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ON QUASI-ALTERNATING KNOTS WITH SYMMETRIC UNION PRESENTATIONS

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Abstract

There are only finitely many alternating symmetric unions for a given partial knot. In this paper, we give a formula for the Q-polynomial of a knot with the symmetric union presentation $D \cup D^*(m)$ and show that, if $2\deg Q(D) > \deg Q(D \cup D^*(\infty))$, then there are only finitely many quasi-alternating knots with the symmetric union presentation $D \cup D^*(m)$ for any knot diagram D. We also give a formula for the Q-polynomial of a knot with the symmetric union presentation $D \cup D^*(m_1, m_2)$.

1. Introduction

A *link* is a closed oriented 1-manifold smoothly embedded in the 3-sphere S^3. A *knot* is a link with one component. A knot with a *symmetric union presentation*, called a *symmetric union* [12, 13], is known to be an example of a *ribbon knot* [15] which bounds a smooth disk in the 4-ball with boundary S^3 with no local maxima. Conversely, every ribbon knot with crossing number ≤ 10 is a symmetric union [4, 13]. Furthermore, it is known that all 2-bridge ribbon knots are symmetric unions, see [14, 16].

Alternating links represent a class of links which is of central importance in classical knot theory. They have been subject to entensive study and have been generalized into several directions. In particular, the study of the Heegaard Floer homology of branched double covers along alternating links led to the definition of quasi-alternating links [20]; an interesting class of links defined recursively on diagrams which share many homological properties with alternating links. It is worth mentioning here that it can be easily proved that there are only finitely many alternating symmetric unions for a given partial knot. (See Proposition 4.1). The question that we consider in this research work is the following:

Question. Is the number of quasi-alternating symmetric unions finite for a given partial knot?
Indeed, this question is related to the following conjecture of Greene [5].

Conjecture. There are only finitely many quasi-alternating links with a given determinant.
If the conjecture is true, then the answer to the question above is affimative since the determinant of a knot with a symmetric union presentation is determined by that of the partial knot [13].

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Let $Q(D)$ be the *Q-polynomial* of a knot with a diagram D. In this paper, we prove the following results.

Theorem 1.1. *Let D be a knot diagram. If $2\deg Q(D) > \deg Q(D \cup D^*(\infty))$, then there are only finitely many quasi-alternating knots with the symmetric union presentation $D \cup D^*(m)$.*

Corollary 1.2. *Let D be a knot diagram. If $Q(D) \neq 1$ and $D \cup D^*(\infty)$ is a trivial link diagram, then there are only finitely many quasi-alternating knots with the symmetric union presentation $D \cup D^*(m)$.*

Throughout the rest of this paper, the notation for prime knots up to 10 crossings is due to Rolfsen's book [23]. Here is an outline of this paper. In Section 2, we shall give the definitions of symmetric unions and quasi-alternating links. In Section 3, we shall define the Q-polynomial of a link and give a formula of the Q-polynomial for knots with symmetric union presentations with one twist region. In Section 4, we shall study quasi-alternating symmetric unions and prove Theorem 1.1 and Corollary 1.2. In Section 5, we shall study examples of knots with a symmetric union presentation with one twist region. In Section 6, we shall give a formula for the Q-polynomial of a knot with a symmetric union presentation with two twist regions.

2. Definitions

Definition 2.1. Let $\mathbb{R}^3$ be the Euclidean 3-dimensional space with x, y and z-axes. A *symmetric union* in $\mathbb{R}^3 \subset S^3$ is a knot in S^3, defined as follows. We denote the tangle made of m half-twists by an integer $m \in \mathbb{Z}$ and the horizontal trivial tangle by ∞ as in Figure 1. We take a knot $\hat{K}$ in $\mathbb{R}^3_- = \{(x, y, z)|x < 0\}$ and its mirror image $\hat{K}^*$ in $\mathbb{R}^3_+ = \{(x, y, z)|x > 0\}$ such that $\hat{K}$ and $\hat{K}^*$ are symmetric with respect to the yz-plane $\mathbb{R}^2_{yz}$ as in Figure 2(a). Here we consider a diagram of a knot in the xz-plane $\mathbb{R}^2_{xz}$ and we denote the diagrams of $\hat{K}$ and $\hat{K}^*$ by D and D^* respectively. We regard each disk-arc pairs of $T_0, T_1, \ldots, T_k$ as in Figure 2(a) as a diagram of the tangle 0. Then we replace the tangles $T_0, T_1, \ldots, T_k$ with tangles ∞, $m_1, m_2, \ldots, m_k$ as in Figure 2(b). Here we assume that $m_i \neq \infty$ $(1 \leq i \leq k)$. The resulting diagram is called a *symmetric union presentation* and we denote it by $D \cup D^*(m_1, \ldots, m_k)$. The tangles are called the *twist regions* of the symmetric union presentation. The knot $\hat{K}$ is called the *partial knot* for the symmetric union presentation. Note that $D \cup D^*(m_1, \ldots, m_k)$ represents a knot and the knot is called a *symmetric union*.

