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PSEUDO-RIEMANNIAN GEODESIC ORBIT METRICS ON CERTAIN COMPACT HOMOGENEOUS SPACES

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Abstract

A pseudo-Riemannian metric is called geodesic orbit if its geodesics are the orbits of one-parameter subgroups of the group of isometries. In this paper, we study pseudo-Riemannian geodesic orbit metrics on compact homogeneous spaces. First we obtain a sufficient and necessary condition for a pseudo-Riemannian metric to be geodesic orbit. Then we show that every Tamaru's homogeneous space admits a two-parameter family of pseudo-Riemannian geodesic orbit metrics. Finally, we obtain a complete description of pseudo-Riemannian geodesic orbit metrics on spheres. In particular, we prove that a $\mathrm{Sp}(n+1)$-invariant pseudo-Riemannian geodesic orbit metric on $\mathrm{S}^{4n+3} = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)$ must be $\mathrm{Sp}(n+1)\mathrm{Sp}(1)$-invariant.

1. Introduction

Let (M, g) be a connected (pseudo-)Riemannian manifold and G be a subgroup of the full group of isometries $I(M, g)$. A geodesic $\gamma : \mathbb{R} \to M$ is called G-homogeneous if there exists a vector $X \in \mathfrak{g}$ such that $\gamma(t) = \exp(tX) \cdot \gamma(0)$, where $\mathfrak{g}$ denotes the Lie algebra of G and exp denotes the exponential map of $\mathfrak{g}$. The notion of a homogeneous geodesic plays a fundamental role in the theory of geodesic orbit manifolds (i.e., a (pseudo-)Riemannian manifold whose geodesics are all G-homogeneous). In [13], Kowalski and Vanhecke started a systematical study on Riemannian geodesic orbit manifolds and presented a fundamental geodesic lemma for a geodesic to be G-homogeneous. Since then, many excellent works have been done. In particular, it is worth to mention that, Gordon in [11] claimed that the classification of Riemannian geodesic orbit manifolds can be reduced to three special cases: (1) Riemannian geodesic orbit nilmanifolds (i.e., a nilpotent Lie group with a left invariant Riemannian metric), (2) compact Riemannian geodesic orbit manifolds, and (3) a Riemannian geodesic orbit manifold admitting a transitive non-compact semisimple Lie group of isometries. Later, in [17], Nikonorov found that Gordon's reduction is not generally correct by showing some examples of Riemannian geodesic orbit solvmanifolds that are different from nilmanifolds (see Examples 6 and 7 in [17]). Recently, Gordon and Nikonorov [12] corrected an error (Theorem 1.15 of [11]) to a slightly weakened version of this statement

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(Theorem 3.1 of [12]) and so the study of an arbitrary Riemannian geodesic orbit manifold largely reduced to the study of the nilradical and two homogeneous spaces generated by the compact and non-compact parts of the Levi group of its isometry group. Moreover, Tamaru [20] classified the compact and non-compact Riemannian geodesic orbit manifolds fibered over irreducible symmetric spaces, we list the classification in [20] in Table 1. Nikonorov [16] obtained a complete classification of Riemannian geodesic orbit metrics on spheres and constructed some explicit geodesic vectors. Chen, Nikolayevsky and Nikonorov [7] classified all G-invariant Riemannian geodesic orbit metrics on a compact and simply connected homogeneous space G/H, where G is (almost) effective and H is a simple Lie group. For more information about homogeneous geodesics and related topics, we refer the readers to [2, 3] and the references therein. More recently, the geodesic orbit property has been extensively studied in Finsler setting [1, 6, 9, 22, 23, 24, 25, 26].

The situation is more complicated for pseudo-Riemannian geodesic orbit manifolds, see [21]. For example, Nikolayevsky and Wolf [15] showed that a geodesic orbit Lorentz nilmanifold need not be two step nilpotent. Moreover, unlike the Riemannian case, it seems not an evident fact that every compact homogeneous space admits a pseudo-Riemannian geodesic orbit metric. In this paper, motivated by Tamaru and Nikonorov's results on compact Riemannian geodesic orbit manifolds, we are going to study pseudo-Riemannian geodesic orbit metrics on compact homogeneous spaces. Now we introduce the main results of this paper. Let $M = G/H$ be a compact homogeneous space with $\mathbf{B}$-orthogonal reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, where $\mathbf{B}$ is an $\mathrm{Ad}(G)$-invariant positive definite inner product on $\mathfrak{g}$ (always exists) and $\mathfrak{h}$ denotes the Lie algebra of H. Then G-invariant pseudo-Riemannian metrics on G/H are in one-to-one correspondence with $\mathrm{Ad}(H)$-invariant indefinite inner products on $\mathfrak{m}$. We obtain a characterization of pseudo-Riemannian metrics on compact homogeneous spaces to be geodesic orbit, which generalizes Proposition 2 of [27] to the pseudo-Riemannian setting.

Theorem 1.1. *A G-invariant pseudo-Riemannian metric g on G/H is geodesic orbit with respect to G if and only if for every $T \in \mathfrak{m}$, there exist a vector $Z \in \mathfrak{h}$ and a constant $c \in \mathbb{R}$ such that*

$$[A(T), T + Z] = cA(T),$$

where A denotes the metric endomorphism of g defined by

$$\langle X, Y\rangle = \mathbf{B}(A(X), Y), \forall X, Y \in \mathfrak{m},$$

$\langle\cdot,\cdot\rangle$ is the indefinite inner product on $\mathfrak{m}$ determined by g.

As applications of Theorem 1.1 to Tamaru's homogeneous spaces (see Table 1), we prove that every Tamaru's homogeneous space admits a two-parameter family of pseudo-Riemannian geodesic orbit metrics. Moreover, we obtain a complete description of pseudo-Riemannian naturally reductive, weakly symmetric and geodesic orbit metrics on spheres and mainly prove the following result, a pseudo-Riemannian version of Theorem 1 of [16].

Theorem 1.2. *A $\mathrm{Sp}(n+1)$-invariant pseudo-Riemannian metric g on $\mathrm{S}^{4n+3} = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)$ is geodesic orbit with respect to $\mathrm{Sp}(n+1)$ if and only if it is $\mathrm{Sp}(n+1)\mathrm{Sp}(1)$-invariant.*

This paper is organized as follows. In Section 2, we recall some basic facts about pseudo-

Riemannian geodesic orbit, naturally reductive and weakly symmetric manifolds and prove Theorem 1.1. In Section 3, we review the Tamaru's classification of Riemannian geodesic orbit manifolds fibered over irreducible symmetric spaces and show that every such space admits a two-parameter family of pseudo-Riemannian geodesic orbit metrics. In Section 4, we study pseudo-Riemannian geodesic orbit metrics on spheres and prove Theorem 1.2.

2. Pseudo-Riemannian geodesic orbit metrics on compact homogeneous spaces

In this section we discuss the characterization of pseudo-Riemannian geodesic orbit metrics on compact homogeneous spaces. First we recall the definition of pseudo-Riemannian geodesic orbit manifolds, which is a generalization of Riemannian geodesic orbit manifolds.

DEFINITION 2.1. Let (M, g) be a connected pseudo-Riemannian manifold and G be a subgroup of the full group of isometries $I(M, g)$. (M, g) is called a geodesic orbit manifold with respect to G if every geodesic of (M, g) is an orbit of a one-parameter subgroup of G. That is, if $\gamma : \mathbb{R} \to M$ is a geodesic of (M, g), then there exists a vector $X \in \mathfrak{g}$ such that $\gamma(t) = \exp(tX) \cdot \gamma(0)$, where $\mathfrak{g}$ denotes the Lie algebra of G and exp denotes the exponential map of $\mathfrak{g}$.

