_{w_j})$ and ${}^tL$ its transposed operator. Let $0 < \epsilon \le 1$. Then we have from (3.9)

$$\int e^{i\hbar^{-1}\psi(t,s;x,w)}\chi(\epsilon w)p(x,w)f(x-\sqrt{\rho}w)dw = \int e^{i\hbar^{-1}m|w|^2/2}({}^tL)^l\{e^{i\hbar^{-1}\phi(t,s;x,\sqrt{\rho}w)}\chi(\epsilon w)p(x,w)f(x-\sqrt{\rho}w)\}dw \tag{3.11}$$

for $l = 0, 1, 2, \ldots$. Noting $f \in \mathcal{S}$, we see from (3.1) and (3.10)

$$|({}^tL)^l\{e^{i\hbar^{-1}\phi(t,s;x,\sqrt{\rho}w)}\chi(\epsilon w)p(x,w)f(x-\sqrt{\rho}w)\}| \le C_{l,N}\langle w\rangle^{-l}\langle x;\sqrt{\rho}w\rangle^{l(2M'+1)}\langle x;w\rangle^{M}\langle x-\sqrt{\rho}w\rangle^{-N}$$

for any $N = 0, 1, 2, \ldots$, where $C_{l,N}$ is a constant independent of $0 < \epsilon \le 1$. So using $\langle x; y\rangle \le \langle x\rangle\langle y\rangle$ and $\langle x+y\rangle^{-1} \le \sqrt{2}\langle x\rangle\langle y\rangle^{-1}$, we have

$$|({}^tL)^l\{e^{i\hbar^{-1}\phi}\chi(\epsilon w)p(x,w)f(x-\sqrt{\rho}w)\}| \le C'_{l,N}\langle x\rangle^{l(2M'+1)+M+N}\langle\sqrt{\rho}w\rangle^{l(2M'+1)-N}\langle w\rangle^{M-l}.$$

Take l and N so that $l \ge M + n + 1$ and $N \ge l(2M' + 1)$. Then

$$|({}^tL)^l\{e^{i\hbar^{-1}\phi(t,s;x,\sqrt{\rho}w)}\chi(\epsilon w)p(x,w)f(x-\sqrt{\rho}w)\}| \le C\langle x\rangle^{l(2M'+1)+M+N}\langle w\rangle^{-(n+1)}, \quad 0 \le s \le t \le T \tag{3.12}$$

with a constant C independent of $0 < \epsilon \le 1$. Hence, applying the Lebesgue dominated convergence theorem to (3.11), we see from (3.8) and (3.9) that $P(t,s)f$ is continuous in $0 \le s \le t \le T$ and $x \in R^n$. Noting (3.10), we can prove in the same way that $\partial_x^\alpha(P(t,s)f)$ for all α are also continuous. □

Let x, y, and z in R^n and $0 \le s \le t \le T$. We set for $0 \le \sigma_2 \le \sigma_1 \le 1$

$$\begin{cases} \tau(\sigma) = \tau(\sigma_1,\sigma_2) = t - \sigma_1(t-s) \in R, \\ \gamma(\sigma) = \gamma(\sigma_1,\sigma_2;x,y,z) = z + \sigma_1(x-z) + \sigma_2(y-x) \in R^n. \end{cases} \tag{3.13}$$

We also set

$$\begin{aligned} B_{jk} &= -B_{kj}, & 1 \le k < j \le n, \\ B_{jj} &= 0, & j = 1, 2, \ldots, n. \end{aligned} \tag{3.14}$$

Lemma 3.2. *Let $p(x,w)$ be a function satisfying* (3.1). *Let $f \in \mathcal{S}$. Then for any $0 < \epsilon \le 1$ and $0 \le s < t \le T$ we have*

(3.15) $$P(t,s)^*|\chi(\epsilon\cdot)|^2 P(t,s)f = \left(\frac{m}{2\pi\hbar(t-s)}\right)^n \int f(y)dy \int |\chi(\epsilon z)|^2 \times \left(\exp i(x-y)\cdot\frac{m\Phi}{\hbar(t-s)}\right)\overline{p\left(z,\frac{z-x}{\sqrt{t-s}}\right)}p\left(z,\frac{z-y}{\sqrt{t-s}}\right)dz,$$

where $\Phi = \Phi(t,s;x,y,z) = (\Phi_1,\ldots,\Phi_n)$ *and*

(3.16) $$\begin{aligned}\Phi_j &= z_j - \frac{x_j+y_j}{2} + \frac{(t-s)}{m}\int_0^1 A_j(s,y+\theta(x-y))d\theta \\ &\quad - \frac{(t-s)}{m}\sum_{k=1}^n (z_k-x_k)\int_0^1\int_0^{\sigma_1} B_{jk}(\tau(\sigma),\gamma(\sigma))d\sigma_2 d\sigma_1 \\ &\quad - \frac{(t-s)^2}{m}\int_0^1\int_0^{\sigma_1} E_j(\tau(\sigma),\gamma(\sigma))d\sigma_2 d\sigma_1 \\ &\quad - \frac{(t-s)}{m}\int_0^1 (\partial_{y_j}F)(t,s;z,y+\theta(x-y))d\theta.\end{aligned}$$

Proof. It follows from (3.2) and (3.3) that

(3.17) $$\begin{aligned}P(t,s)^*|\chi(\epsilon\cdot)|^2 P(t,s)f &= \left(\frac{m}{2\pi\hbar(t-s)}\right)^n \int f(y)dy \\ &\times \int |\chi(\epsilon z)|^2\{\exp -i\hbar^{-1}(\Psi(t,s;z,x)-\Psi(t,s;z,y))\} \\ &\times \overline{p\left(z,\frac{z-x}{\sqrt{t-s}}\right)}p\left(z,\frac{z-y}{\sqrt{t-s}}\right)dz.\end{aligned}$$

We write from (1.17)

(3.18) $$\begin{aligned}&\Psi(t,s;z,x)-\Psi(t,s;z,y) \\ &= S_c(q^{t,s}_{z,x}) - S_c(q^{t,s}_{z,y}) + F(t,s;z,x) - F(t,s;z,y).\end{aligned}$$

Set $\mathbf{A} = (-V, A)$, $\mathbf{x} = (t,x)$, and $\boldsymbol{q}^{t,s}_{x,y}(\theta) = (\theta, q^{t,s}_{x,y}(\theta))$. Then we can write from (2.3)

$$S_c(q^{t,s}_{x,y}) = \frac{m|x-y|^2}{2(t-s)} + \int_{\boldsymbol{q}^{t,s}_{x,y}} \mathbf{A}\cdot d\mathbf{x}.$$

So, using the Stokes theorem, we have

$$\begin{aligned}&S_c(q^{t,s}_{z,x}) - S_c(q^{t,s}_{z,y}) \\ &= -(x-y)\cdot\frac{m}{t-s}\left(z-\frac{x+y}{2}\right) - \int_{\boldsymbol{q}^{s,s}_{x,y}} \mathbf{A}\cdot d\mathbf{x} - \iint_{\boldsymbol{\Delta}} d(\mathbf{A}\cdot d\mathbf{x}),\end{aligned}$$

where $\boldsymbol{\Delta}$ is the 2-dimensional plane with oriented boundary consisting of $-\boldsymbol{q}^{s,s}_{x,y}$, $\boldsymbol{q}^{t,s}_{z,y}$, and $-\boldsymbol{q}^{t,s}_{z,x}$. Noting that $\sigma = (\sigma_1,\sigma_2)$ in (3.13) are coordinates with positive orientation on $\boldsymbol{\Delta}$, we obtain as in section 3 of [9] and as in the proof of Lemma 2.2 in [10]

$$
\begin{aligned}
(3.19)\qquad & S_c(q^{t,s}_{z,x}) - S_c(q^{t,s}_{z,y}) \\
&= -(x-y)\cdot \frac{m}{t-s}\left(z - \frac{x+y}{2}\right) - (x-y)\cdot \int_0^1 A(s, y+\theta(x-y))d\theta \\
&\quad + \sum_{j=1}^{n}(x_j - y_j)\sum_{k=1}^{n}(z_k - x_k)\int_0^1\int_0^{\sigma_1} B_{jk}(\tau(\sigma),\gamma(\sigma))d\sigma_2 d\sigma_1 \\
&\quad + (t-s)(x-y)\cdot\int_0^1\int_0^{\sigma_1} E(\tau(\sigma),\gamma(\sigma))d\sigma_2 d\sigma_1 .
\end{aligned}
$$