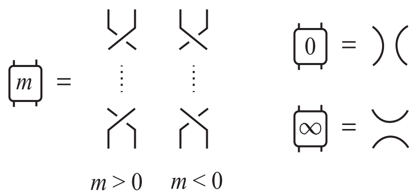

Fig. 1. The notation of tangles

We define the *determinant* of a link L as $|\Delta_L(-1)|$, where $\Delta_L(t)$ is the *Alexander polynomial* of L [23]. This determinant will be denoted hereafter by $\det(L)$. Now we shall give the

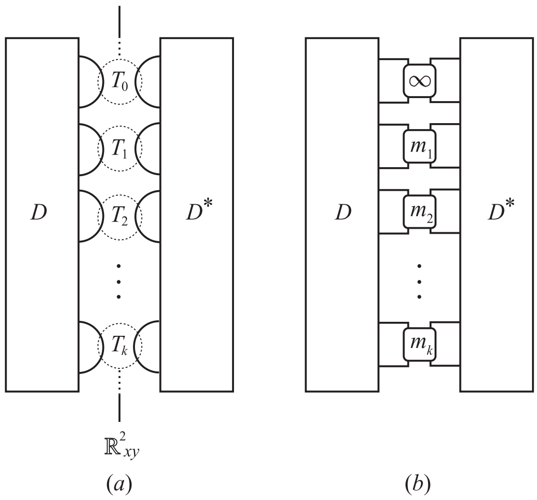

Fig. 2. A symmetric union

recursive definition of quasi-alternating links.

Definition 2.2. The set $\mathbf{S}$ of *quasi-alternating links* is the smallest set of links such that
(1) the unknot O belongs to $\mathbf{S}$,
(2) if L is a link with a diagram containing a crossing for which the two resolutions L_0 and L_∞ belong to $\mathbf{S}$, $\det(L_0) \geq 1$, $\det(L_\infty) \geq 1$, and
$\det(L) = \det(L_0) + \det(L_\infty)$, then L belongs to $\mathbf{S}$. (The links L, L_0 and L_∞ are shown in Figure 3.)

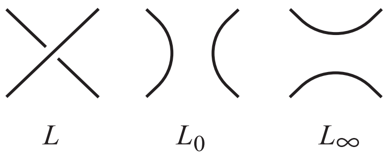

Fig. 3. Three links which are identical except in a small ball.

Remark 2.3. For prime knots with up to 10 crossings, there are exactly 21 symmetric unions. The knots 6_1, 8_8, 8_9, 9_{27}, 9_{41}, 10_3, 10_{22}, 10_{35}, 10_{42}, 10_{48}, 10_{75}, 10_{87}, 10_{99} and 10_{123} are alternating. The knots 8_{20}, 10_{129}, 10_{137} and 10_{155} are not alternating but quasi-alternating. The knots 9_{46}, 10_{140}, 10_{153} are not quasi-alternating [17].

3. The Q-polynomial of symmetric unions

In [1], Brandt, Lickorish and Millett introduced a link invariant $Q_L(x)$. For any link L, $Q_L(x)$ is a Laurent polynomial which can be defined by $Q_O(x) = 1$ and a recursive relation on link diagrams as follows:

$$Q_{L_+}(x) + Q_{L_-}(x) = x(Q_{L_0}(x) + Q_{L_\infty}(x))$$

where L_+, L_-, L_0 and L_∞ are four links which are identical except in a small ball where they are as in Figure 4.

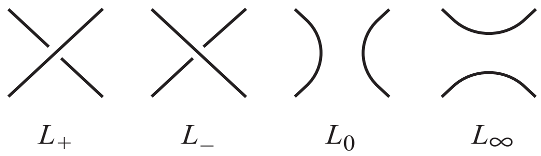

Fig.4. Four links which are identical except in a small ball.

We call $Q_L(x)$ the *Q-polynomial* of L. We also denote the Q-polynomial of a knot with a diagram D by $Q(D)$. Let $S_m(= S_m(x))$ be the m-th *Chebyshev polynomial* of the first kind which is defined inductively by $S_{-1}(x) = 0$, $S_0(x) = 1$ and $S_k(x) = xS_{k-1}(x) - S_{k-2}(x)$. Let $F_m = \dfrac{1 - S_{|m|-1} + S_{|m|-2}}{2x^{-1} - 1}$.