Now let G be a compact Lie group and H be a closed subgroup of G, which has no nontrivial normal subgroup of G. As usual we denote the Lie algebras of G and H by $\mathfrak{g}$ and $\mathfrak{h}$ respectively. As H is compact, $\mathfrak{g}$ has an Ad(H)-invariant reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{m}$ is a subspace of $\mathfrak{g}$ satisfying $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. In this case, one can identify $\mathfrak{m}$ with the tangent space $T_o(G/H)$ of G/H at the point $o = eH$ via the mapping $\pi : X \to \frac{d}{dt}\big|_{t=0} \exp(tX) \cdot o$. Moreover, under this identification, G-invariant pseudo-Riemannian metrics on homogeneous space G/H are in one-to-one correspondence with Ad(H)-invariant indefinite inner products on $\mathfrak{m}$.

Lemma 2.2 ([10], Lemma 2.1)**.** *Notation as above. Let G/H be a compact homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Then a G-invariant pseudo-Riemannian metric g on G/H is geodesic orbit with respect to G if and only if for every $T \in \mathfrak{m}$, there exist $Z = Z(T) \in \mathfrak{h}$ and $c = c(T) \in \mathbb{R}$ such that*

$$\langle [T + Z, T']_{\mathfrak{m}}, T \rangle = c\langle T, T' \rangle \tag{2.1}$$

holds for all $T' \in \mathfrak{m}$, where $\langle \cdot, \cdot \rangle$ is the indefinite inner product on $\mathfrak{m}$ associated to g and $[\cdot, \cdot]_{\mathfrak{m}}$ is the projection to $\mathfrak{m}$ with respect to the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$.

REMARK 2.3. Take $T' = T$ in (2.1), we can see that $c = 0$ unless T is a null vector.

Since G is compact, there exists an Ad(G)-invariant positive definite inner product $\mathbf{B}$ on $\mathfrak{g}$. We can choose the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ by requiring $\mathbf{B}(\mathfrak{h}, \mathfrak{m}) = 0$. In this situation, for an Ad(H)-invariant indefinite inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{m}$ associated to a G-invariant pseudo-Riemannian metric g on G/H, there exists a unique Ad(H)-equivariant, symmetric and nondegenerate endomorphism $A : \mathfrak{m} \to \mathfrak{m}$, called the metric endomorphism of g, such that

$$\langle X, Y \rangle = \mathbf{B}(A(X), Y), \forall X, Y \in \mathfrak{m}.$$

Proof of Theorem 1.1. By Lemma 2.2, a G-invariant pseudo-Riemannian metric g on G/H is geodesic orbit with respect to G if and only if for every $T \in \mathfrak{m}$, there exist $Z \in \mathfrak{h}$ and $c \in \mathbb{R}$ such that

$$\langle [T+Z, T']_{\mathfrak{m}}, T\rangle = c\langle T, T'\rangle, \forall T' \in \mathfrak{m}.$$

Notice that

$$\begin{aligned} & \langle [T+Z, T']_{\mathfrak{m}}, T\rangle - c\langle T, T'\rangle \\ = \ & \mathbf{B}([T+Z, T'], A(T)) - c\mathbf{B}(A(T), T') \\ = \ & \mathbf{B}(T', [A(T), T+Z]) - c\mathbf{B}(A(T), T') \\ = \ & \mathbf{B}(T', [A(T), T+Z] - cA(T)), \end{aligned}$$

which implies that g is geodesic orbit if and only if $[A(T), T+Z] - cA(T) \in \mathfrak{h}$.

On the other hand, since the metric endomorphism A is $\mathrm{Ad}(H)$-equivariant and symmetric with respect to $\mathbf{B}$, for every $X \in \mathfrak{h}$, we have

$$\begin{aligned} & \mathbf{B}([A(T), T+Z], X) = \mathbf{B}([A(T), T], X) \\ = \ & -\mathbf{B}(T, [A(T), X]) = -\mathbf{B}(T, A([T, X])) \\ = \ & -\mathbf{B}(A(T), [T, X]) = \mathbf{B}([T, A(T)], X). \end{aligned}$$

Thus $\mathbf{B}([A(T), T], X) = 0$ and $[A(T), T+Z] - cA(T) \in \mathfrak{m}$. As a result, $[A(T), T+Z] - cA(T) \in \mathfrak{h}$ if and only if $[A(T), T+Z] = cA(T)$. This completes the proof. □

Recall that a G-invariant pseudo-Riemannian metric g on homogeneous space G/H is said to be naturally reductive with respect to G if there is an $\mathrm{Ad}(H)$-invariant decomposition (not necessarily $\mathbf{B}$-orthogonal) $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ such that

(2.2) $$\langle [T', T]_{\mathfrak{m}}, T\rangle = 0, \forall T, T' \in \mathfrak{m},$$

where $\langle \cdot, \cdot \rangle$ is the indefinite inner product on $\mathfrak{m}$ induced by the pseudo-Riemannian metric g on G/H. We can also replace equation (2.2) by

(2.3) $$\langle [T, T']_{\mathfrak{m}}, T''\rangle + \langle [T, T'']_{\mathfrak{m}}, T'\rangle = 0, \forall T, T', T'' \in \mathfrak{m}.$$

It is easily seen that a pseudo-Riemannian naturally reductive metric must be geodesic orbit. The following result is due to Ovando [18].

Theorem 2.4 ([18], Theorem 2.2). *Let $(G/H, g)$ be a compact pseudo-Riemannian naturally reductive homogeneous space with respect to a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Assume $\mathfrak{g} = \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$, then there exists a unique* $\mathrm{Ad}(G)$*-invariant symmetric nondegenerate bilinear form* $\mathbf{Q}$ *on* $\mathfrak{g}$ *such that*

$$\mathbf{Q}(\mathfrak{h}, \mathfrak{m}) = 0, \quad \mathbf{Q}|_{\mathfrak{m}} = \langle \cdot, \cdot \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the indefinite inner product on $\mathfrak{m}$ associated to g.

Now we present a sufficient condition for a special class of G-invariant pseudo-Riemannian metrics on G/H to be naturally reductive. It can be viewed as a pseudo-Riemannian version of Theorem 3 of [28].