Hence we can prove Lemma 3.2 from (3.17)-(3.19). □

4. Stability of $G(t, s)$

We write $\Phi = \Phi(t, s; x, y, z)$ defined by (3.16) as

$$
\begin{aligned}
(4.1)\qquad \Phi(t,s;x,y,z) &= z - \frac{x+y}{2} + \frac{(t-s)}{m}\int_0^1 A(s, y+\theta(x-y))d\theta \\
&\quad - \frac{(t-s)}{m}B'(t,s;x,y,z) - \frac{(t-s)^2}{m}E'(t,s;x,y,z) \\
&\quad - \frac{(t-s)}{m}F'(t,s;x,y,z),
\end{aligned}
$$

where $E' = (E'_1, \dots, E'_n)$, $B' = (B'_1, \dots, B'_n)$, and $F' = (F'_1, \dots, F'_n)$.

Lemma 4.1. *Assume* (1.7) *and* (1.8) *in* Theorem. *Let α, β and γ be multi-indices such that $|\alpha + \beta + \gamma| \geq 1$. Then we have for $j = 1, 2, \dots, n$*

$$(4.2)\qquad |\partial_x^\alpha \partial_y^\beta \partial_z^\gamma E'_j| \leq C_{\alpha,\beta,\gamma},$$

$$(4.3)\qquad |\partial_x^\alpha \partial_y^\beta \partial_z^\gamma B'_j| \leq C_{\alpha,\beta,\gamma},$$

$$(4.4)\qquad |\partial_x^\alpha \partial_y^\beta \partial_z^\gamma F'_j| \leq C_{\alpha,\beta,\gamma},\ \ 0 \leq s \leq t \leq T, x, y, z \in R^n.$$

Proof. The inequalities (4.2) and (4.3) can be proved from (1.7). These have been already proved in page 28 of [9] and Lemma 3.1 of [10]. We can easily prove (4.4) from (1.8), noting (3.6). □

Proposition 4.2. *Assume* (1.7) *and* (1.8). *Then we have*:
(1) *There exist constants $\rho^* > 0$ and $c_0 > 0$ such that the mapping*: $R^n \ni z \to \xi = \Phi(t, s; x, y, z) \in R^n$ *is homeomorphic and* $\det \partial\Phi/\partial z \geq c_0$ *for all $0 \leq t - s \leq \rho^*$, x, and y. We write its inverse mapping as $R^n \ni \xi \to z = z(t, s; x, y, \xi) = (z_1, \dots, z_n) \in R^n$.*

(2) *We have for* $j = 1, 2, \ldots, n$

$$|\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma z_j(t, s; x, y, \xi)| \le C_{\alpha,\beta,\gamma}, \quad |\alpha + \beta + \gamma| \ge 1, \\ 0 \le t - s \le \rho^*, \ \ x, y, \xi \in R^n. \tag{4.5}$$

Proof. It follows from (4.1) that

$$\frac{\partial \Phi}{\partial z} = I_n - \frac{(t-s)}{m}\frac{\partial B'}{\partial z} - \frac{(t-s)^2}{m}\frac{\partial E'}{\partial z} - \frac{(t-s)}{m}\frac{\partial F'}{\partial z}, \tag{4.6}$$

where I_n is the identity matrix. So, using Theorem 1.22 in [17], we can prove the statement (1) from Lemma 4.1.

We note that we have from (1.8) for $j = 1, 2, \ldots, n$

$$|\partial_x^\alpha A_j(t, x)| \le C_\alpha, \quad |\alpha| \ge 1 \tag{4.7}$$

in $[0, T] \times R^n$. So we obtain together with Lemma 4.1

$$|\partial_x^\alpha \partial_y^\beta \partial_z^\gamma \Phi| \le C_{\alpha,\beta,\gamma}, \quad |\alpha + \beta + \gamma| \ge 1, \ \ 0 \le s \le t \le T, \ \ x, y, z \in R^n. \tag{4.8}$$

Hence we can prove (2) from (1). □

We fix $\rho^* > 0$ determined in Proposition 4.2 hereafter.

Theorem 4.3. *Assume* (1.7) *and* (1.8). *Let* $G(t, s)$ *be the operator on* $\mathcal{S}$ *defined by* (1.18) *and* $0 \le t - s \le \rho^*$. *Then* $G(t, s)$ *can be extended to a bounded operator on* L^2. *In addition, there exists a constant* $K \ge 0$ *such that*

$$\|G(t, s)f\| \le e^{K(t-s)}\|f\|, \quad 0 \le t - s \le \rho^*, f \in L^2. \tag{4.9}$$

Proof. The proof below is analogous to that of Theorem 3.7 in [9] and of Theorem 3.3 in [10]. Let $t = s$. Then (4.9) is clear. Let $0 < t - s \le \rho^*$. It follows from Remark 3.1 and Lemma 3.2 that

$$G(t, s)^* |\chi(\epsilon\cdot)|^2 G(t, s) f \\ = \left(\frac{m}{2\pi\hbar(t-s)}\right)^n \int f(y)dy \int |\chi(\epsilon z)|^2 \left(\exp i(x-y)\cdot \frac{m\Phi}{\hbar(t-s)}\right) dz.$$

We can make the change of variables: $R^n \ni z \to \xi = \Phi(t, s; x, y, z) \in R^n$ from Proposition 4.2. Then

$$G(t,s)^*|\chi(\epsilon\cdot)|^2 G(t,s)f = \left(\frac{m}{2\pi\hbar(t-s)}\right)^n \int f(y)dy \times \int |\chi(\epsilon z(t,s;x,y,\xi))|^2 \left(\exp i(x-y)\cdot\frac{m\xi}{\hbar(t-s)}\right) \det\frac{\partial z}{\partial\xi} d\xi. \tag{4.10}$$

We can prove from (4.6), Lemma 4.1, and Proposition 4.2 that

$$\det\frac{\partial z}{\partial\xi} = 1 + (t-s)b(t,s;x,y,\xi), \tag{4.11}$$

$$|\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma b(t,s;x,y,\xi)| \le C_{\alpha,\beta,\gamma} \quad \text{for all } \alpha, \beta, \text{ and } \gamma. \tag{4.12}$$

Consequently we can write

$$G(t,s)^*|\chi(\epsilon\cdot)|^2 G(t,s)f = \left(\frac{1}{2\pi}\right)^n \int f(y)dy \int |\chi(\epsilon z(t,s;x,y,\xi))|^2 \times e^{i(x-y)\cdot\eta}\big(1+(t-s)b(t,s;x,y,\xi)\big)d\eta, \quad \xi = \frac{\hbar(t-s)\eta}{m}. \tag{4.13}$$