Proposition 3.1. *For any integer m, we have*

$$Q(D \cup D^*(m)) = \Big(\frac{x}{2}S_{|m|-1} - S_{|m|-2}\Big)(Q(D))^2 + \Big(\frac{x}{2}S_{|m|-1} + F_m\Big)Q(D \cup D^*(\infty)).$$

Proof. First we prove the following formula.

$$Q(D \cup D^*(m)) = S_{|m|-1}Q(D \cup D^*(1)) - S_{|m|-2}Q(D \cup D^*(0)) + F_mQ(D \cup D^*(\infty)).$$

In the case when $m = 0$, we have

$$S_{-1}Q(D \cup D^*(1)) - S_{-2}Q(D \cup D^*(0)) + F_0Q(D \cup D^*(\infty)) = Q(D \cup D^*(0))$$

since $S_{-1} = F_0 = 0$ and $S_{-2} = -1$. In the case when $m \geq 1$, we shall proceed by induction on m as follows. In the case when $m = 1$, we have

$$S_0Q(D \cup D^*(1)) - S_{-1}Q(D \cup D^*(0)) + F_1Q(D \cup D^*(\infty)) = Q(D \cup D^*(1))$$

since $S_{-1} = F_1 = 0$ and $S_0 = 1$. Assume that the formula holds in the case when $m \leq k - 1$ (≥ 1). In the case when $m = k$, by the assumption, we have

$$\begin{aligned}
Q(D \cup D^*(k)) &= xQ(D \cup D^*(k-1)) - Q(D \cup D^*(k-2)) + xQ(D \cup D^*(\infty)) \\
&= x(S_{k-2}Q(D \cup D^*(1)) - S_{k-3}Q(D \cup D^*(0)) + F_{k-1}Q(D \cup D^*(\infty))) \\
&\quad - (S_{k-3}Q(D \cup D^*(1)) - S_{k-4}Q(D \cup D^*(0)) + F_{k-2}Q(D \cup D^*(\infty))) \\
&\quad + xQ(D \cup D^*(\infty)) \\
&= (xS_{k-2} - S_{k-3})Q(D \cup D^*(1)) - (xS_{k-3} - S_{k-4})Q(D \cup D^*(0)) \\
&\quad + (xF_{k-1} - F_{k-2} + x)Q(D \cup D^*(\infty)). \\
&= S_{k-1}Q(D \cup D^*(1)) - S_{k-2}Q(D \cup D^*(0)) + G(x)Q(D \cup D^*(\infty)),
\end{aligned}$$

where $G(x) = xF_{k-1} - F_{k-2} + x$.

Now we consider the diagram of the unknot for D as shown in Figure 5.

Then we have $1 = S_{k-1} - S_{k-2} + G(x)(2x^{-1} - 1)$. So we obtain that $G(x) = \dfrac{1 - S_{k-1} + S_{k-2}}{2x^{-1} - 1}$.

In the case when $m \leq -1$, we obtain the formula since the mirror image of $D \cup D^*(m)$ is $D \cup D^*(-m)$, $Q(D \cup D^*(m)) = Q(D \cup D^*(-m))$ and $F_m = F_{-m}$.

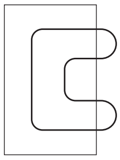

Fig. 5. A diagram of the unknot

Next, by the definition of Q-polynomial, we have

$$Q(D \cup D^*(1)) + Q(D \cup D^*(-1)) = x(Q(D \cup D^*(0)) + Q(D \cup D^*(\infty))).$$

Since $D \cup D^*(1)$ is the mirror image of $D \cup D^*(-1)$, we have $Q(D \cup D^*(1)) = Q(D \cup D^*(-1))$ and $Q(D \cup D^*(1)) = \frac{x}{2}(Q(D \cup D^*(0)) + Q(D \cup D^*(\infty)))$. By substituting this formula into the formula shown above, we obtain the result. (Note that $Q(D \cup D^*(0)) = (Q(D))^2$ since $D \cup D^*(0)$ is a diagram of the connected sum of the partial knot and its mirror image.) This completes the proof. □

We denote the maximum degree of $Q_L(x)$ (or $Q(D)$) by $\deg Q_L(x)$ (or $\deg Q(D)$) for a link L (or the diagram D).

Corollary 3.2. *If* $2\deg Q(D) > \deg Q(D \cup D^*(\infty))$, *then* $\deg Q(D \cup D^*(m)) = |m| + 2\deg Q(D)$.