Theorem 2.5 ([28], Theorem 3)**.** *Let G/H be a compact homogeneous space with $\mathbf{B}$-orthogonal reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Assume G is semisimple and H is connected. If $\mathfrak{m}$ has an* $\mathrm{Ad}(H)$*-invariant* $\mathbf{B}$*-orthogonal decomposition $\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2$ with $[\mathfrak{h}, \mathfrak{m}_2] = 0$ and $[\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{m}_2$, then the G-invariant pseudo-Riemannian metrics $g_{a,b}$ on G/H corresponding to $a\mathbf{B}|_{\mathfrak{m}_1} + b\mathbf{B}|_{\mathfrak{m}_2}$ ($ab < 0, a, b \in \mathbb{R}$) are naturally reductive with respect to $G \times K$, where K is the connected subgroup of G with the Lie algebra $\mathfrak{k} \cong \mathfrak{m}_2$.*

Proof. The proof is similar to that of Theorem 3 of [28]. Let $\bar{G} = G \times K$. For any $(g, k) \in \bar{G}$, g operates on G/H by left translation with g and k operates on G/H by right translation with k^{-1}, then the isotropy subgroup of this action at the point eH is $\bar{H} = H \times K$ with embedding $(h, k) \to (hk, k)$. The reductive decomposition of the Lie algebra is

$$\bar{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{k} = \bar{\mathfrak{h}} + \bar{\mathfrak{m}}(s) = (\mathfrak{h}, 0) + \bar{\mathfrak{k}} + \bar{\mathfrak{m}}_1 + \bar{\mathfrak{m}}_2(s),$$

where $\bar{\mathfrak{k}} = \{(X, X)|X \in \mathfrak{m}_2\}$, $\bar{\mathfrak{m}}_1 = (\mathfrak{m}_1, 0)$, $\bar{\mathfrak{m}}_2(s) = \{(sX, (s-1)X)|X \in \mathfrak{m}_2\}$, $s \in \mathbb{R}$. We need to find a real number $s \in \mathbb{R}$ for $g_{a,b}$ to be naturally reductive with respect to the above decomposition.

Clearly, both $\bar{\mathfrak{m}}_1$ and $\bar{\mathfrak{m}}_2(s)$ are $\mathrm{Ad}(\bar{H})$-invariant subspaces. Notice that the isomorphism between the tangent spaces $\varphi : \bar{\mathfrak{m}}_1 + \bar{\mathfrak{m}}_2(s) \to \mathfrak{m}_1 + \mathfrak{m}_2$ is given by

$$\varphi((X, 0) + (sY, (s-1)Y)) = X + Y, \forall X \in \mathfrak{m}_1, Y \in \mathfrak{m}_2.$$

Hence the $\mathrm{Ad}(\bar{H})$-invariant indefinite inner product $\langle\cdot,\cdot\rangle_{a,b}$ on $\bar{\mathfrak{m}}_1 + \bar{\mathfrak{m}}_2(s)$ induced by the pseudo-Riemannian metric $g_{a,b}$ on G/H is given as follows:

$$\begin{aligned}
&\langle (X, 0), (Y, 0)\rangle_{a,b} = a\mathbf{B}(X, Y), X, Y \in \mathfrak{m}_1,\\
&\langle \bar{\mathfrak{m}}_1, \bar{\mathfrak{m}}_2(s)\rangle_{a,b} = 0,\\
&\langle (sX, (s-1)X), (sY, (s-1)Y)\rangle_{a,b} = b\mathbf{B}(X, Y), X, Y \in \mathfrak{m}_2.
\end{aligned}$$

Note that $[\mathfrak{h}, \mathfrak{m}_2] = 0$ and $[\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{m}_2$, one easily has that $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1$. Now for every $X, Y \in \mathfrak{m}_1$ and $Z \in \mathfrak{m}_2$, by a direct computation we have

$$\begin{aligned}
&\langle [(X, 0), (Y, 0)]_{\bar{\mathfrak{m}}(s)}, (sZ, (s-1)Z)\rangle_{a,b}\\
&+\langle [(X, 0), (sZ, (s-1)Z)]_{\bar{\mathfrak{m}}(s)}, (Y, 0)\rangle_{a,b}\\
= \;&\langle ([X, Y]_{\mathfrak{m}_1}, 0) + (s[X, Y]_{\mathfrak{m}_2}, (s-1)[X, Y]_{\mathfrak{m}_2}), (sZ, (s-1)Z)\rangle_{a,b}\\
&+s\langle ([X, Z], 0), (Y, 0)\rangle_{a,b}\\
= \;&b\mathbf{B}([X, Y]_{\mathfrak{m}_2}, Z) + sa\mathbf{B}([X, Z], Y)\\
= \;&(b - sa)\mathbf{B}([X, Y], Z).
\end{aligned}$$

So let $s = \frac{b}{a}$ and by a similar computation as above, we see that $g_{a,b}$ is naturally reductive with respect to $G \times K$. Namely, the associated indefinite inner product $\langle\cdot,\cdot\rangle_{a,b}$ on $\bar{\mathfrak{m}}_1 + \bar{\mathfrak{m}}_2(s)$ satisfies equation (2.3). This completes the proof of the theorem. □

Another important class of pseudo-Riemannian geodesic orbit manifolds consist of pseudo-Riemannian weakly symmetric manifolds introduced by Selberg [19], Chen and Wolf [8].

Definition 2.6. Let (M, g) be a connected pseudo-Riemannian manifold. Suppose that for every $x \in M$ and every nonzero tangent vector $\xi \in T_xM$, there is an isometry $\phi = \phi_{x,\xi}$ of (M, g) such that $\phi(x) = x$ and $d\phi(\xi) = -\xi$. Then we say that (M, g) is a pseudo-Riemannian weakly symmetric manifold. In particular, a pseudo-Riemannian weakly symmetric manifold is homogeneous.

Definition 2.7. Let G be a Lie group and H be a closed subgroup of G. The pair (G, H) is called a weakly symmetric pair if there exists an automorphism θ of G such that
(i) $\theta(H) \subset H$ and there exists $h \in H$ such that $\theta^2 = \mathrm{Ad}(h)$ (i.e., $\theta^2(g) = hgh^{-1}, g \in G$).
(ii) $H\theta(g)H = Hg^{-1}H$ for all $g \in G$.

Theorem 2.8 ([4, 8]). *Let (G, H) be a weakly symmetric pair and $M = G/H$, then every G-invariant pseudo-Riemannian metric on M is weakly symmetric and geodesic orbit with respect to G.*

At the last of this section, we study the isometry group of the indefinite inner product on the tangent space of a pseudo-Riemannian geodesic orbit manifold. We first need a technical lemma.

Lemma 2.9. *Let $\mathfrak{g}$ be a Lie algebra and $\langle\cdot,\cdot\rangle$ be an indefinite inner product on $\mathfrak{g}$ satisfying the condition of* Lemma 2.2*(the isotropy subalgebra $\mathfrak{h}$ is assumed trivial). Namely, for every $T \in \mathfrak{g}$, there exists a constant $c(T) \in \mathbb{R}$ such that*

$$\langle [T, T'], T\rangle = c(T)\langle T, T'\rangle$$

holds for all $T' \in \mathfrak{g}$. Then we have

$$\langle [T, T'], T\rangle = 0, \forall T, T' \in \mathfrak{g}.$$

Proof. Fix a non-null vector $T \in \mathfrak{g}$, then $c(T) = 0$ and $\langle [T, T'], T\rangle = 0$ holds for all $T' \in \mathfrak{g}$. Set $V_T = \{Y \in \mathfrak{g} | \langle Y, T\rangle = 0\}$, then $\mathfrak{g} = \mathbb{R}T + V_T$. Moreover, for every $Y \in V_T$, $\langle [Y, T], Y\rangle = c(Y)\langle Y, T\rangle = 0$. Now for every $X = \gamma T + Y \in \mathfrak{g}$, $\gamma \in \mathbb{R}$, $Y \in V_T$, we obtain

$$\langle [X, T], X\rangle = \langle [\gamma T + Y, T], \gamma T + Y\rangle = \langle [Y, T], \gamma T\rangle + \langle [Y, T], Y\rangle = 0,$$

which implies that $\mathrm{ad}(T)$ is skew-symmetric.