Noting (4.5), we can prove from (4.13) for $f \in \mathcal{S}$

$$\lim_{\epsilon\to 0} G(t,s)^*|\chi(\epsilon\cdot)|^2 G(t,s)f = f + (t-s)\left(\frac{1}{2\pi}\right)^n \mathrm{Os}-\iint e^{i(x-y)\cdot\eta} b\left(t,s;x,y,\frac{\hbar(t-s)\eta}{m}\right) f(y)dyd\eta \tag{4.14}$$

in the topology of $\mathcal{S}$. The second term on the right-hand side of (4.14) is a pseudo-differential operator with double symbol (cf. [12]). It follows from (4.12) that we can apply the Calderón-Vaillancourt theorem (cf. [12]) to this term. Then there exists a constant $K \ge 0$ such that the L^2-norm of this term is bounded by $2K(t-s)\|f\|$ for all $f \in \mathcal{S}$. Consequently we have

$$\lim_{\epsilon\to 0}\|\chi(\epsilon\cdot)G(t,s)f\|^2 = \lim_{\epsilon\to 0}(G(t,s)^*|\chi(\epsilon\cdot)|^2 G(t,s)f, f) \le \mathrm{e}^{2K(t-s)}\|f\|^2$$

and so by the Fatou lemma

$$\|G(t,s)f\| \le \mathrm{e}^{K(t-s)}\|f\|, \quad f \in \mathcal{S}.$$

Hence we can easily complete the proof of Theorem 4.3. □

The corollary below follows from Theorem 4.3.

Corollary 4.4. *Assume* (1.7) *and* (1.8). *Let* $|\Delta| \le \rho^*$. *Then we have for all* $f \in L^2$

$$\|G(t,t_{\mu-1})G(t_{\mu-1},t_{\mu-2})\cdots G(t_2,t_1)G(t_1,0)f\| \le e^{Kt}\|f\|, \quad 0 \le t \le T. \tag{4.15}$$

REMARK 4.1. Let $F(t, s; x, y)$ be a function satisfying (4.4), where

$$F'_j = \int_0^1 \left(\frac{\partial F}{\partial y_j}\right)(t, s; z, y + \theta(x - y))d\theta$$

and define $\Psi(t, s; x, y)$ by $S_c(q^{t,s}_{x,y}) + F(t, s; x, y)$ and $G(t, s)$ by (1.18), respectively. Assume (4.2), (4.3), and (4.7). Then we can prove that the same results as in Theorem 4.3 and Corollary 4.4 hold, following our proofs.

5. Boundedness of integral operators

Lemma 5.1. *Assume* (1.7) *and* (1.8). *Let* $z_j(t, s; x, y, \xi)$ $(j = 1, 2, \dots, n)$ *be the function defined in* Proposition 4.2. *Then*

$$\frac{1}{\sqrt{\rho}}\left(z_j\left(t, s; x, x + \sqrt{\rho}y, \frac{\hbar\rho\eta}{m} + \frac{\hbar\sqrt{\rho}\eta'}{m}\right) - x_j\right)$$

can be extended to be continuous in $0 \le t - s \le \rho^*, x, y, \eta,$ *and* η' *in* R^n, *where* $\rho = t - s$. *We also have for* $j = 1, 2, \dots, n$

$$\left|z_j\left(t, s; x, x + \sqrt{\rho}y, \frac{\hbar\rho\eta}{m} + \frac{\hbar\sqrt{\rho}\eta'}{m}\right) - x_j\right| \le C\sqrt{\rho}(1 + |x| + |y| + |\eta| + |\eta'|), \tag{5.1}$$

$$\left|\partial^{\alpha}_{\eta}\partial^{\alpha'}_{\eta'}\partial^{\beta}_{x}\partial^{\beta'}_{y}\left(z_j\left(t, s; x, x + \sqrt{\rho}y, \frac{\hbar\rho\eta}{m} + \frac{\hbar\sqrt{\rho}\eta'}{m}\right) - x_j\right)\right| \le C_{\alpha,\alpha',\beta,\beta'}\sqrt{\rho}, \tag{5.2}$$

$$|\alpha + \alpha' + \beta + \beta'| \ge 1,\ \ 0 \le t - s \le \rho^*,\ \ x, y, \eta, \eta' \in R^n.$$

Proof. Let $z = z(t, s; x, x + \sqrt{\rho}y, \hbar\rho\eta/m + \hbar\sqrt{\rho}\eta'/m)$. Then we have from (4.1)

$$\begin{aligned}\frac{\hbar\rho\eta}{m} + \frac{\hbar\sqrt{\rho}\eta'}{m} &= z - \frac{2x + \sqrt{\rho}y}{2} + \frac{\rho}{m}\int_0^1 A(s, x + (1 - \theta)\sqrt{\rho}y)d\theta \\ &\quad - \frac{\rho}{m}B'(t, s; x, x + \sqrt{\rho}y, z) - \frac{\rho^2}{m}E'(t, s; x, x + \sqrt{\rho}y, z) \\ &\quad - \frac{\rho}{m}F'(t, s; x, x + \sqrt{\rho}y, z)\end{aligned}$$

and so

$$\frac{z - x}{\sqrt{\rho}} = \frac{y}{2} + \frac{\hbar\sqrt{\rho}\eta}{m} + \frac{\hbar\eta'}{m} - \frac{\sqrt{\rho}}{m}\int_0^1 A d\theta + \frac{\sqrt{\rho}}{m}B' + \frac{\rho^{3/2}}{m}E' + \frac{\sqrt{\rho}}{m}F'. \tag{5.3}$$

Hence we can easily complete the proof, using (4.7), Lemma 4.1, and Proposition 4.2. □

We can prove Lemma 5.2, Proposition 5.3, and Theorem 5.4 below as in the proofs of Lemma 4.2, Proposition 4.3, and Theorem 4.4 in [10], respectively. So we give a little rough sketch of their proofs.

Lemma 5.2. *Assume* (1.7) *and* (1.8). *Let* $p(x, w)$ *be a function satisfying* (3.1) *and* $P(t, s)$ *the operator defined by* (3.2). *Let* $0 \le t - s \le \rho^*$ *and set*

$$(5.4)\quad q(t, s; x, \eta) = \left(\frac{1}{2\pi}\right)^n \mathrm{Os} - \iint e^{-iy\cdot\eta'} \overline{p\left(z, \frac{z-x}{\sqrt{\rho}}\right)} p\left(z, \frac{z-x-\sqrt{\rho}y}{\sqrt{\rho}}\right) \times \det \frac{\partial z}{\partial \xi}\left(t, s; x, x+\sqrt{\rho}y, \frac{\hbar\rho\eta}{m} + \frac{\hbar\sqrt{\rho}\eta'}{m}\right) dy d\eta',$$

where $z = z(t, s; x, x + \sqrt{\rho}y, \hbar\rho\eta/m + \hbar\sqrt{\rho}\eta'/m)$. *Then we have*:
(1) *For any* α *and* β *there exists a constant* $C_{\alpha,\beta}$ *such that*

$$(5.5)\quad |\partial_\eta^\alpha \partial_x^\beta q(t, s; x, \eta)| \le C_{\alpha,\beta}\langle x; \eta\rangle^{2M}.$$

(2) *We have for* $f \in \mathcal{S}$

$$(5.6)\quad \|P(t, s)f\|^2 = (Q(t, s; x, D_x)f, f),$$

where $Q(t,s;x,D_x)f$ *is the pseudo-differential operator* $(2\pi)^{-n}\int e^{ix\cdot\eta} q(t, s; x, \eta)\hat{f}(\eta)d\eta$.