Proof. The case when $m = 0$ is obvious since $Q(D \cup D^*(0)) = (Q(D))^2$. In the case when $m = \pm 1$, by Proposition 3.1, we have

$$Q(D \cup D^*(\pm 1)) = \frac{x}{2}((Q(D))^2 + Q(D \cup D^*(\infty))),$$

since $S_0 = 1$ and $S_{-1} = F_{\pm 1} = 0$. Then by the assumption, we have $\deg Q(D \cup D^*(\pm 1)) = 1 + 2\deg Q(D)$. In the case when $|m| \geq 2$, by Proposition 3.1, we have

$$Q(D \cup D^*(m)) = \left(\frac{x}{2} S_{|m|-1} - S_{|m|-2}\right)(Q(D))^2 + \left(\frac{x}{2} S_{|m|-1} + F_m\right) Q(D \cup D^*(\infty)).$$

By the assumption, we obtain that

$$\begin{aligned} \deg Q(D \cup D^*(m)) &= \deg\left(\left(\frac{x}{2} S_{|m|-1} - S_{|m|-2}\right)(Q(D))^2 + \left(\frac{x}{2} S_{|m|-1} + F_m\right) Q(D \cup D^*(\infty))\right) \\ &= \deg\left(\left(\frac{x}{2} S_{|m|-1} - S_{|m|-2}\right)(Q(D))^2\right) = |m| + 2\deg Q(D). \end{aligned}$$

(Note that $\deg S_{|m|} = |m|$ and $\deg F_m = |m| - 1$ if $|m| \geq 2$.) □

4. Alternating knots and quasi-alternating knots

Recall that the *crossing number* of a knot K is the minimum number of crossings in any diagram of K, denoted by $c(K)$ [3, 15, 10].

Proposition 4.1. *There are only finitely many alternating symmetric unions for a given partial knot.*

Proof. Let $\overline{K}$ be a knot with a symmetric union presentation with K as a partial knot. Then by [13, Theorem 2.6], we have $\det(\overline{K}) = \det(K)^2$. Suppose that $\overline{K}$ is alternating. Then by

[11] and [18], we have $\deg Q_{\overline{K}}(x) = c(\overline{K}) - 1$. Since any alternating knot is quasi-alternating, by [21], we have $\deg Q_{\overline{K}}(x) < \det(\overline{K}) = \det(K)^2$. Thus we obtain that $c(\overline{K}) - 1 < \det(K)^2$. □

Remark 4.2. Let $\overline{K}$ be a knot with a symmetric union presentation with K as a partial knot. If K is the trefoil knot 3_1, then, by using the last inequality in the proof of Proposition 4.1, we conclude that $\overline{K}$ is alternating if and only if $\overline{K}$ is one of 6_1, its mirror image or the square knot (which is the connected sum of 3_1 and its mirror image).

Proposition 4.3. *Let K be a non-trivial knot with a symmetric union presentation such that the determinant of the partial knot is equal to one. Then K is not quasi-alternating.*

Proof. By [13, Theorem 2.6], we have $det(K) = 1$. Suppose that K is quasi-alternating. If K satisfies $det(K) = 1$, then K is the unknot by the definition. This is contrary to the assumption. □

By Proposition 4.3, we know that the number of quasi-alternating knots with symmetric union presentations with the unknot as the partial knot is one.

Example 4.4. Let K be the pretzel knot $P(q, p, -q)$ $(p \geq 2, q \geq 1)$ as in Figure 6.

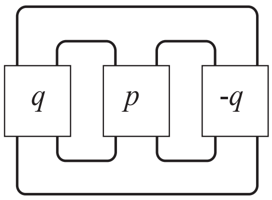

Fig. 6. The pretzel knot $P(q, p, -q)$

We note that $P(q, p, -q)$ has a symmetric union presentation with one twist region with the *torus knot* $T(2, q)$ [10] as the partial knot. By a result of Greene [5], we know that K is quasi-alternating if and only if $q > p$. So if we fix q, then we only have a finite number of quasi-alternating symmetric unions as the pretzel knot.