Finally, assume $T \in \mathfrak{g}$ is a null vector. One can always find another null vector $S \in \mathfrak{g}$ such that $\langle S, T\rangle = 1$. Notice that $\langle 2T + S, 2T + S\rangle = 4$ and $\langle T + S, T + S\rangle = 2$, by the above arguments, we have

$$\langle [X, T], X\rangle = \langle [X, 2T + S], X\rangle - \langle [X, T + S], X\rangle = 0, \forall X \in \mathfrak{g}.$$

This asserts that for all $T \in \mathfrak{g}$, $\mathrm{ad}(T)$ is skew-symmetric, which completes the proof of the lemma. □

Now assume G is a Lie group (not necessarily compact) and H is a compact subgroup of G, which has no nontrivial normal subgroup of G. Let H_0 be the unit component of H and $N_G(H_0)$ be the normalizer of H_0 in G. Note that $N_G(H_0)$ has a well-defined action on the homogeneous space G/H by $g(xH) = gxg^{-1}H$, $\forall g \in N_G(H_0)$, $x \in G$. In [17], Nikonorov proved that the inner product, generating the metric of a Riemannian geodesic orbit manifold, is not only $\mathrm{Ad}(H)$-invariant but also $\mathrm{Ad}(N_G(H_0))$-invariant (see Corollary 4 of [17]). We prove that this statement is still true in pseudo-Riemannian case.

Theorem 2.10. *Notation as above. Let $(G/H, g)$ be a pseudo-Riemannian geodesic orbit manifold with respect to G. Then the indefinite inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{m}$ is not only* $\mathrm{Ad}(H)$-*invariant but also* $\mathrm{Ad}(N_G(H_0))$-*invariant.*

Proof. Let $\mathbf{K}$ be the Killing form of $\mathfrak{g}$, then $\mathbf{K}$ is negative definite on $\mathfrak{h}$ (see Lemma 2 of [17]). So the Lie algebra $\mathfrak{g}$ of G has a $\mathbf{K}$-orthogonal decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{h}$ denotes the Lie algebra of H, $\mathfrak{m}$ is a subspace of $\mathfrak{g}$ and we identify it with the tangent space $T_o(G/H)$ as described above. Let $\langle\cdot,\cdot\rangle$ be the indefinite inner product on $\mathfrak{m}$ determined by g. Obviously, $\langle\cdot,\cdot\rangle$ is $\mathrm{Ad}(H)$-invariant, we will show that it is also $\mathrm{Ad}(N_G(H_0))$-invariant.

According to Proposition 11 of [17], the Lie algebra $N_{\mathfrak{g}}(\mathfrak{h})$ of $N_G(H_0)$ is given by

$$N_{\mathfrak{g}}(\mathfrak{h}) = \{X \in \mathfrak{g} | [X, \mathfrak{h}] \subset \mathfrak{h}\} = C_{\mathfrak{g}}(\mathfrak{h}) + [\mathfrak{h}, \mathfrak{h}],$$

where $C_{\mathfrak{g}}(\mathfrak{h}) = \{X \in \mathfrak{g} | [X, \mathfrak{h}] = 0\}$ denotes the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$. Notice that

$$\mathbf{K}(\mathfrak{h}, [C_{\mathfrak{g}}(\mathfrak{h}), \mathfrak{g}]) = \mathbf{K}([\mathfrak{h}, C_{\mathfrak{g}}(\mathfrak{h})], \mathfrak{g}) = 0,$$

one has $[C_{\mathfrak{g}}(\mathfrak{h}), \mathfrak{g}] \subset \mathfrak{m}$. Hence $C_{\mathfrak{g}}(\mathfrak{h})$ keeps $\mathfrak{m}$ invariant and consequently $\mathrm{Ad}(N_G(H_0))$ keeps $\mathfrak{m}$ invariant. It is clear that $C_{\mathfrak{g}}(\mathfrak{h}) = C_{\mathfrak{g}}(\mathfrak{h}) \cap \mathfrak{h} + C_{\mathfrak{g}}(\mathfrak{h}) \cap \mathfrak{m}$, so to prove the theorem, it is sufficient to prove that for every $T \in C_{\mathfrak{g}}(\mathfrak{h}) \cap \mathfrak{m}$, $\mathrm{ad}(T)|_{\mathfrak{m}}$ is skew-symmetric with respect to $\langle\cdot,\cdot\rangle$.

By Proposition 9 of [17], regarding $\mathfrak{m}$ as an $\mathrm{ad}(\mathfrak{h})$-module, we have the $\mathbf{K}$-orthogonal decomposition

$$\mathfrak{m} = C_{\mathfrak{g}}(\mathfrak{h}) \cap \mathfrak{m} + [\mathfrak{h}, \mathfrak{m}].$$

Moreover, this decomposition is also $\langle\cdot,\cdot\rangle$-orthogonal, since

$$\langle C_{\mathfrak{g}}(\mathfrak{h}) \cap \mathfrak{m}, [\mathfrak{h}, \mathfrak{m}]\rangle = -\langle [\mathfrak{h}, C_{\mathfrak{g}}(\mathfrak{h}) \cap \mathfrak{m}], \mathfrak{m}\rangle = 0.$$

Hence the restriction of $\langle\cdot,\cdot\rangle$ to $C_{\mathfrak{g}}(\mathfrak{h}) \cap \mathfrak{m}$ is nondegenerate. We note also that

$$[\mathfrak{h}, [C_{\mathfrak{g}}(\mathfrak{h}) \cap \mathfrak{m}, C_{\mathfrak{g}}(\mathfrak{h}) \cap \mathfrak{m}]] \subset [[\mathfrak{h}, C_{\mathfrak{g}}(\mathfrak{h}) \cap \mathfrak{m}], C_{\mathfrak{g}}(\mathfrak{h}) \cap \mathfrak{m}] = 0,$$

therefore $[C_{\mathfrak{g}}(\mathfrak{h}) \cap \mathfrak{m}, C_{\mathfrak{g}}(\mathfrak{h}) \cap \mathfrak{m}] \subset C_{\mathfrak{g}}(\mathfrak{h})$ and consequently $[C_{\mathfrak{g}}(\mathfrak{h}) \cap \mathfrak{m}, C_{\mathfrak{g}}(\mathfrak{h}) \cap \mathfrak{m}] \subset C_{\mathfrak{g}}(\mathfrak{h}) \cap \mathfrak{m}$. This asserts that $C_{\mathfrak{g}}(\mathfrak{h}) \cap \mathfrak{m}$ is a Lie subalgebra of $\mathfrak{g}$ and $(C_{\mathfrak{g}}(\mathfrak{h}) \cap \mathfrak{m}, \langle\cdot,\cdot\rangle|_{C_{\mathfrak{g}}(\mathfrak{h}) \cap \mathfrak{m}})$ satisfies the condition of Lemma 2.9. Hence for every $T \in C_{\mathfrak{g}}(\mathfrak{h}) \cap \mathfrak{m}$, $\mathrm{ad}(T)|_{C_{\mathfrak{g}}(\mathfrak{h}) \cap \mathfrak{m}}$ is skew-symmetric. Now for every $X \in [\mathfrak{h}, \mathfrak{m}]$, there exist a vector $Z \in \mathfrak{h}$ and $c(X) \in \mathbb{R}$ such that

$$\langle [T, X + Z]_{\mathfrak{m}}, X\rangle = -c(X)\langle T, X\rangle = 0$$

holds for all $T \in C_{\mathfrak{g}}(\mathfrak{h}) \cap \mathfrak{m}$. Obviously, $[C_{\mathfrak{g}}(\mathfrak{h}) \cap \mathfrak{m}, [\mathfrak{h}, \mathfrak{m}]] \subset [\mathfrak{h}, \mathfrak{m}]$, so the above equality says that $\langle [T, X], X\rangle = 0$, $\forall T \in C_{\mathfrak{g}}(\mathfrak{h}) \cap \mathfrak{m}$. This implies that $\mathrm{ad}(T)|_{\mathfrak{m}}$ is skew-symmetric, which completes the proof of the theorem. □

3. Pseudo-Riemannian geodesic orbit metrics on Tamaru's homogeneous spaces

Let G/H be a compact homogeneous space and K be an intermediate closed subgroup of G, $H < K < G$. Consider the $\mathbf{B}$-orthogonal decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} = \mathfrak{h} + \mathfrak{m}_2 + \mathfrak{m}_1, \mathfrak{k} = \mathfrak{h} + \mathfrak{m}_2,$$

where $\mathfrak{k}$ denotes the Lie algebra of K.