Proof. We use the integration by parts with respect to y and η' in (5.4). Then we get the statement (1) from (4.5), (5.1), and (5.2).

We consider (2). Let $f \in \mathcal{S}$. At first let $0 < t - s \le \rho^*$. As in the proof of (4.13) we can easily show from Lemma 3.2

$$(5.7)\quad \begin{aligned} &P(t, s)^* |\chi(\epsilon\cdot)|^2 P(t, s)f \\ &= \left(\frac{1}{2\pi}\right)^n \int f(y)dy \int |\chi(\epsilon z)|^2 e^{i(x-y)\cdot\eta} \overline{p\left(z, \frac{z-x}{\sqrt{\rho}}\right)} p\left(z, \frac{z-y}{\sqrt{\rho}}\right) \times \det\frac{\partial z}{\partial\xi}\left(t, s; x, y, \frac{\hbar\rho\eta}{m}\right) d\eta \\ &\equiv \left(\frac{1}{2\pi}\right)^n \int f(y)dy \int e^{i(x-y)\cdot\eta} \tilde{q}_\epsilon(t, s; x, y, \eta)d\eta, \end{aligned}$$

where $z = z(t, s; x, y, \hbar\rho\eta/m)$. The right-hand side above is a pseudo-differential operator with double symbol. Set

$$(5.8)\quad q_\epsilon(t, s; x, \eta) = \left(\frac{1}{2\pi}\right)^n \mathrm{Os} - \iint e^{-iy\cdot\eta'} \tilde{q}_\epsilon\left(t, s; x, x+\sqrt{\rho}y, \eta + \frac{\eta'}{\sqrt{\rho}}\right) dy d\eta'.$$

Then since

$$q_\epsilon(t,s;x,\eta)=\left(\frac{1}{2\pi}\right)^n \mathrm{Os}-\iint e^{-iy\cdot\eta'}\tilde{q}_\epsilon(t,s;x,x+y,\eta+\eta')dyd\eta',$$

we see from Theorem 2.5 of chapter 2 in [12] that the right-hand side of (5.7) is equal to $Q_\epsilon(t,s;x,D_x)f$. Consequently we have $P(t,s)^*|\chi(\epsilon\cdot)|^2P(t,s)f = Q_\epsilon(t,s;x,D_x)f$ and so

$$\|\chi(\epsilon\cdot)P(t,s)f\|^2=(Q_\epsilon(t,s;x,D_x)f,f). \tag{5.9}$$

We write from (5.7) and (5.8)

$$\begin{aligned} q_\epsilon(t,s;x,\eta) = {}& \left(\frac{1}{2\pi}\right)^n \mathrm{Os}-\iint e^{-iy\cdot\eta'}|\chi(\epsilon z)|^2\overline{p\left(z,\frac{z-x}{\sqrt{\rho}}\right)} \\ & \times p\left(z,\frac{z-x-\sqrt{\rho}y}{\sqrt{\rho}}\right)\det\frac{\partial z}{\partial\xi}\left(t,s;x,x+\sqrt{\rho}y,\frac{\hbar\rho\eta}{m}+\frac{\hbar\sqrt{\rho}\eta'}{m}\right)dyd\eta', \end{aligned}$$

where $z = z(t,s;x,x+\sqrt{\rho}y,\hbar\rho\eta/m+\hbar\sqrt{\rho}\eta'/m)$. Hence, letting ϵ tend to 0, we can show as in the proof of the statement (1) that the right-hand side of (5.9) converges to $(Q(t,s;x,D_x)f,f)$. Thus we could prove $P(t,s)f\in L^2$ and (5.6).

Let $t=s$. It follows from (4.1) and (5.3) that

$$z(s,s;x,x,0)=x,\quad \det\left(\frac{\partial z}{\partial\xi}\right)(s,s;x,y,\xi)=1,$$

and

$$\lim_{t\to s}\frac{1}{\sqrt{\rho}}\left(z\left(t,s;x,x+\sqrt{\rho}y,\frac{\hbar\rho\eta}{m}+\frac{\hbar\sqrt{\rho}\eta'}{m}\right)-x\right)=\frac{y}{2}+\frac{\hbar\eta'}{m}.$$

Substituting these equations into (5.4), we have

$$\begin{aligned} q(s,s;x,\eta) &= \left(\frac{1}{2\pi}\right)^n \mathrm{Os}-\iint e^{-iy\cdot\eta'}\overline{p\left(x,\frac{y}{2}+\frac{\hbar\eta'}{m}\right)}p\left(x,\frac{-y}{2}+\frac{\hbar\eta'}{m}\right)dyd\eta' \\ &= \left(\frac{m}{2\pi\hbar}\right)^n\left|\mathrm{Os}-\iint e^{i\hbar^{-1}m|w|^2/2}p(x,w)dw\right|^2. \end{aligned}$$

Hence we see (5.6) from (3.2). □

Proposition 5.3. *Assume* (1.7) *and* (1.8). *Let* $p(x,w)$ *be a function satisfying* (3.1). *Then we have*

$$\|P(t,s)f\|\le C\|f\|_{B^M},\quad 0\le t-s\le\rho^*,\ f\in B^M \tag{5.10}$$

for a constant C.

Proof. There exist a constant $\mu_M \geq 0$ and a $w_M(x,\eta)$ such that we have

$$|\partial_\eta^\alpha \partial_x^\beta w_M(x,\eta)| \leq C_{\alpha,\beta}\langle x;\eta\rangle^{-M}$$

for all α and β and

$$W_M(x,D_x) = (\mu_M + \langle x\rangle^M + \langle D_x\rangle^M)^{-1} \quad \text{on } \mathcal{S} \tag{5.11}$$

(Lemma 2.3 in [8]). Let $Q(t,s;x,D_x)$ be the operator determined in Lemma 5.2. Then we can prove

$$\|W_M(x,D_x)Q(t,s;x,D_x)f\| \leq \text{Const.}\|f\|_{B^M} \tag{5.12}$$

from (5.5), using Lemmas 2.1 and 2.5 in [8]. Consequently we have

$$\begin{aligned}(Q(t,s;x,D_x)f,f) &= (W_M(x,D_x)Qf, (\mu_M + \langle x\rangle^M + \langle D_x\rangle^M)f) \\ &\leq \text{Const.}\|f\|_{B^M}^2, \quad 0 \leq t-s \leq \rho^*, \ \ f \in B^M.\end{aligned}$$

Hence we can prove (5.10) from (5.6). □

REMARK 5.1. Let $F(t,s;x,y)$ and $\Psi(t,s;x,y)$ be the functions stated in Remark 4.1 and $p(x,w)$ a function satisfying (3.1). Let's define $P(t,s)$ by (3.2) in general. Assume (4.2), (4.3), and (4.7). Then we can prove (5.10), following our proof.