Proof of Theorem 1.1. By Corollary 3.2, we have $\deg Q(D \cup D^*(m)) = |m| + 2\deg Q(D)$. By a result in [21], if $D \cup D^*(m)$ represents a quasi-alternating knot, then we know that $|m| + 2\deg Q(D) = Q(D \cup D^*(m)) < \det(D \cup D^*(m)) = \det(D)^2$. Thus we have $|m| < \det(D)^2 - 2\deg Q(D)$. This completes the proof. □

Proof of Corollary 1.2. By the assumption, we know that $\deg Q(D \cup D^*(\infty)) = \deg(2x^{-1} - 1) = 0$. Since $Q(D) \neq 1$ if and only if $\deg Q(D) > 0$, if $Q(D) \neq 1$, then by Theorem 1.1, we obtain the result. □

5. Knots with symmetric union presentation with one twist region

Let K_n^m $(m, n \in \mathbb{Z}, n \geq 1)$ be the knot with symmetric union presentation $D_n \cup D_n^*(m)$ as shown in Figure 7. (Note that D_n represents the *twist knot* $W(n)$ with n twists.)

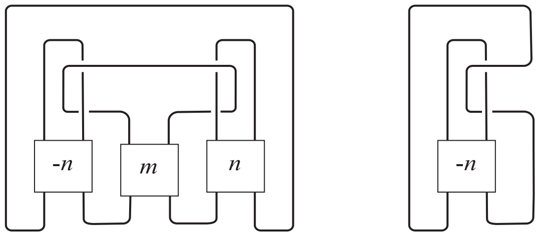

Fig. 7. A symmetric union K_n^m and its partial knot $W(n)$.

Proposition 5.1. *K_1^m is quasi-alternating if and only if $|m| \leq 2$.*

Proof. We know that K_1^m is equivalent to the pretzel knot $P(3, m, -3)$ as shown in Figure 8.

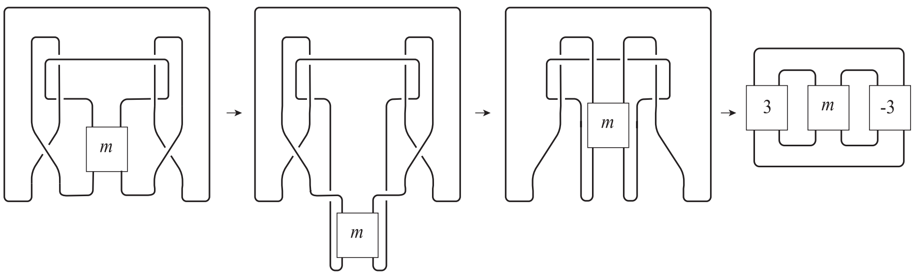

Fig. 8. The knot K_1^m.

In the case when $m \geq 2$, as in Example 4.4, we know that $P(3, m, -3)$ is quasi-alternating if and only if $m \leq 2$. In the case when $m \leq -2$, since $P(3, m, -3)$ is the mirror image of $P(3, -m, -3)$, we also know that $P(3, m, -3)$ is quasi-alternating if and only if $|m| \leq 2$ by the same way. In the case when $m = \pm 1$, K_1^m is either 6_1 or its mirror image. So it is quasi-alternating. The knot K_1^0 is the connected sum of an alternating knot and its mirror image. □

In general, we have the following.

Proposition 5.2. *If K_n^m is quasi-alternating then $|m| \leq 4n^2 + 2n - 3$.*

Proof. Since D_n is a *reduced alternating diagram* with $n + 2$ crossings, we know that $c(W(n)) = c(D_n) = n + 2$. (See [10, Chapter 8] for example.) Then we have $\deg Q(D_n) = c(W(n)) - 1 = (n + 2) - 1 = n + 1 > 0$ and $\deg Q(D \cup D^*(\infty)) = \deg(2x^{-1} - 1) = 0$. Thus, by Corollary 3.2, we have $\deg Q(K_n^m) = |m| + 2n + 2$. On the other hand, it is easily seen that $\det(W(n)) = 2n + 1$. So we have $\det(K_n^m) = (2n + 1)^2$ by [13, Theorem 2.6]. Note that a (non-trivial) symmetric union is not $T(2, q)$ since the *signature* of a slice knot is zero and the sigunature of $T(2, q)$ $(q > 1)$ is non-zero [19]. Then, by a result of [24], if K_n^m is quasi-alternating, then we know that $|m| + 2n + 2 = \deg Q(K_n^m) \leq \det(K_n^m) - 2 = (2n + 1)^2 - 2 = 4n^2 + 4n - 1$. Thus we have $|m| \leq 4n^2 + 2n - 3$. □

REMARK 5.3. In the case when $n = 2$, we know that if K_2^m is quasi-alternating then $|m| \leq 17$ by Proposition 5.2. In fact, K_2^1 is (the mirror image of) 8_8 which is alternating. The knot K_2^2 is 10_{137} which is not alternating, but 10_{137} is quasi-alternating by a result of [2]. The knot K_2^3 is $11n50$ [17] which is not quasi-alternating by a result of [6]. The knot K_2^4 is $12n145$ which is quasi-alternating by a result of [8]. We expect that K_2^m is not quasi-alternating if $|m| \geq 5$. In fact, K_2^m is a special case of *Kanenobu knots* [9] which are considered in Section 6.