In [11], Gordon studied the geodesic orbit property of G-invariant Riemannian metrics $g_{a,b}$ on G/H generated by $a\mathbf{B}|_{\mathfrak{m}_1} + b\mathbf{B}|_{\mathfrak{m}_2}$, $a, b > 0$.

Theorem 3.1 ([11], Theorem 3.3)**.** *The G-invariant Riemannian metric $g_{a,b}$ is geodesic orbit with respect to G for every $a, b > 0$ if and only if, for every $X \in \mathfrak{m}_1$ and $Y \in \mathfrak{m}_2$, there exists $Z \in \mathfrak{h}$ such that* $[Z, Y] = 0$ *and* $[Z, X] = [Y, X]$.

Later in [20], Tamaru classified the triples (G, K, H) whose Lie algebras satisfy Theorem 3.1 and (G, K) are compact effective irreducible symmetric pairs. These triples of Lie algebras are listed in Table 1. Now we state the main result of this section.

Theorem 3.2. *Let (G, K, H) be a triple listed in* Table 1. *Then for every $a, b \in \mathbb{R}$, $ab < 0$, the G-invariant pseudo-Riemannian metric $g_{a,b}$ on G/H generated by $a\mathbf{B}|_{\mathfrak{m}_1} + b\mathbf{B}|_{\mathfrak{m}_2}$ is geodesic orbit with respect to G.*

Proof. It is easily seen that the metric endomorphism of $g_{a,b}$ is

$$A = a\mathrm{Id}|_{\mathfrak{m}_1} + b\mathrm{Id}|_{\mathfrak{m}_2}.$$

Then for every $T = X + Y \in \mathfrak{m}$, $X \in \mathfrak{m}_1$, $Y \in \mathfrak{m}_2$, by Theorem 3.1 we can choose a $Z \in \mathfrak{h}$ such that $[Z, Y] = 0$ and $[Z, X] = \frac{b-a}{a}[Y, X]$. A direct computation shows that

$$[A(T), T + Z] = [aX + bY, X + Y + Z] = (a - b)[X, Y] + a[X, Z] = 0,$$

which asserts that $g_{a,b}$ is geodesic orbit with respect to G, according to Theorem 1.1. □

Remark 3.3. From the proof of Theorem 3.2, the constant $c = c(T)$ associated to a geodesic vector $T + Z$ is necessarily equal to 0.

4. Pseudo-Riemannian geodesic orbit metrics on spheres

In this section, we study pseudo-Riemannian geodesic orbit metrics on spheres. We will give a complete description of pseudo-Riemannian geodesic orbit, naturally reductive and weakly symmetric metrics on spheres. These results are listed in the last column of Table 2. Spheres can be viewed as a special class of homogeneous spaces. Borel in [5] and Montgomery and Samelson in [14] classified the compact connected Lie groups that admit an effective transitive action on spheres. In Table 2 we list all homogeneous spheres G/H where G is a compact connected Lie group with an effective action on G/H. We also give the isotropy representations of the homogeneous spaces.

• Cases 1, 2, 3. The isotropy representations are irreducible, there are no invariant pseudo-Riemannian metrics.

• Case 4. (Spin(9), Spin(7)) is a weakly symmetric pair [8] and the isotropy representation of Spin(9)/Spin(7) has two irreducible components. The family of pseudo-Riemannian invariant metrics on Spin(9)/Spin(7) is two-parametric. All these pseudo-Riemannian metrics are weakly symmetric but not naturally reductive, according to Theorem 2.4.

• Case 5. $(\mathrm{U}(n + 1), \mathrm{U}(n))$ is a weakly symmetric pair and the family of pseudo-

Table 1. Tamaru's homogeneous spaces.

	$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{h}$	Cond.
1.1	$\mathfrak{so}(2n+1)$	$\mathfrak{so}(2n)$	$\mathfrak{u}(n)$	$n \geq 2$
1.2	$\mathfrak{so}(4n+1)$	$\mathfrak{so}(4n)$	$\mathfrak{su}(2n)$	$n \geq 1$
1.3	$\mathfrak{so}(8)$	$\mathfrak{so}(7)$	$\mathfrak{g}_2$	
1.4	$\mathfrak{so}(9)$	$\mathfrak{so}(8)$	$\mathfrak{so}(7)$	
1.5	$\mathfrak{su}(n+1)$	$\mathfrak{u}(n)$	$\mathfrak{su}(n)$	$n \geq 2$
1.6	$\mathfrak{su}(2n+1)$	$\mathfrak{u}(2n)$	$\mathfrak{u}(1) \oplus \mathfrak{sp}(n)$	$n \geq 2$
1.7	$\mathfrak{su}(2n+1)$	$\mathfrak{u}(2n)$	$\mathfrak{sp}(n)$	$n \geq 2$
1.8	$\mathfrak{sp}(n+1)$	$\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$	$\mathfrak{u}(1) \oplus \mathfrak{sp}(n)$	$n \geq 1$
1.9	$\mathfrak{sp}(n+1)$	$\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$	$\mathfrak{sp}(n)$	$n \geq 1$
2.1	$\mathfrak{su}(2r+n)$	$\mathfrak{su}(r) \oplus \mathfrak{su}(r+n) \oplus \mathbb{R}$	$\mathfrak{su}(r) \oplus \mathfrak{su}(r+n)$	$r \geq 2, n \geq 1$
2.2	$\mathfrak{so}(4r+2)$	$\mathfrak{u}(2r+1)$	$\mathfrak{su}(2r+1)$	$r \geq 2$
2.3	$\mathfrak{e}_6$	$\mathbb{R} \oplus \mathfrak{so}(10)$	$\mathfrak{so}(10)$	
3.1	$\mathfrak{so}(9)$	$\mathfrak{so}(7) \oplus \mathfrak{so}(2)$	$\mathfrak{g}_2 \oplus \mathfrak{so}(2)$	
3.2	$\mathfrak{so}(10)$	$\mathfrak{so}(8) \oplus \mathfrak{so}(2)$	$\mathfrak{spin}(7) \oplus \mathfrak{so}(2)$	
3.3	$\mathfrak{so}(11)$	$\mathfrak{so}(8) \oplus \mathfrak{so}(3)$	$\mathfrak{spin}(7) \oplus \mathfrak{so}(3)$	

Table 2. Homogeneous spheres.