Theorem 5.4. *Assume* (1.7), (1.8)*, and*

$$|\partial_x^\alpha V(t,x)| \leq C_\alpha \langle x\rangle^{b^*}, \quad |\alpha| \geq 1 \tag{5.13}$$

in $[0,T]\times R^n$ *for a constant* $b^* \geq 0$. *Set* $M^* = \max(b^*, 1)$. *Let* $p(x,w)$ *be a function satisfying* (3.1). *Then we have for* $l = 0,1,2,\ldots$

$$\|P(t,s)f\|_{B^l} \leq C_l\|f\|_{B^{M+lM^*}}, \quad 0 \leq t-s \leq \rho^*, f \in B^{M+lM^*}. \tag{5.14}$$

Proof. Let's use (3.8). Then we can write

$$\partial_x^\alpha(P(t,s)f) = \sum_{\beta\leq\alpha} P_\beta(t,s)(\partial_x^{\alpha-\beta}f), \tag{5.15}$$

where $\beta \leq \alpha$ means $\beta_j \leq \alpha_j$ $(j = 1,2,\ldots,n)$. We have from (3.6) and (4.7)

$$|\partial_x^\alpha \partial_y^\beta F(t,s;x,y)| \leq C_{\alpha,\beta}(1+|x|+|y|), \quad |\alpha+\beta| \geq 1. \tag{5.16}$$

So we get from (3.9) together with (4.7) and (5.13)

$$|\partial_w^{\alpha'}\partial_x^{\beta'}\psi(t,s;x,w)| \le C_{\alpha',\beta'}\langle x;\sqrt{\rho}w\rangle^{M^*}$$

for all α' and $|\beta'| \ge 1$. Consequently we have

$$|\partial_w^{\alpha'}\partial_x^{\beta'}p_\beta(t,s;x,w)| \le C_{\alpha',\beta'}\langle x;w\rangle^{M+|\beta|M^*} \tag{5.17}$$

for all α' and β'.

Let $W_a = W_a(x, D_x)$ ($a \ge 0$) be the operator defined by (5.11). We can write $W_a(\partial_x^\alpha f) = W_a\partial_x^\alpha W_{a+|\alpha|}^{-1}(W_{a+|\alpha|}f)$. As in the proof of (5.12) we get together with Lemma 2.4 in [8]

$$\begin{aligned}\|\partial_x^\alpha f\|_{B^a} &\le \text{Const.}\|W_a(\partial_x^\alpha f)\| \\ &\le \text{Const.}\|W_{a+|\alpha|}f\| \\ &\le \text{Const.}\|f\|_{B^{a+|\alpha|}}.\end{aligned} \tag{5.18}$$

Using Proposition 5.3, we have from (5.15), (5.17), and (5.18)

$$\begin{aligned}\|\langle\cdot\rangle^k\partial_x^\alpha(P(t,s)f)\| &= \left\|\sum_{\beta\le\alpha}\langle\cdot\rangle^k P_\beta(t,s)\partial_x^{\alpha-\beta}f\right\| \\ &\le \text{Const.}\sum_{\beta\le\alpha}\|\partial_x^{\alpha-\beta}f\|_{B^{M+|\beta|M^*+k}} \\ &\le \text{Const.}\sum_{\beta\le\alpha}\|f\|_{B^{M+|\beta|M^*+k+|\alpha-\beta|}} \\ &\le \text{Const.}\|f\|_{B^{M+|\alpha|M^*+k}}.\end{aligned}$$

From this we can easily complete the proof. □

Corollary 5.5. *Let $p(x,w)$ be a function satisfying* (3.1). *We see under the assumptions of* Theorem 5.4 *that $P(t,s)f$ for $f \in B^{M+lM^*}$ ($l = 0,1,2,\ldots$) is a B^l-valued continuous function in $0 \le t-s \le \rho^*$.*

Proof. Let $0 \le t-s \le \rho^*$ and $g \in \mathcal{S}$. As in the proof of (5.18) we have

$$\|\partial_x^\alpha(\langle x\rangle^{n/2+1}P(t,s)g)\| \le \|P(t,s)g\|_{B^{n/2+1+|\alpha|}}.$$

Hence we have together with the Sobolev inequality and (5.14)

$$\begin{aligned}&|\langle x\rangle^{n/2+1}P(t,s)g| \\ &\le \text{Const.}\sum_{|\alpha|\le n/2+1}\|\partial_x^\alpha(\langle x\rangle^{n/2+1}P(t,s)g)\|\end{aligned}$$

$$\leq \text{Const.}\|P(t,s)g\|_{B^{n+2}}$$
$$\leq \text{Const.}\|g\|_{B^{M+(n+2)M^*}} < \infty.$$

So $|P(t,s)g|$ is bounded by a function in $L^2(R^n)$ uniformly in $0 \leq t-s \leq \rho^*$. Consequently, applying the Lebesgue dominated convergence theorem to $P(t,s)g$, we see from Lemma 3.1 that $P(t,s)g$ is an L^2-valued continuous function in $0 \leq t-s \leq \rho^*$. We have from (5.14)

$$\|P(t',s')f - P(t,s)f\| \leq \|P(t',s')g - P(t,s)g\| + 2C_0\|f-g\|_{B^M}$$

for $f \in B^M$ and $g \in \mathcal{S}$. Hence we can easily prove that $P(t,s)f$ for $f \in B^M$ is also an L^2-valued continuous function in $0 \leq t-s \leq \rho^*$. In the same way we can prove the statement in general. □

6. Convergence of $G_{\overline{\epsilon},\overline{\epsilon}'}(\Delta)$ as $|\overline{\epsilon}| + |\overline{\epsilon}'| \to 0$

Let $G_\epsilon(t,s)$ be the operator defined by (1.14). We proved (1.16) in Proposition 2.2. We can write (1.16) as

(6.1)
$$G_\epsilon(t,s)f = \begin{cases} \left(\sqrt{\dfrac{m}{2\pi i\hbar(t-s)}}\right)^n \displaystyle\int (\exp i\hbar^{-1}\Psi(t,s;x,y)) \\ \qquad\qquad \times\, p_\epsilon\left(t,s;x,\dfrac{x-y}{\sqrt{t-s}}\right) f(y)dy, \quad s<t, \\ f, \quad s=t. \end{cases}$$

Then we can easily prove the following from (3.4).

Lemma 6.1. *Assume* (4.7) *for* $j = 1,2,\ldots,n$. *Then there exist constants* $C_{\alpha,\beta}$ *independent of* $0 < \epsilon \leq 1$, $0 \leq s < t \leq T$, *and* $(x,w) \in R^{2n}$ *for all* α *and* β *such that*

(6.2)
$$|\partial_w^\alpha \partial_x^\beta p_\epsilon(t,s;x,w)| \leq C_{\alpha,\beta}.$$

We also have

(6.3)
$$\lim_{\epsilon\to 0} \partial_w^\alpha \partial_x^\beta (p_\epsilon(t,s;x,w) - 1) = 0 \quad \textit{pointwisely}$$

for all α *and* β.

Proposition 6.2. *Assume* (1.7) *and* (1.8). *Let* $0 \leq t-s \leq \rho^*$. *Then we have*:
(1) *There exists a constant* C *independent of* $0 < \epsilon \leq 1$ *such that*

(6.4)
$$\|G_\epsilon(t,s)f\| \leq C\|f\|, \quad f \in L^2.$$

(2) *We have*

$$\lim_{\epsilon\to 0}\|G_\epsilon(t,s)f - G(t,s)f\| = 0, \quad f\in L^2. \tag{6.5}$$

Proof. The statement (1) follows from Lemma 6.1 and Proposition 5.3. We consider (2). Let $t>s$. We have from (1.18) and (6.1)

$$G_\epsilon(t,s)f - G(t,s)f = \left(\sqrt{\frac{m}{2\pi i\hbar(t-s)}}\right)^n \int (\exp i\hbar^{-1}\Psi(t,s;x,y)) \times\left\{p_\epsilon\left(t,s;x,\frac{x-y}{\sqrt{t-s}}\right)-1\right\} f(y)dy.$$