6. Knots with a symmetric union presentation with two twist regions

In this section, we consider knots with the symmetric union presentation $D \cup D^*(m_1, m_2)$.

Proposition 6.1. *If* $|m_1|, |m_2| \geq 2$*, then we have*

$$\begin{aligned}
Q(D \cup D^*(m_1, m_2)) = {} & S_{|m_1|-1}S_{|m_2|-1}Q\Big(D \cup D^*\Big(\frac{m_1}{|m_1|}, \frac{m_2}{|m_2|}\Big)\Big) \\
& + \Big(S_{|m_1|-2}S_{|m_1|-2} - \frac{x}{2}\Big(S_{|m_1|-1}S_{|m_2|-2} + S_{|m_1|-2}S_{|m_2|-1}\Big)\Big)Q(D \cup D^*(0, 0)) \\
& + \Big(\frac{x}{2}\Big(F_{m_1}S_{|m_2|-1} - S_{|m_1|-1}S_{|m_2|-2}\Big) - F_{m_1}S_{|m_2|-2}\Big)Q(D \cup D^*(\infty, 0)) \\
& + \Big(\frac{x}{2}\Big(F_{m_2}S_{|m_1|-1} - S_{|m_1|-2}S_{|m_2|-1}\Big) - F_{m_2}S_{|m_1|-2}\Big)Q(D \cup D^*(0, \infty)) \\
& + \Big(\frac{x}{2}\Big(F_{m_2}S_{|m_1|-1} + F_{m_1}S_{|m_2|-1}\Big) + F_{m_1}F_{m_2}\Big)Q(D \cup D^*(\infty, \infty)).
\end{aligned}$$

Proof. First, we consider the case when $m_1, m_2 \geq 2$. By using the same method as in the proof of Proposition 3.1, we have
$Q(D \cup D^*(m_1, m_2)) = S_{m_1-1}Q(D \cup D^*(1, m_2)) - S_{m_1-2}Q(D \cup D^*(0, m_2)) + F_{m_1}Q(D \cup D^*(\infty, m_2))$.
(Here we consider a diagram of the unknot as in Figure 9 in place of the diagram as in Figure 5.)

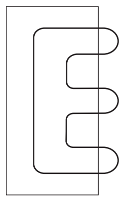

Fig.9. A diagram of the unknot

In the same way
$Q(D \cup D^*(1, m_2)) = S_{m_2-1}Q(D \cup D^*(1, 1)) - S_{m_2-2}Q(D \cup D^*(1, 0)) + F_{m_2}Q(D \cup D^*(1, \infty))$,
$Q(D \cup D^*(0, m_2)) = S_{m_2-1}Q(D \cup D^*(0, 1)) - S_{m_2-2}Q(D \cup D^*(0, 0)) + F_{m_2}Q(D \cup D^*(0, \infty))$,
and $Q(D \cup D^*(\infty, m_2)) = S_{m_2-1}Q(D \cup D^*(\infty, 1)) - S_{m_2-2}Q(D \cup D^*(\infty, 0)) + F_{m_2}Q(D \cup D^*(\infty, \infty))$.
Then we obtain that
$Q(D \cup D^*(m_1, m_2)) = S_{m_1-1}S_{m_2-1}Q(D \cup D^*(1, 1)) - S_{m_1-1}S_{m_2-2}Q(D \cup D^*(1, 0))$
$+ F_{m_2}S_{m_1-1}Q(D \cup D^*(1, \infty)) - S_{m_1-2}S_{m_2-1}Q(D \cup D^*(0, 1)) + S_{m_1-2}S_{m_2-2}Q(D \cup D^*(0, 0))$
$- F_{m_2}S_{m_1-2}Q(D \cup D^*(0, \infty)) + F_{m_1}S_{m_2-1}Q(D \cup D^*(\infty, 1)) - F_{m_1}S_{m_2-2}Q(D \cup D^*(\infty, 0))$

$+ F_{m_1} F_{m_2} Q(D \cup D^*(\infty, \infty))$.