	G/H	iso. rep	Cond.	
1	$\mathrm{SO}(n+1)/\mathrm{SO}(n)$	irreducible	$n \geq 1$	
2	$\mathrm{G}_2/\mathrm{SU}(3)$	irreducible		
3	$\mathrm{Spin}(7)/\mathrm{G}_2$	irreducible		
4	$\mathrm{Spin}(9)/\mathrm{Spin}(7)$	$\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2$		w.s.
5	$\mathrm{U}(n+1)/\mathrm{U}(n)$	$\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{m}_1$	$n \geq 1$	w.s., n.r.
6	$\mathrm{SU}(n+1)/\mathrm{SU}(n)$	$\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{m}_1$	$n \geq 2$	w.s., g.o.
7	$\mathrm{SU}(2)$	trivial		
8	$\mathrm{Sp}(n+1)/\mathrm{Sp}(n)$	$\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{m}_1$	$n \geq 1$	
9	$\mathrm{Sp}(n+1)\mathrm{Sp}(1)/\mathrm{Sp}(n)\mathrm{diag}(\mathrm{Sp}(1))$	$\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2$	$n \geq 1$	w.s., n.r.
10	$\mathrm{Sp}(n+1)\mathrm{U}(1)/\mathrm{Sp}(n)\mathrm{diag}(\mathrm{U}(1))$	$\mathfrak{m} = \widetilde{\mathfrak{m}}_1 + \widetilde{\mathfrak{m}}_2 + \widetilde{\mathfrak{m}}_3$	$n \geq 1$	w.s.

Riemannian invariant metrics on $\mathrm{U}(n+1)/\mathrm{U}(n)$ is two-parametric. Every such metric is naturally reductive and weakly symmetric [8].

• Case 6. This case is just the case 1.5 of Table 1. Hence every $\mathrm{SU}(n+1)$-invariant pseudo-Riemannian metric on $\mathrm{SU}(n+1)/\mathrm{SU}(n)$ is geodesic orbit with respect to $\mathrm{SU}(n+1)$. Moreover, note that every $\mathrm{SU}(n+1)$-invariant pseudo-Riemannian metric on $\mathrm{SU}(n+1)/\mathrm{SU}(n)$ is $\mathrm{U}(n+1)$-invariant. Hence every such metric is weakly symmetric and naturally reductive with respect to $\mathrm{U}(n+1)$ (not $\mathrm{SU}(n+1)$), see Theorem 2.5.

• Case 7. We will show that $\mathrm{SU}(2)$ admits no left invariant pseudo-Riemannian geodesic orbit metrics with respect to $\mathrm{SU}(2)$.

Lemma 4.1. *Assume A is an invertible linear isomorphism on $\mathfrak{su}(2)$. If for every $T \in \mathfrak{su}(2)$, there exists $c \in \mathbb{R}$ satisfying*

$$[A(T), T] = cA(T),$$

then $A = \gamma \mathrm{Id}$ *for some* $\gamma \in \mathbb{R}$.

Proof. Since $[A(T), T] = cA(T)$, T is in the normalizer $N(\mathbb{R} \cdot A(T))$ of $\mathbb{R} \cdot A(T)$ in $\mathfrak{su}(2)$. As $\mathbb{R} \cdot A(T)$ is the maximal torus subalgebra of $\mathfrak{su}(2)$, the normalizer $N(\mathbb{R} \cdot A(T))$ must be one-dimensional subalgebra of $\mathfrak{su}(2)$. Consequently, $N(\mathbb{R} \cdot A(T)) = \mathbb{R} \cdot A(T)$ and there exists $c \in \mathbb{R}$ such that $A(T) = cT$. Since T is an arbitrary element in $\mathfrak{su}(2)$ we have $A = \gamma \mathrm{Id}$ for some $\gamma \in \mathbb{R}$. □

Theorem 4.2. *Let g be any left invariant pseudo-Riemannian metric on* SU(2)*, then it is not geodesic orbit with respect to* SU(2)*.*

Proof. Let A be the metric endomorphism corresponding to the left invariant pseudo-Riemannian metric g on SU(2). By Theorem 1.1, g is geodesic orbit with respect to SU(2) if and only if for every $T \in \mathfrak{su}(2)$, there exists $c \in \mathbb{R}$ such that $[A(T), T] = cA(T)$. The theorem follows from Lemma 4.1. □

• Cases 8 and 9. Case 9 is a special situation of case 8. We adopt the notation as in [16]. Let $\mathbb{H} = \mathbb{R} + \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$ be the field of quaternions, where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the quaternionic units in $\mathbb{H}$. That is, $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$, $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$, $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$, $\mathbf{ii} = \mathbf{jj} = \mathbf{kk} = -1$. For $u = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \in \mathbb{H}$, $x_0, x_1, x_2, x_3 \in \mathbb{R}$, define $\mathrm{Re}(u) = x_0$ and $\bar{u} = x_0 - x_1\mathbf{i} - x_2\mathbf{j} - x_3\mathbf{k}$. In the Lie algebra $\mathfrak{sp}(n+1)$ of Lie group $\mathrm{Sp}(n+1)$, we define

$$\mathbf{B}(X, Y) = -\frac{1}{2}\mathrm{tr}\,(\mathrm{Re}(XY)), \quad X, Y \in \mathfrak{sp}(n+1).$$

It is easy to see that $\mathbf{B}$ is an $\mathrm{Ad}(\mathrm{Sp}(n+1))$-invariant inner product on the Lie algebra $\mathfrak{sp}(n+1)$. So we have a $\mathbf{B}$-orthogonal reductive decomposition of $\mathfrak{sp}(n+1)$:

$$\mathfrak{sp}(n+1) = \mathfrak{sp}(n) + \mathfrak{m} = \mathfrak{sp}(n) + \mathfrak{m}_0 + \mathfrak{m}_1,$$

where $\mathfrak{m}_0 = \mathbb{R}\mathbf{i}G_1 + \mathbb{R}\mathbf{j}G_1 + \mathbb{R}\mathbf{k}G_1$, G_1 denotes the matrix with $\sqrt{2}$ in the (1, 1)-th entry, and zeros elsewhere,

$$\mathfrak{m}_1 = \left\{ \left(\begin{array}{cc} 0 & \alpha \\ -\bar{\alpha}' & 0_n \end{array} \right) \middle| \alpha = (u_1, u_2, \ldots, u_n) \in \mathbb{H}^n \right\}, \quad \bar{\alpha} = (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n).$$

Every $\mathrm{Sp}(n+1)$-invariant pseudo-Riemannian metric g on S^{4n+3} can be generated by a metric endomorphism $A = \tilde{A} + a\mathrm{Id}|_{\mathfrak{m}_1}$ for some $a \in \mathbb{R}$ and some symmetric nondegenerate operator $\tilde{A} : \mathfrak{m}_0 \to \mathfrak{m}_0$. In particular, when $\tilde{A} = b\mathrm{Id}|_{\mathfrak{m}_0}$ for some $b \in \mathbb{R}$, the corresponding $\mathrm{Sp}(n+1)$-invariant pseudo-Riemannian metric g on S^{4n+3} is $\mathrm{Sp}(n+1)\mathrm{Sp}(1)$-invariant and naturally reductive with respect to $\mathrm{Sp}(n+1)\mathrm{Sp}(1)$, according to Theorem 2.5. Moreover, the pair $(\mathrm{Sp}(n+1)\mathrm{Sp}(1), \mathrm{Sp}(n)\mathrm{diag}(\mathrm{Sp}(1)))$ is also a weakly symmetric pair.