Let's apply Lemma 5.2 to $G_\epsilon(t,s)f - G(t,s)f$. Then we have from (6.2)

$$\|G_\epsilon(t,s)f - G(t,s)f\|^2 = (\Gamma_\epsilon(t,s;x,D_x)f,f), \tag{6.6}$$

$$|\partial_\eta^\alpha\partial_x^\beta\gamma_\epsilon(t,s;x,\eta)| \le C_{\alpha,\beta}<\infty, \tag{6.7}$$

for all α and β, where $C_{\alpha,\beta}$ are independent of $0<\epsilon\le 1$, $0\le t-s\le\rho^*$, and $(x,\eta)\in R^{2n}$. In addition, we can easily prove

$$\lim_{\epsilon\to 0}\partial_\eta^\alpha\partial_x^\beta\gamma_\epsilon(t,s;x,\eta) = 0 \quad \text{pointwisely} \tag{6.8}$$

for all α and β, noting (4.5), Lemma 5.1, (6.2), and (6.3). Using Lemma 2.2 in [8], we get from (6.7) and (6.8)

$$\lim_{\epsilon\to 0}\|\Gamma_\epsilon(t,s;x,D_x)f\| = 0.$$

Hence we can prove (6.5) from (6.6). □

Theorem 6.3. *Assume* (1.7) *and* (1.8). *Let* $|\Delta|\le\rho^*$. *Then we have for* $f\in L^2$

$$\lim_{|\overline{\epsilon}|+|\overline{\epsilon}'|\to 0} G_{\overline{\epsilon},\overline{\epsilon}'}(\Delta)f = G(t,t_{\mu-1})\cdots G(t_1,0)f \quad \textit{in } L^2 \tag{6.9}$$

and so (1.20).

Proof. We have from (1.15)

$$\begin{aligned}
&G_{\overline{\epsilon},\overline{\epsilon}'}(\Delta)f - G(t,t_{\mu-1})\cdots G(t_1,0)f\\
&= \sum_{j=1}^{\mu} G_{\epsilon_{\mu-1}}(t,t_{\mu-1})\chi(\epsilon'_{\mu-1}\cdot)G_{\epsilon_{\mu-2}}(t_{\mu-1},t_{\mu-2})\cdots\chi(\epsilon'_j\cdot)\{G_{\epsilon_{j-1}}(t_j,t_{j-1})\\
&\quad -G(t_j,t_{j-1})\}G(t_{j-1},t_{j-2})\cdots G(t_1,0)f + \sum_{j=1}^{\mu-1} G_{\epsilon_{\mu-1}}(t,t_{\mu-1})\chi(\epsilon'_{\mu-1}\cdot)
\end{aligned}$$

$$\times G_{\epsilon_{\mu-2}}(t_{\mu-1}, t_{\mu-2}) \cdots G_{\epsilon_j}(t_{j+1}, t_j)\{\chi(\epsilon'_j \cdot) - 1\} G(t_j, t_{j-1}) \cdots G(t_1, 0) f$$

and so from (4.9) and (6.4)

$$\text{(6.10)} \qquad \|G_{\overline{\epsilon}, \overline{\epsilon}'}(\Delta) f - G(t, t_{\mu-1}) \cdots G(t_1, 0) f\|$$
$$\leq \text{Const.} \sum_{j=1}^{\mu} \|\{G_{\epsilon_{j-1}}(t_j, t_{j-1}) - G(t_j, t_{j-1})\} G(t_{j-1}, t_{j-2}) \cdots G(t_1, 0) f\|$$
$$+ \text{Const.} \sum_{j=1}^{\mu-1} \|\{\chi(\epsilon'_j \cdot) - 1\} G(t_j, t_{j-1}) \cdots G(t_1, 0) f\|.$$

Using Proposition 6.2 and the Lebesgue dominated convergence theorem, then we can prove (6.9). □

7. Proof of Theorem

Lemma 7.1. *We assume that there exists a constant $M'' \geq 0$ satisfying*

$$\text{(7.1)} \qquad |\partial_x^\alpha V(t, x)| + \sum_{j=1}^{n} \sum_{k=0}^{1} |\partial_t^k \partial_x^\alpha A_j(t, x)| \leq C_\alpha \langle x \rangle^{M''}$$

for all α in $[0, T] \times R^n$. Let $H(t)$ be the Hamiltonian operator defined by (1.10). *Then there exists a continuous function $r(t, s; x, w)$ in $0 \leq s \leq t \leq T$, x, and w in R^n satisfying* (3.1) *for an $M \geq 0$ such that for $f \in \mathcal{S}$*

$$\text{(7.2)} \quad \left(i\hbar \frac{\partial}{\partial t} - H(t)\right) G(t, s) f$$
$$= \sqrt{t-s} \left(\sqrt{\frac{m}{2\pi i\hbar(t-s)}}\right)^n \int (\exp i\hbar^{-1} \Psi(t, s; x, y)) r\left(t, s; x, \frac{x-y}{\sqrt{t-s}}\right) f(y) dy$$
$$\equiv \sqrt{t-s} R(t, s) f, \quad 0 \leq s < t \leq T.$$

Proof. We note (3.6). Then we have

$$\text{(7.3)} \qquad \partial_{x_j} F(t, s; x, y) = \rho p_1 \left(t, s; x, \frac{x-y}{\sqrt{\rho}}\right),$$

$$\text{(7.4)} \qquad \Delta_x F(t, s; x, y) = \sqrt{\rho} p_2 \left(t, s; x, \frac{x-y}{\sqrt{\rho}}\right),$$

and

$$\text{(7.5)} \qquad \partial_t F(t, s; x, y) = \sqrt{\rho} p_3 \left(t, s; x, \frac{x-y}{\sqrt{\rho}}\right), \quad \rho = t - s,$$

using the Taylor formula for the proof of (7.5). Here $p_j(t,s;x,w)$ $(j = 1,2,3)$ are functions satisfying (3.1) for an M.

We get by direct calculations

$$(7.6)\quad \left(i\hbar\frac{\partial}{\partial t} - H(t)\right)G(t,s)f = -\left(\sqrt{\frac{m}{2\pi i\hbar(t-s)}}\right)^n \int(\exp i\hbar^{-1}\Psi(t,s;x,y)) \times\left(r_1(t,s;x,y) + \frac{i\hbar}{2m}r_2(t,s;x,y)\right) f(y)dy,$$

where

$$r_1 = \partial_t\Psi(t,s;x,y) + \frac{1}{2m}\sum_{j=1}^n\left(\partial_{x_j}\Psi - A_j(t,x)\right)^2 + V(t,x),$$

$$r_2 = \frac{nm}{t-s} - \Delta_x\Psi + \sum_j \partial_{x_j}A_j(t,x)$$

and so from (1.17)

$$\begin{aligned}(7.7)\quad r_1 &= \left\{\partial_t S_c(q^{t,s}_{x,y}) + \frac{1}{2m}\sum_{j=1}^n\left(\partial_{x_j}S_c(q^{t,s}_{x,y}) - A_j(t,x)\right)^2 + V(t,x)\right\} \\ &\quad +\left\{\partial_t F(t,s;x,y) + \frac{1}{m}\sum_{j=1}^n\left(\partial_{x_j}S_c(q^{t,s}_{x,y}) - A_j(t,x)\right)\partial_{x_j}F(t,s;x,y)\right. \\ &\quad\quad \left. + \frac{1}{2m}\sum_{j=1}^n(\partial_{x_j}F)^2\right\} \\ &\equiv I_1 + I_2,\end{aligned}$$

$$\begin{aligned}(7.8)\quad r_2 &= \left\{\frac{nm}{t-s} - \Delta_x S_c(q^{t,s}_{x,y}) + \sum_j \partial_{x_j}A_j(t,x)\right\} - \Delta_x F(t,s;x,y) \\ &\equiv J - \Delta_x F(t,s;x,y).\end{aligned}$$