Now, by the definition of Q-polynomial, we have

$$Q(D \cup D^*(1,0)) = \frac{x}{2}\Big(Q(D \cup D^*(0,0)) + Q(D \cup D^*(\infty, 0))\Big),$$

$$Q(D \cup D^*(1,\infty)) = \frac{x}{2}\Big(Q(D \cup D^*(0,\infty)) + Q(D \cup D^*(\infty, \infty))\Big),$$

$$Q(D \cup D^*(0,1)) = \frac{x}{2}\Big(Q(D \cup D^*(0,0)) + Q(D \cup D^*(0, \infty))\Big),$$

and $Q(D \cup D^*(\infty,1)) = \frac{x}{2}\Big(Q(D \cup D^*(\infty,0)) + Q(D \cup D^*(\infty, \infty))\Big)$.

Then by applying these equations to the formula obtained above, we obtain the result.

In the case when $m_1 \leq -2$ and $m_2 \geq 2$, we can obtain the formula by the same method. We can settle the case $m_1, m_2 \leq -2$ and the case when $m_2 \leq -2$ and $m_1 \geq 2$ from the case $m_1, m_2 \geq 2$ and the case $m_1 \leq -2$ and $m_2 \geq 2$, since $Q(D \cup D^*(m_1, m_2)) = Q(D \cup D^*(-m_1, -m_2))$. □

Corollary 6.2. *If $D \cup D^*(\infty, 0), D \cup D^*(0, \infty)$ and $D \cup D^*(\infty, \infty)$ are trivial link diagrams and* $\deg Q\Big(D \cup D^*\big(\frac{m_1}{|m_1|}, \frac{m_2}{|m_2|}\big)\Big) > \deg Q(D \cup D^*(0,0)) > 0$, *then we have*

$$\deg Q(D \cup D^*(m_1, m_2)) = |m_1| + |m_2| - 2 + \deg Q\Big(D \cup D^*\big(\frac{m_1}{|m_1|}, \frac{m_2}{|m_2|}\big)\Big).$$

Proof. Let

$$A_1 = S_{|m_1|-1} S_{|m_2|-1} Q\Big(D \cup D^*\big(\frac{m_1}{|m_1|}, \frac{m_2}{|m_2|}\big)\Big),$$

$$A_2 = \Big(S_{|m_1|-2} S_{|m_1|-2} - \frac{x}{2}\big(S_{|m_1|-1} S_{|m_2|-2} + S_{|m_1|-2} S_{|m_2|-1}\big)\Big) Q(D \cup D^*(0,0)),$$

$$A_3 = \Big(\frac{x}{2}\big(F_{m_1} S_{|m_2|-1} - S_{|m_1|-1} S_{|m_2|-2}\big) - F_{m_1} S_{|m_2|-2}\Big) Q(D \cup D^*(\infty, 0)),$$

$$A_4 = \Big(\frac{x}{2}\big(F_{m_2} S_{|m_1|-1} - S_{|m_1|-2} S_{|m_2|-1}\big) - F_{m_2} S_{|m_1|-2}\Big) Q(D \cup D^*(0, \infty)),$$

$$\text{and } A_5 = \Big(\frac{x}{2}\big(F_{m_2} S_{|m_1|-1} + F_{m_1} S_{|m_2|-1}\big) + F_{m_1} F_{m_2}\Big) Q(D \cup D^*(\infty, \infty)).$$

By the assumption, we have $Q(D \cup D^*(\infty, 0)) = Q(D \cup D^*(0, \infty)) = 2x^{-1} - 1$ and $Q(D \cup D^*(\infty, \infty)) = 4x^{-2} - 4x^{-1} + 1$. Then, we have

$$\deg A_1 = |m_1| + |m_2| - 2 + \deg Q\Big(D \cup D^*\big(\frac{m_1}{|m_1|}, \frac{m_2}{|m_2|}\big)\Big),$$

$$\deg A_2 = |m_1| + |m_2| - 2 + \deg Q(D \cup D^*(0,0)),$$

$$\deg A_3 \leq |m_1| + |m_2| - 1,$$

$$\deg A_4 \leq |m_1| + |m_2| - 1,$$

$$\text{and } \deg A_5 \leq |m_1| + |m_2| - 1.$$

(Here we use the fact that $S_{|m|}$ has leading coefficient $2^{|m|-1}$ to obtain the formula of $\deg A_2$.) Thus by the assumption and Proposition 6.1, we obtain that

$$\deg Q(D \cup D^*(m_1, m_2)) = \deg A_1 = |m_1| + |m_2| - 2 + \deg Q\Big(D \cup D^*\big(\frac{m_1}{|m_1|}, \frac{m_2}{|m_2|}\big)\Big).$$ □

EXAMPLE 6.3. We consider the following symmetric unions $D \cup D^*(m_1, m_2)$ $(m_1, m_2 \in \mathbb{Z})$ (Figure 10) which are called *Kanenobu knots* [9].