Proof of Theorem 1.2. Assume g is geodesic orbit with respect to $\mathrm{Sp}(n+1)$, then according to Theorem 1.1, for every $T \in \mathfrak{m}_0$, there exists $Z \in \mathfrak{sp}(n)$ and $c \in \mathbb{R}$ such that

$$[\tilde{A}(T), T + Z] = c\tilde{A}(T). \tag{4.4}$$

Since $\tilde{A}(T) \in \mathfrak{m}_0$ and $[\tilde{A}(T), Z] = 0$, we obtain that equation (4.4) is equivalent to $[\tilde{A}(T), T] =$

$c\tilde{A}(T)$. By Lemma 4.1 and the fact that $\mathfrak{m}_0$ is isomorphic to $\mathfrak{su}(2)$, we get $\tilde{A} = \gamma \mathrm{Id}$ for some $\gamma \in \mathbb{R}$. Consequently, g is $\mathrm{Sp}(n+1)\mathrm{Sp}(1)$-invariant.

On the other hand, we consider any $\mathrm{Sp}(n+1)\mathrm{Sp}(1)$-invariant pseudo-Riemannian metric on S^{4n+3}. It is generated by a metric endomorphism of the type $A = a\mathrm{Id}|_{\mathfrak{m}_1} + b\mathrm{Id}|_{\mathfrak{m}_0}$, $a, b \in \mathbb{R}$, $ab < 0$. It coincides with the case 1.9 of Table 1, hence by Theorem 3.2, every $\mathrm{Sp}(n+1)\mathrm{Sp}(1)$-invariant pseudo-Riemannian metric on S^{4n+3} is geodesic orbit with respect to $\mathrm{Sp}(n+1)$. □

• Case 10. The family of $\mathrm{Sp}(n+1)\mathrm{U}(1)$-invariant pseudo-Riemannian metrics on S^{4n+3} is three-parametric. Every such metric is weakly symmetric and we have a three-parameter family of pseudo-Riemannian geodesic orbit metrics. In the following, we give an explicit description of geodesic vectors for $\mathrm{Sp}(n+1)\mathrm{U}(1)$-invariant pseudo-Riemannian metrics on S^{4n+3}.

Identify $\mathfrak{sp}(1)$ with $\mathfrak{m}_0$, $\mathbf{B}|_{\mathfrak{sp}(1)}$ is an $\mathrm{Ad}(\mathrm{Sp}(1))$-invariant inner product and we can extend $\mathbf{B}$ to an $\mathrm{Ad}(\mathrm{Sp}(n+1)\mathrm{Sp}(1))$-invariant inner product (also denoted by $\mathbf{B}$) on the Lie algebra $\mathfrak{sp}(n+1)\oplus\mathfrak{sp}(1)$ by assuming $\mathbf{B}((\mathfrak{sp}(n+1), 0), (0, \mathfrak{sp}(1))) = 0$. Let $\mathfrak{u}(1)$ be any Lie subalgebra of $\mathfrak{sp}(1)$ ($\cong \mathfrak{m}_0$) and $\mathfrak{m}_2$ be the $\mathbf{B}$-orthogonal complement of $\mathfrak{u}(1)$ in $\mathfrak{m}_0$. We have a $\mathbf{B}$-orthogonal reductive decomposition of $\mathfrak{sp}(n+1) \oplus \mathfrak{u}(1)$:

$$\mathfrak{sp}(n+1) \oplus \mathfrak{u}(1) = \widetilde{\mathfrak{h}} + \widetilde{\mathfrak{m}}_1 + \widetilde{\mathfrak{m}}_2 + \widetilde{\mathfrak{m}}_3,$$

where $\widetilde{\mathfrak{h}} = \widetilde{\mathfrak{h}}_1 + \widetilde{\mathfrak{h}}_2$, $\widetilde{\mathfrak{h}}_1 = \{(X, 0) \in \mathfrak{sp}(n+1)\oplus\mathfrak{u}(1) | X \in \mathfrak{sp}(n)\}$, $\widetilde{\mathfrak{h}}_2 = \{(X, X) \in \mathfrak{sp}(n+1)\oplus\mathfrak{u}(1) | X \in \mathfrak{u}(1)\}$, $\widetilde{\mathfrak{m}}_1 = \{(X, 0) \in \mathfrak{sp}(n+1) \oplus \mathfrak{u}(1) | X \in \mathfrak{m}_1\}$, $\widetilde{\mathfrak{m}}_2 = \{(X, 0) \in \mathfrak{sp}(n+1) \oplus \mathfrak{u}(1) | X \in \mathfrak{m}_2\}$, $\widetilde{\mathfrak{m}}_3 = \{(X, -X) \in \mathfrak{sp}(n+1) \oplus \mathfrak{u}(1) | X \in \mathfrak{u}(1)\}$. We have the following relations:

$$[\widetilde{\mathfrak{h}}_1, \widetilde{\mathfrak{h}}_2] = 0, \quad [\widetilde{\mathfrak{h}}_1, \widetilde{\mathfrak{m}}_2] = 0, \quad [\widetilde{\mathfrak{h}}_1, \widetilde{\mathfrak{m}}_3] = 0, \quad [\widetilde{\mathfrak{h}}_2, \widetilde{\mathfrak{m}}_3] = 0.$$

It is easy to see that the modules $\widetilde{\mathfrak{m}}_i$, $i = 1, 2, 3$, are $\mathrm{ad}(\widetilde{\mathfrak{h}})$-irreducible. Then every $\mathrm{Sp}(n+1)\mathrm{U}(1)$-invariant pseudo-Riemannian metric g on S^{4n+3} is determined by the metric endomorphism

$$A = x_1 \mathrm{Id}|_{\widetilde{\mathfrak{m}}_1} + x_2 \mathrm{Id}|_{\widetilde{\mathfrak{m}}_2} + x_3 \mathrm{Id}|_{\widetilde{\mathfrak{m}}_3}$$

for some nonzero numbers $x_1, x_2, x_3 \in \mathbb{R}$.

Without losing generality, we may assume $\mathfrak{u}(1) = \mathbb{R}\mathbf{i}G_1$ and then $\mathfrak{m}_2 = \mathbb{R}\mathbf{j}G_1 + \mathbb{R}\mathbf{k}G_1$.