It follows from (2.3) that

$$S_c(q^{t,s}_{x,y}) = \frac{m|x-y|^2}{2(t-s)} + (x-y)\cdot\int_0^1 A(t-\theta\rho, x-\theta(x-y))d\theta - \int_s^t V\left(\theta, y + \frac{\theta-s}{t-s}(x-y)\right)d\theta.$$

So we can prove

$$\partial_t S_c(q^{t,s}_{x,y}) = -\frac{m|x-y|^2}{2(t-s)^2} - V(t,x) + \sqrt{\rho}\,p_4\left(t,s;x,\frac{x-y}{\sqrt{\rho}}\right).$$

Here let's use (2.23) and (2.25) of [9]. Then we have

$$I_1 = \sqrt{\rho} p_5 \left(t, s; x, \frac{x-y}{\sqrt{\rho}} \right), \tag{7.9}$$

$$J = \sqrt{\rho} p_6 \left(t, s; x, \frac{x-y}{\sqrt{\rho}} \right). \tag{7.10}$$

We have from (2.22) of [9]

$$\begin{aligned} \partial_{x_j} S_c(q^{t,s}_{x,y}) - A_j(t,x) = {} & \frac{m}{\sqrt{\rho}} \frac{(x_j - y_j)}{\sqrt{\rho}} - \frac{1}{2}\sqrt{\rho} \sum_{l=1}^{n} \frac{\partial A_j}{\partial x_l}(t,x) \frac{(x_l - y_l)}{\sqrt{\rho}} \\ & + \frac{1}{2}\sqrt{\rho} \sum_{k=1}^{n} \frac{\partial A_k}{\partial x_j}(t,x) \frac{(x_k - y_k)}{\sqrt{\rho}} + \rho p_7 \left(t, s; x, \frac{x-y}{\sqrt{\rho}} \right) \end{aligned}$$

and so together with (7.3) and (7.5)

$$I_2 = \sqrt{\rho} p_8 \left(t, s; x, \frac{x-y}{\sqrt{\rho}} \right). \tag{7.11}$$

Hence we can complete the proof together with (7.4) and (7.6)-(7.10). □

Proposition 7.2 below follows from Lemma 7.1 and Theorem 5.4.

Proposition 7.2. *Assume* (1.7), (1.8), *and* (7.1). *Let* M^* *and* M *be the constants determined in* Theorem 5.4 *and* Lemma 7.1, *respectively. Then we have for* $l = 0, 1, 2, \ldots$

$$\begin{aligned} \left\| \left(i\hbar \frac{\partial}{\partial t} - H(t) \right) G(t,s) f \right\|_{B^l} \le C_l \sqrt{t-s} \| f \|_{B^{M+lM^*}}, \\ 0 < t - s \le \rho^*, \quad f \in B^{M+lM^*}. \end{aligned} \tag{7.12}$$

Theorem 7.3. *We assume* (1.7), (1.8), *and* (7.1). *Then there exist a constant* $l \ge 0$ *and a function* $\lambda(\rho) \ge 0$ *in* $0 \le \rho \le \rho^*$ *such that*

$$\lim_{\rho \to 0} \lambda(\rho) = 0, \tag{7.13}$$

$$\left\| i\hbar \frac{G(t,s)f - f}{t-s} - H(s) f \right\| \le \lambda(t-s) \| f \|_{B^l}, \quad 0 < t-s \le \rho^*, \quad f \in B^l. \tag{7.14}$$

Proof. Noting Lemma 3.1, we can write from Lemma 7.1

$$i\hbar \frac{G(t,s)f - f}{t-s} = i\hbar \int_0^1 \frac{\partial G}{\partial t}(s + \theta\rho, s) f d\theta \tag{7.15}$$

$$= \int_0^1 H(s+\theta\rho)G(s+\theta\rho,s)f d\theta + \sqrt{\rho}\int_0^1 \sqrt{\theta}R(s+\theta\rho,s)f d\theta,$$
$$\rho = t-s, \quad f \in \mathcal{S}$$

and so

$$(7.16) \qquad i\hbar\frac{G(t,s)f-f}{t-s} - H(s)f$$
$$= \int_0^1 H(s+\theta\rho)\{G(s+\theta\rho,s)f-f\}d\theta + \int_0^1 \{H(s+\theta\rho)f - H(s)f\}d\theta$$
$$+\sqrt{\rho}\int_0^1 \sqrt{\theta}R(s+\theta\rho,s)f d\theta.$$

We see from (7.1) and (7.15)

$$(7.17) \qquad \|H(s+\theta\rho)\{G(s+\theta\rho,s)f-f\}\|$$
$$\leq \text{Const.}\|G(s+\theta\rho,s)f-f\|_{B^{l_1}}$$
$$\leq \text{Const.}\rho\|f\|_{B^{l_2}}$$

for some l_1 and $l_2 \geq 0$, using Theorem 5.4. We have

$$\|V(s+\theta\rho,\cdot)f - V(s,\cdot)f\| \leq (\sup|V(s+\theta\rho,\cdot)-V(s,\cdot)|\langle\cdot\rangle^{-(M''+1)})\|f\|_{B^{M''+1}}.$$

It follows from (7.1) that $\lim_{|x|\to\infty}|V(t,x)|\langle x\rangle^{-(M''+1)} = 0$ uniformly in $t \in [0,T]$. So we can easily prove

$$\lim_{\rho\to 0}(\sup|V(s+\theta\rho,\cdot)-V(s,\cdot)|\langle\cdot\rangle^{-(M''+1)}) = 0$$

uniformly in $s \in [0,T]$. Consequently there exists a function $\lambda'(\rho) \geq 0$ such that

$$(7.18) \qquad \lim_{\rho\to 0}\lambda'(\rho) = 0,$$

$$(7.19) \qquad \|V(s+\theta\rho,\cdot)f - V(s,\cdot)f\| \leq \lambda'(t-s)\|f\|_{B^{M''+1}}.$$

Hence, noting the assumption (7.1) on $A(t,x)$, we can prove that there exist a function $\lambda''(\rho) \geq 0$ and an $l_3 \geq 0$ satisfying $\lim_{\rho\to 0}\lambda''(\rho) = 0$ and

$$(7.20) \qquad \|H(s+\theta\rho)f - H(s)f\| \leq \lambda''(t-s)\|f\|_{B^{l_3}}.$$

Thus we can prove Theorem 7.3 from (7.16) together with (7.17) and (7.20). □

Proof of Theorem. We proved the statement (1) of Theorem and (1.20) in Theorem 6.3. Let $H(t)$ be the operator defined by (1.10). We are assuming (1.8) and (1.9).