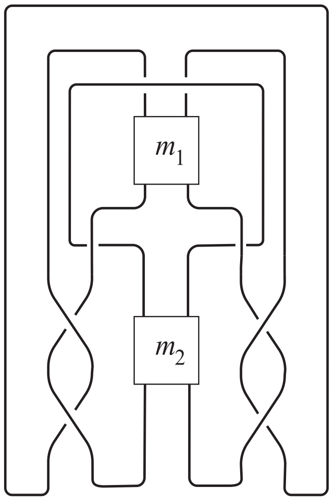

Fig. 10. Kanenobu knots

We note that $D \cup D^*(\infty, 0), D \cup D^*(0, \infty)$ and $D \cup D^*(\infty, \infty)$ are trivial link diagrams. In the case when $m_1, m_2 \geq 2$, we have $\deg Q(D \cup D^*(1, 1)) = \deg Q(10_{155}) = 8 > 6 = \deg Q(D \cup D^*(0, 0))$. Then, by Corollary 6.2, we have

(A) $\deg Q(D \cup D^*(m_1, m_2)) = m_1 + m_2 + 6$.

By a result of [24], if $D \cup D^*(m_1, m_2)$ is quasi-alternating, then we know that $m_1 + m_2 + 6 \leq \det D \cup D^*(m_1, m_2) - 2 = 23$ as in the proof of Proposition 5.2. Thus we have $m_1 + m_2 \leq 17$. In the case when $m_1 \geq 2, m_2 \leq -2$, we have $\deg Q(D \cup D^*(1, -1)) = \deg Q(8_9) = 7 > 6 = \deg Q(D \cup D^*(0, 0))$. Then, by Corollary 6.2, we have

(B) $\deg Q(D \cup D^*(m_1, m_2)) = m_1 + m_2 + 5$.

By a result of [24], if $D \cup D^*(m_1, m_2)$ is quasi-alternating, then we know that $m_1 + m_2 + 5 \leq \det D \cup D^*(m_1, m_2) - 2 = 23$. Thus we have $m_1 + |m_2| \leq 18$. Therefore if $D \cup D^*(m_1, m_2)$ is quasi-alternating, then we know that

(1) $|m_1| + |m_2| \leq 17$ if either $m_1, m_2 \geq 2$ or $m_1, m_2 \leq -2$,

(2) $|m_1| + |m_2| \leq 18$ if either $m_1 \geq 2, m_2 \leq -2$ or $m_2 \geq 2, m_1 \leq -2$.

(Compare these inequalities with a result in [21, Proof of Corollary 3.3].)

REMARK 6.4. The equations (A) and (B) were shown in [7] and [22].

EXAMPLE 6.5. We consider the following symmetric unions $D \cup D^*(m_1, m_2)$ $(m_1, m_2 \in \mathbb{Z})$ (Figure 11). We note that $D \cup D^*(\infty, 0), D \cup D^*(0, \infty)$ and $D \cup D^*(\infty, \infty)$ are trivial link diagrams and we have

$Q(D \cup D^*(1, 1)) = -15 + 16x + 48x^2 - 38x^3 - 48x^4 + 38x^5 + 20x^6 - 22x^7 - 6x^8 + 6x^9 + 2x^{10}$,

$Q(D \cup D^*(1, -1)) = 17 - 16x - 48x^2 + 34x^3 + 54x^4 - 26x^5 - 32x^6 + 6x^7 + 10x^8 + 2x^9$,

and $\det D \cup D^*(m_1, m_2) = 49$.

Then by using the same method as in Example 6.3, we conclude that if $D \cup D^*(m_1, m_2)$ is quasi-alternating then we should have:

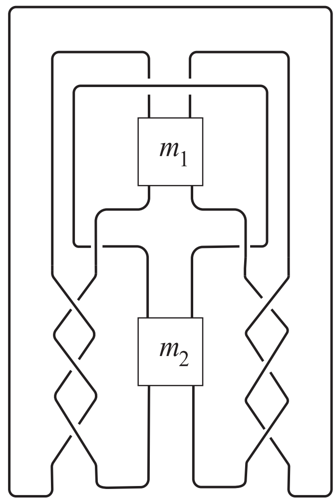

Fig. 11. An example of symmetric union presentation with two twist regions.

(1) $|m_1| + |m_2| \leq 39$ if either $m_1, m_2 \geq 2$ or $m_1, m_2 \leq -2$,
(2) $|m_1| + |m_2| \leq 40$ if either $m_1 \geq 2, m_2 \leq -2$ or $m_2 \geq 2, m_1 \leq -2$.

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