Proposition 4.3. *Notation as above. For every* $T_1 = (V, 0) \in \widetilde{\mathfrak{m}}_1$, $T_2 = ((s\mathbf{j} + t\mathbf{k})G_1, 0) \in \widetilde{\mathfrak{m}}_2$, $T_3 = (r\mathbf{i}G_1, -r\mathbf{i}G_1) \in \widetilde{\mathfrak{m}}_3$, $r, s, t \in \mathbb{R}$, *let* $Z_1 = (U, 0) \in \widetilde{\mathfrak{h}}_1$, $Z_2 = \Big((\frac{x_3}{x_2} - 1)r\mathbf{i}G_1, (\frac{x_3}{x_2} - 1)r\mathbf{i}G_1\Big) \in \widetilde{\mathfrak{h}}_2$, *where* U *satisfies*

$$[U, V] = \left[\Big((\frac{x_3}{x_1} - \frac{x_3}{x_2})r\mathbf{i} + (\frac{x_2}{x_1} - 1)(s\mathbf{j} + t\mathbf{k})\Big)G_1, V\right].$$

Then $T_1 + T_2 + T_3 + Z_1 + Z_2$ *is a geodesic vector.*

Proof. It follows by a direct calculation.

$$\begin{aligned}
& [A(T), T + Z] \\
= \; & (x_1 - x_2)[T_1, T_2] + (x_1 - x_3)[T_1, T_3] + x_1[T_1, Z_1] + x_1[T_1, Z_2] \\
& + x_2[T_2, Z_2] + (x_2 - x_3)[T_2, T_3]
\end{aligned}$$

$$
\begin{aligned}
&= (x_1 - x_2)[(V,0),((s\mathbf{j}+t\mathbf{k})G_1,0)] + (x_1 - x_3)[(V,0),(r\mathbf{i}G_1,-r\mathbf{i}G_1)] \\
&\quad + x_1[(V,0),(U,0)] + x_1\left[(V,0),\left(\left(\frac{x_3}{x_2}-1\right)r\mathbf{i}G_1,\left(\frac{x_3}{x_2}-1\right)r\mathbf{i}G_1\right)\right] \\
&\quad + x_2\left[((s\mathbf{j}+t\mathbf{k})G_1,0),\left(\left(\frac{x_3}{x_2}-1\right)r\mathbf{i}G_1,\left(\frac{x_3}{x_2}-1\right)r\mathbf{i}G_1\right)\right] \\
&\quad + (x_2 - x_3)[((s\mathbf{j}+t\mathbf{k})G_1,0),(r\mathbf{i}G_1,-r\mathbf{i}G_1)] \\
&= ((x_1 - x_2)[V,(s\mathbf{j}+t\mathbf{k})G_1] + (x_1 - x_3)[V, r\mathbf{i}G_1] + x_1[V,U] \\
&\quad + x_1\left[V,\left(\frac{x_3}{x_2}-1\right)r\mathbf{i}G_1\right] + x_2\left[(s\mathbf{j}+t\mathbf{k})G_1,\left(\frac{x_3}{x_2}-1\right)r\mathbf{i}G_1\right] \\
&\quad + (x_2 - x_3)[(s\mathbf{j}+t\mathbf{k})G_1, r\mathbf{i}G_1],0) \\
&= \left(\left[V,\left((x_1-x_2)(s\mathbf{j}+t\mathbf{k}) + (x_1-x_3)r\mathbf{i} + x_1\left(\frac{x_3}{x_2}-1\right)r\mathbf{i}\right)G_1\right] + x_1[V,U],0\right) \\
&= 0. \qquad \square
\end{aligned}
$$

Remark 4.4. For the explicit matrix form of U in Proposition 4.3, one can see Lemma 3 of [16] or Lemma 6.2 of [1].

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References

[1] H. An, Z. Yan and S. Zhang: *S-curvature and geodesic orbit property of invariant (α_1, α_2)-metrics on spheres*, Bull. Korean Math. Soc. **60** (2023), 33–46.

[2] A. Arvanitoyeorgos: *Homogeneous manifolds whose geodesics are orbits: recent results and some open problems*, Irish Math. Soc. Bull. **79** (2017), 5–29.

[3] V. Berestovskiĭ and Y. Nikonorov: Riemannian Manifolds and Homogeneous Geodesics, Springer-Verlag, 2020.

[4] J. Berndt, O. Kowalski and L. Vanhecke: *Geodesics in weakly symmetric spaces*, Ann. Glob. Anal. Geom. **15** (1997), 153–156.

[5] A. Borel: *Some remarks about Lie groups transitive on spheres and tori*, Bull. Amer. Math. Soc. **55** (1949), 580–587.

[6] H. Chen, S. Zhang and F. Zhu: *On Randers geodesic orbit spaces*, Differential Geom. Appl. **85** (2022), Paper No. 101939, 17pp.

[7] Z. Chen, Y. Nikolayevsky and Y. Nikonorov: *Compact geodesic orbit spaces with a simple isotropy group*, Ann. Global Anal. Geom. **63** (2023), no. 1, Paper No. 7, 34pp.

[8] Z. Chen and J.A. Wolf: *Pseudo-Riemannian weakly symmetric manifolds*, Ann. Glob. Anal. Geom. **41** (2012), 381–390.

[9] S. Deng: Homogeneous Finsler Spaces, Springer-Verlag, New York, USA, 2012.

[10] Z. Dušek and O. Kowalski: *Light-like homogeneous geodesics and the geodesic lemma for any signature*, Publ. Math. Debrecen. **71** (2007), 245–252.

[11] C.S. Gordon: *Homogeneous Riemannian manifolds whose geodesics are orbits*, in: Topics in Geometry in Memory of Joseph D'Arti (ed. S. Gindikin), Birkhäuser, 1996, 155–174.

[12] C.S. Gordon and Y. Nikonorov: *Geodesic orbit Riemannian structures on $\mathbb{R}^n$*, J. Geom. Phys. **134** (2018), 235–243.

[13] O. Kowalski and L. Vanhecke: *Riemannian manifolds with homogeneous geodesics*, Boll. Un. Math. Ital. Ser. VII B **5** (1991), 189–246.

[14] D. Montgomery and H. Samelson: *Transformation groups of spheres*, Ann. Math. **44** (1943), 454–470.
[15] Y. Nikolayevsky and J.A. Wolf: *The structure of geodesic orbit Lorentz nilmanifolds*, J. Geom. Anal. **33** (2023), Paper No. 82, 12pp.
[16] Y. Nikonorov: *Geodesic orbit Riemannian metrics on spheres*, Vladikavkaz. Mat. Zh. **15** (2013), 67–76.
[17] Y. Nikonorov: *On the structure of geodesic orbit Riemannian spaces*, Ann. Glob. Anal. Geom. **52** (2017), 289–311.
[18] G. Ovando: *Naturally reductive pseudo-Riemannian spaces*, J. Geom. Phys. **61** (2011), 157–171.
[19] A. Selberg: *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc. **20** (1956), 47–87.
[20] H. Tamaru: *Riemannian g.o. spaces fibered over irreducible symmetric spaces*, Osaka J. Math. **36** (1999), 835–851.
[21] J.A. Wolf: *Families of geodesic orbit spaces and related pseudo-Riemannian manifolds*, J. Geom. Anal. **33** (2023), Paper No. 214, 7pp.
[22] M. Xu: *Geodesic orbit spheres and constant curvature in Finsler geometry*, Diff. Geom. Appl. **61** (2018), 197–206.
[23] Z. Yan: *Some Finsler spaces with homogeneous geodesics*, Math. Nachr. **290** (2017), 474–481.
[24] Z. Yan and S. Deng: *Finsler spaces whose geodesics are orbits*, Differential Geom. Appl. **36** (2014), 1–23.
[25] L. Zhang and M. Xu: *Standard homogeneous* (α_1, α_2)*-metrics and geodesic orbit property*, Math. Nachr. **295** (2022), 1443–1453.
[26] S. Zhang and Z. Yan: *Geodesic orbit Randers metrics on spheres*, Adv. Geom. **21** (2021), 273–280.
[27] N.P. Souris: *Geodesic orbit metrics in compact homogeneous manifolds with equivalent isotropy submodules*, Transform. Groups. **23** (2018), 1149–1165.
[28] W. Ziller: *The Jacobi equation on naturally reductive compact Riemannian homogeneous spaces*, Comment. Math. Helv. **52** (1977), 573–590.

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