Consequently we see from Theorem of [8] that for any $f \in B^a$ $(-\infty < a < \infty)$ there exists a unique solution $U(t,s)f$, B^a-valued continuous and B^{a-2}-valued continuously differentiable in $0 \le s,\ t \le T$, of the Schrödinger equation (1.1) and that we have

$$\begin{cases} \|U(t,s)f\|_{B^a} \le C_a(T)\|f\|_{B^a}, \quad a \neq 0, \\ \|U(t,s)f\| = \|f\|. \end{cases} \tag{7.21}$$

Hence we can prove the following as in the proof of Theorem 7.3. There exist a constant $l' \ge 0$ and a function $\lambda_u(\rho) \ge 0$ in $0 \le \rho \le \rho^*$ such that

$$\lim_{\rho \to 0} \lambda_u(\rho) = 0, \tag{7.22}$$

$$\left\| i\hbar \frac{U(t,s)f - f}{t-s} - H(s)f \right\| \le \lambda_u(t-s)\|f\|_{B^{l'}}, 0 \le s < t \le T, \ \ f \in B^{l'}. \tag{7.23}$$

We can write

$$\begin{aligned} & i\hbar G(t,s)f - i\hbar U(t,s)f \\ &= (t-s)\left\{ i\hbar \frac{G(t,s)f - f}{t-s} - H(s)f \right\} - (t-s)\left\{ i\hbar \frac{U(t,s)f - f}{t-s} - H(s)f \right\}. \end{aligned}$$

So we have from (7.14) and (7.23)

$$\|G(t,s)f - U(t,s)f\| \le \text{Const.}(t-s)\tilde{\lambda}(t-s)\|f\|_{B^a}, \quad 0 \le t-s \le \rho^*, \tag{7.24}$$

where $\tilde{\lambda}(\rho) = \max(\lambda(\rho), \lambda_u(\rho))$ and $a = \max(l, l')$. We also have from (7.13) and (7.22)

$$\lim_{\rho \to 0} \tilde{\lambda}(\rho) = 0. \tag{7.25}$$

Let $f \in B^a$ and $|\Delta| \le \rho^*$. We write by (1.20)

$$\begin{aligned} & G(\Delta)f - U(t,0)f \\ &= G(t,t_{\mu-1})\cdots G(t_1,0)f - U(t,t_{\mu-1})\cdots U(t_1,0)f \\ &= \sum_{j=1}^{\mu} G(t,t_{\mu-1})\cdots G(t_{j+1},t_j)\big(G(t_j,t_{j-1}) - U(t_j,t_{j-1})\big)U(t_{j-1},0)f. \end{aligned}$$

So we have from Theorem 4.3 and (7.24)

$$\begin{aligned} \|G(\Delta)f - U(t,0)f\| &\le \text{Const.} \sum_{j=1}^{\mu} e^{K(t-t_j)}(t_j - t_{j-1})\tilde{\lambda}(t_j - t_{j-1})\|U(t_{j-1},0)f\|_{B^a} \\ &\le \text{Const.} e^{KT} \sup_{0 \le \rho \le |\Delta|} \tilde{\lambda}(\rho) \sum_{j=1}^{\mu} (t_j - t_{j-1})\|U(t_{j-1},0)f\|_{B^a}. \end{aligned}$$

Consequently we obtain together with (7.21)

$$(7.26)\quad \|G(\Delta)f - U(t,0)f\| \le \text{Const.}e^{KT}T \sup_{0\le\rho\le|\Delta|} \tilde{\lambda}(\rho)\|f\|_{B^a}, \quad 0 \le t \le T,\ \ f \in B^a.$$

Hence it follows from (7.25) that as $|\Delta| \to 0$, $G(\Delta)f$ for $f \in B^a$ converges to $U(t,0)f$ in L^2 uniformly in $t \in [0,T]$.

Let $f \in L^2$ and $|\Delta| \le \rho^*$. Using Corollary 4.4 and (7.21), we see for any $g \in B^a$

$$(7.27)\quad \|G(\Delta)f - U(t,0)f\| \le \|G(\Delta)g - U(t,0)g\| + (1 + e^{KT})\|f - g\|, \quad 0 \le t \le T.$$

Hence we can easily prove that $G(\Delta)f$ converges to $U(t,0)f$ in L^2 uniformly in $t \in [0,T]$ as $|\Delta| \to 0$. Thus we could complete the proof of Theorem. □

REMARK 7.1. As in [9] and [10] we assume

$$|\partial_t \partial_x^\alpha V(t,x)| \le C_\alpha \langle x\rangle^{c^*}, \quad |\alpha| \ge 1$$

for a constant $c^* \ge 0$ besides the assumptions of Theorem. Then, following our proof, we can easily see that we can take Const.$\sqrt{\rho}$ as $\lambda(\rho)$ in (7.14) and take Const.ρ as $\lambda_u(\rho)$ in (7.23). Consequently we obtain (7.26) where $\sup_{0\le\rho\le|\Delta|} \tilde{\lambda}(\rho)$ is replaced by $\sqrt{|\Delta|}$.

REMARK 7.2. In [9] and [10] we studied the path integral in configuration space defined by

$$(7.28)\qquad \lim_{|\Delta|\to 0} \prod_{j=1}^{\mu} \left(\sqrt{\frac{m}{2\pi i\hbar(t_j - t_{j-1})}}\right)^n \text{Os} - \int_{R^n} \cdots \int_{R^n} (\exp i\hbar^{-1} S_c(q_\Delta)) \times f(x^{(0)})dx^{(0)}dx^{(1)} \cdots dx^{(\mu-1)},$$

where $q_\Delta = q_\Delta(x^{(0)}, x^{(1)}, \ldots, x^{(\mu-1)}, x) \in (R^n)^{[0,t]}$ is the piecewise linear function defined in introduction. In [10] we proved statements similar to those of Theorem under the assumption (1.7) without (1.8) and (1.9) for arbitrary potentials V and A.

References

[1] V.I. Arnold: Mathematical methods of classical mechanics, Springer, Berlin, 1978.

[2] A. Dynin: *A rigorous path integral construction in any dimension*, Lett. Math. Phys. **44** (1998), 317–329.

[3] R.P. Feynman: *Space-time approach to non-relativistic quantum mechanics*, Rev. Mod. Phys. **20** (1948), 367–387.

[4] R.P. Feynman: *An operator calculus having applications in quantum electrodynamics*, Phys. Rev. **84** (1951), 108–128.
[5] R.P. Feynman and A.R. Hibbs: Quantum mechanics and path integrals, McGraw-Hill, New York, 1965.
[6] C. Garrod: *Hamiltonian path integral methods*, Rev. Mod. Phys. **38** (1966), 483–494.
[7] K. Gawędzki: *Construction of quantum-mechanical dynamics by means of path integrals in phase space*, Rep. Math. Phys. **6** (1974), 327–342.
[8] W. Ichinose: *A note on the existence and* $\hbar$*-dependency of the solution of equations in quantum mechanics*, Osaka J. Math. **32** (1995), 327–345.
[9] W. Ichinose: *On the formulation of the Feynman path integral through broken line paths*, Commun. Math. Phys. **189** (1997), 17–33.
[10] W. Ichinose: *On convergence of the Feynman path integral formulated through broken line paths*, Rev. Math. Phys. **11** (1999), 1001–1025.
[11] K. Itô: *Generalized uniform complex measures in the hilbertian metric space with their application to the Feynman integral*, 145–161 in Proc. 5th Berkeley symposium on Mathematical Statistics and Probability Vol. 2, Univ. of California Press, Berkeley, 1967.
[12] H. Kumano-go: Pseudo-differential operators, MIT Press, Cambridge, 1981.
[13] P.D. Lax and R.D. Richtmyer: *Survey of the stability of linear finite difference equations*, Commun. Pure Appl. Math. **9** (1956), 267–293.
[14] A. Messiah: Mécanique quantique, Dunod, Paris, 1959.
[15] P. Ramond: Field theory: A modern primier, Addison-Wesley, Massachusetts, 1990.
[16] L.S. Schulman: Techniques and applications of path integration, John Wiley and Sons, New York, 1981.
[17] J.T. Schwartz: Nonlinear functional analysis, Gordon and Breach Science Publishers, New York, London, Paris, Montreux, Tokyo, 1969.
[18] W. Tobocman: *Transition amplitudes as sums over histories*, Nuovo Cimento, **3** (1956), 1213–1229.

Department of Mathematical Science
Shinshu University
Matsumoto 390-8621, Japan